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On a theorem of de Franchis

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1 Introduction

Let $X$ be a compact Riemann surface of genus $g (> 1)$. De Franchis [1] stated the following:

**Theorem 1 (de Franchis)** (a) For a fixed compact Riemann surface $Y$ of genus $> 1$, the number of nonconstant holomorphic maps $X \to Y$ is finite.

(b) There are only finitely many compact Riemann surfaces $Y_i$ of genus $> 1$ which admit a nonconstant holomorphic map from $X$.

The second statement (b) is often attributed to Severi. After knowing the finiteness of maps, we may ask if there exists a upper bound depending only on some topological invariant, for example, the genus $g$. Related to the statement (a), the author [4] showed that the bound is smaller than $(cg)^{2g}$ for some constant $c$.

Now, we consider a bound for holomorphic maps when $Y$ is not fixed, that is, we estimate the number of all nonconstant holomorphic maps from $X$ to other Riemann surfaces. Let $f_i : X \to Y_i$ be nonconstant holomorphic maps for $i = 1, 2$. We say that $f_1$ and $f_2$ are isomorphic if and only if there is a conformal map $h : Y_1 \to Y_2$ such that $h \circ f_1 = f_2$. Let $\mathcal{I}_\gamma(X)$ denote the set of all isomorphic classes of nonconstant holomorphic maps into compact Riemann surfaces of genus $\gamma > 1$, and denote $\mathcal{I}(X) = \bigcup_{g > \gamma > 1} \mathcal{I}_\gamma(X)$. By the theorem of de Franchis, we see that $\# \mathcal{I}(X)$ is finite. In 1983 Howard and Sommese [2] first showed that there is a bound on $\# \mathcal{I}(X)$ depending only on $g$.

Let

$$M(g) = \max_X \{\# \mathcal{I}(X)\},$$
where the maximum is taken over all Riemann surfaces $X$ of genus $g$. It is an interesting problem to determine the exact rate of growth of $M(g)$. The author [5] showed

$$M(g) \leq (cg)^{5g}$$

for some constant $c$ and it was the best upper bound depending only on $g$.

In this note we will improve the bound and show

$$M(g) \leq (cg)^{2g}$$

for some constant $c$.

On the other hand, Kani [3] also constructed a sequence of Riemann surfaces of genera $g_1 < g_2 < \ldots < g_n < \ldots$, such that the number of isomorphic classes of nonconstant holomorphic maps of each Riemann surface is larger than $\exp(c(\log(g_n))^2)$ for some constant $c > 0$ (independent of $n$). It implies that $M(g)$ cannot be bounded by any polynomial in $g$.

## 2 The bound

In the following, we will refer to [5] for all of the notation and lemmata. In [5], the leading term of the upper bound was depend on Lemma 3 (p.3060) and the Proposition (p.3062). We improve them as follows.

**Lemma 3'** Let $f_1 : X \rightarrow Y_1$ be a holomorphic map of degree $d$, and $f_1 : J(X) \rightarrow J(Y_1)$ be the homomorphism induced by $f_1$. Take an arbitrary $u \in t_{f_1}(J(Y_1))$. Then, the number of isomorphic classes of holomorphic maps $f_i : X \rightarrow Y_i$ of degree $d$ such that the dual map $t_{f_i} : \overline{J(Y_i)} \rightarrow J(X)$ of the induced homomorphism $f_i$ satisfies $u \in t_{f_i}(\overline{J(Y_i)})$ is at most $\left( \frac{2g - 2}{d} \right) \times \left( \frac{4g - 4}{d} \right)$.

In [5], the conclusion was $\left( \frac{2g - 2}{d} \right) \times (2g-1)^d$ which is now replaced by $\left( \frac{2g - 2}{d} \right) \times \left( \frac{4g - 4}{d} \right)$.

**Proof.** The assumption means that there exist holomorphic differentials $\phi_1$ on $Y_1$ and $\phi_i$ on $Y_i$ such that their pull backs satisfy $f_{1*}\phi_1 = f_{i*}\phi_i$. 

Then, for a zero $p_{01}$ of $\phi_1$, the number of possible $f_i^{-1}(p_{01})$ (counting multiplicities) that can occur is at most $\binom{2g-2}{d}$. After determining $\phi = f_1 \cdot \phi_1$ and $f_i^{-1}(p_{01})$, we can show that there are at most $\binom{4g-4}{d}$ possible isomorphic classes of holomorphic maps of degree $d$ as follows.

Let $f_i : X \to Y_i$ be holomorphic maps ($i = 1, 2$). Suppose that there are holomorphic differentials $\phi_1$ and $\phi_2$ on $Y_1$ and $Y_2$, respectively, with $f_1 \cdot \phi_1 = f_2 \cdot \phi_2$, and there is a zero $p_{01}$ (resp. $p_{02}$) of $\phi_1$ (resp. $\phi_2$) satisfying $f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. We put $\phi = f_1 \cdot \phi_1 = f_2 \cdot \phi_2$.

Let $\tilde{p}_0 \in f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. Take a sufficiently small neighbourhood $U_{\tilde{p}_0}$ (resp. $U_{p_{01}}$) of $\tilde{p}_0$ (resp. $p_{01}$) so that there is no zero of $\phi$ (resp. $\phi_i$) on $U_{\tilde{p}_0}$ (resp. $U_{p_{01}}$) except $\tilde{p}_0$ (resp. $p_{01}$), and that $f_i(U_{\tilde{p}_0}) \subset U_{p_{0i}}$ ($i = 1, 2$). We may take a local coordinate $z$ (resp. $z_i$) on $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) such that $z(\tilde{p}_0) = 0$ (resp. $z_i(p_{0i}) = 0$) and the differential is written as

$$\phi = z^m dz \quad \text{(resp. } \phi_i = z_i^n dz_i).$$

Recalling that $f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$, we see $n_1 = n_2$ and we will denote it by $n$ for brevity. We take two real lines $\gamma_i : [0, a) \to U_{p_{0i}}$ with $\gamma_i(t) = t \in \mathbb{R}$ in the local coordinates $z_i$ ($i = 1, 2$). For an arbitrary $\tilde{p} \in U_{\tilde{p}_0} \setminus \{\tilde{p}_0\}$,

$$\int_0^{\tilde{p}} z^m dz = \int_0^{f_1(\tilde{p})} z_1^n dz_1 = \int_0^{f_2(\tilde{p})} z_2^n dz_2,$$

hence the number of possible positions for the set of lifts of $\gamma_1$ (thus also those of $\gamma_2$) in $U_{\tilde{p}_0}$ is at most $m + 1$. Accordingly, the total number of possible positions for the set of all the lifts of $\gamma_1$ is at most $\binom{4g-4}{d}$.

Let $\{\tilde{p}_{0j}\}_{j=1}^N = f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. Suppose that, for every $\tilde{p}_{0j} \in f_1^{-1}(p_{01})$, $U_{\tilde{p}_{0j}} \cap f_1^{-1}(\gamma_1) = U_{\tilde{p}_{0j}} \cap f_2^{-1}(\gamma_2)$, that is, the set of lifts of $\gamma_1$ coincide with that of $\gamma_2$. Then, it is easy to see that we can define a local conformal map $h : f_1(U_{\tilde{p}_{0j}}) \to f_2(U_{\tilde{p}_{0j}})$ such that $h \circ f_1|_{U_{\tilde{p}_{0j}}} = f_2|_{U_{\tilde{p}_{0j}}}$. We want to extend it to a global conformal map from $Y_1$ to $Y_2$, and actually it is possible. Indeed, for an arbitrary point $p \in Y_1$, we will draw a curve $c$ from $p_{01}$ to $p$ avoiding branch points of $f_1$ other than possibly at $p_{01}$ and $p$. Let $\tilde{c}$ and $\tilde{c}'$ be two lifts of $c$ by $f_1$. Then, we see that $f_2(\tilde{c}) = f_2(\tilde{c}')$ since $h \circ f_1$ is well-defined near $\tilde{p}_{0j}$ ($j = 1, \ldots, N$). It implies that $h$ is well-defined on $Y_1$. It is easy to see that $h$ is invertible. □

**Proposition** Let $f_i : X \to Y_i$ be nonconstant holomorphic maps, and $F_i$ be the rational representations of the endomorphisms associated with $f_i$ ($i = 1, 2$). Suppose
that, for some $k < 2g$,

$$\begin{align*}
{^t}\mathcal{F}_1a_1 = \ldots = {^t}\mathcal{F}_1a_{k-1} &= 0, \\
{^t}\mathcal{F}_2a_1 = \ldots = {^t}\mathcal{F}_2a_{k-1} &= 0,
\end{align*}$$

and that there exists some integer $l > 2g - 2$ such that $t\mathcal{F}_1a_k \equiv t\mathcal{F}_2a_k \pmod{l}$ holds. Then $t\mathcal{F}_1a_k = t\mathcal{F}_2a_k$.

If, in addition, $Y_1$ and $Y_2$ are of the same genus $\gamma$, then the assumption $l > 2g - 2$ can be replaced by $l > (2g - 2)/(\gamma - 1)$.

In [5], we assumed $l > (2g - 2)^2$. But in Proposition', we only need $l > 2g - 2$.

**Proof.** Let $D = {\mathcal{F}_1} - {\mathcal{F}_2}$. Then, $D$ is the rational representation of some endomorphism of $J(X)$. By an easy calculation, we see $tD' = tD$. We note that $tD'x, a_1, \ldots, a_{k-1}$ are linearly independent for any vector $x \in \mathbb{R}^{2g}$ if $tD'x$ is not zero. Indeed, using Lemma 1 in [5], we see $(tD'x, a_j)_X = (x, {^t}Da_j)_X = 0$ for $j = 1, \ldots, k-1$ by the assumption. Thus, $tD'x, a_1, \ldots, a_{k-1}$ are linearly independent. By the assumption, $tD'a_k \equiv 0 \pmod{l}$ thus the vector $tD'a_k$ can be written in the form $tD'a_k = l \times n$, where $n \in \mathbb{Z}^{2g}$. Thus, if it is not 0, then

$$||{^t}Da_k|| \geq l \lambda_k.$$ 

We also have

$$||{^t}Da_k|| \leq ||{^t}\mathcal{F}_1a_k|| + ||{^t}\mathcal{F}_2a_k|| \leq d_1||a_k|| + d_2||a_k||,$$

where $d_i$ is the degree of $f_i$ ($i = 1, 2$). The first inequality is just the triangle inequality, and the second one is obtained by Lemma 2.

Therefore, we have

$$||{^t}Da_k|| \leq ||a_k||(d_1 + d_2) = (d_1 + d_2)\lambda_k.$$ 

By Riemann-Hurwitz formula, $d_i \leq g - 1$ and we see that $tDa_k$ must be 0 since $l > 2(g - 1)$.

A little modification of above argument lead us to the conclusion for the case $Y_1$ and $Y_2$ are of the same genus $\gamma$. $\square$

Now we will get the improved bound. Just the same consideration as in[5, p.3063], we have

$$\#\mathcal{I}_\gamma(X) < \sum_{d > 1}(2g - 2\gamma + 1) \times \left\{ (\frac{2g - 2}{\gamma - 1}) + 1 \right\}^{2g} \times \left( \frac{2g - 2}{d} \right) \times \left( \frac{4g - 4}{d} \right).$$
Observing \( \binom{m}{d} \leq 2^m \), we see that the right hand side is smaller than

\[
\left\{ \left( \frac{2g-2}{\gamma-1} \right) + 1 \right\}^{2g} \times 2^{2g-2} \times 2^{4g-4} \times (2g - 2\gamma + 1)(g - \gamma)/\gamma - 1).
\]

Summing up for all possible \( \gamma \), we get

\[ M(g) \leq (cg)^{2g} \]

for some constant \( c \).

**References**


