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Kyoto University
Complexified Penner’s coordinates and its applications

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1 Penner’s $\lambda$-lengths

1.1 A coordinate-system for Teichmüller space

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, a model of hyperbolic plane and

$$SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$ 

Then $PSU(1, 1)$ is the group of orientation preserving hyperbolic motions of $\mathbb{D}$.

Let $G = G_{g,n}$ be the punctured surface group of type $(g, n)$, where $2g - 2 + n > 0$:

$$G = \langle a_1, b_1, ..., a_g, b_g, d_1, ..., d_n : (\prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}) d_1 \cdots d_n = 1 \rangle.$$

A point of the Teichmüller space $\mathcal{T} = \mathcal{T}_{g,n}$ is a class of faithful Fuchsian representations of $G$ into $PSU(1, 1)$ which have finite covolume. We denote points in $\mathcal{T}$ by marked groups $\Gamma_m$, where $\Gamma$ is a Fuchsian group and $m : G \rightarrow \Gamma$ is an isomorphism.

Elements $D_1, ..., D_n$ in $\Gamma_m \in \mathcal{T}$ corresponding to $d_1, ..., d_n$ are parabolic. Choose a horocycle $H_k$ invariant under $D_k$ such that action of $D_k$ on $H_k$ is the translation of length one. Then the identification of $\Gamma_m$ with $(\Gamma_m, H_1, ..., H_n)$ gives the following statement.

$\mathcal{T}_{g,n}$ is naturally embedded in the decorated Teichmüller space $\tilde{\mathcal{T}}_{g,n}$.

Therefore, by restricting them to this embedded subspace, Penner’s $\lambda$-length coordinates for $\tilde{\mathcal{T}}_{g,n}$ give also global coordinates for the Teichmüller space $\mathcal{T}_{g,n}$.

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1.2 Distance between horocycles

Let \( p \) be a point of the unit circle. A horcycle \( h \) at \( p \) is a Euclidean circle in \( \mathbb{D} \) tangent at \( p \) to the unit circle. The point \( p \) is called the base point of \( h \).

Let \( h_1 \) and \( h_2 \) be horocycles based at different points \( p_1 \) and \( p_2 \) and \( \gamma \) the hyperbolic line between \( p_1 \) and \( p_2 \). Define

\[
\lambda = e^{\delta/2},
\]

where \( \delta \) is the signed length of the portion of the geodesic \( \gamma \) intercepted between the two horocycles \( h_1 \) and \( h_2 \). \( \delta > 0 \) if \( h_1 \) and \( h_2 \) are disjoint and \( \delta < 0 \) otherwise. In this way we can assign a positive number \( \lambda \) to the pair \((h_1, h_2)\).

1.3 \( \lambda \)-length of an ideal arc

Let \( S \) be the oriented closed surface of genus \( g \), \( P = \{p_1, \ldots, p_n\} \) a set of \( n \) points. An ideal arc \( c \) of \((S, P)\) is a path joining two points \( p_i \) and \( p_j \) in \( S - P \). The ideal arc \( c \) is simple if \( c \cap (S - P) \) is a simple arc.

Let \( \Gamma_m \in \mathcal{T}_{g,n} \), then there exists an orientation preserving homeomorphism

\[
f : S - P \rightarrow \mathbb{D}/\Gamma
\]

inducing \( m \). Let \( \gamma \) be the geodesic representative in the homotopy class of \( f(c) \) for the Poincaré metric of the punctured surface \( \mathbb{D}/\Gamma \). By the identification of \( \Gamma_m \) with \( (\Gamma_m, H_1, \ldots, H_n) \), the horocycles at the endpoints of \( \gamma \) defines the \( \lambda \)-length \( \lambda(c, \Gamma_m) \).

Let \( \Delta = \{c_1, c_2, \ldots, c_q\}, q = 6g - 6 + 3n \), be an ideal triangulation of \((S, P)\). Then

**Theorem 1** (Penner [1])

\[
\lambda_\Delta = \prod_{i=1}^{q} \lambda(c_i) : \mathcal{T}_{g,n} \rightarrow (\mathbb{R}_+)^q
\]

is an embedding.
The image of $\lambda_\Delta$ is a real algebraic variety determined by $n$ polynomials. A component of $S - \cup_{j=1}^{q}c_{j}$ is called a triangle in $\Delta$. The image of $\lambda_\Delta$ is a real algebraic variety determined by zero loci of $n$ algebraic equations $D_1, ..., D_n$, where $D_k$ is easily obtained by triangles abutting on the $k$th puncture $p_k$.

\[ D_k(\lambda_1, ..., \lambda_q) = \sum_{i=1}^{N} \frac{\lambda(e_i)}{\lambda(a_i)\lambda(b_i)} - 1. \]  

1.4 The Ptolemy identity

Let $\Delta = \{c_1, c_2, ..., c_q\}$ be an ideal triangulation of $(S, P)$. Let $e \in \Delta$ and $T_1$ and $T_2$ be triangles being on the different sides of $e$. It is possible that $T_1 = T_2$. Lift $T_1 \cup e \cup T_2$ to a quadrangle $Q = \tilde{T}_1 \cup \tilde{e} \cup \tilde{T}_2$ in $D$. Then $\tilde{e}$ is a diagonal of $Q$. Let $\tilde{f}$ be the other diagonal and project $\tilde{f}$ to an ideal arc $f$ in $T_1 \cup e \cup T_2$. Then

\[ \Delta' = (\Delta - \{e\}) \cup \{f\} \]

is another ideal triangulation of $(S, P)$. We say that $\Delta'$ arises from $\Delta$ by the elementary move on $e$. 

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Let \((\tilde{a}, \tilde{b}, \tilde{c})\) be the sides of \(\tilde{T}_1\) and \((\tilde{c}, \tilde{d}, \tilde{e})\) be the sides of \(\tilde{T}_2\). Suppose that \(\tilde{a}\) and \(\tilde{c}\) are opposite sides of \(Q\). Let \(a, b, c, d \in \Delta \cap \Delta'\) be the projections of \(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\). The following theorems are proved in Penner’s paper:

**Theorem 2** (the Ptolemy identity, Penner [1])

*The \(\lambda\)-lengths function satisfy the identity*

\[
\lambda(a)\lambda(c) + \lambda(b)\lambda(d) = \lambda(e)\lambda(f)
\]

This theorem describes the coordinate-change between \(\lambda_\Delta(T)\) and \(\lambda_\Delta'(T)\):

\[
\lambda_\Delta' \circ \lambda_\Delta^{-1}(\cdots, \lambda(a), \lambda(b), \lambda(c), \lambda(d), \lambda(e), \cdots) = (\cdots, \lambda(a), \lambda(b), \lambda(c), \lambda(d), \frac{\lambda(a)\lambda(c) + \lambda(b)\lambda(d)}{\lambda(e)}, \cdots)
\]

**Theorem 3** (Penner [1]) *For arbitrary ideal triangulations \(\Delta\) and \(\Delta'\) of \((S, P)\), there exists a finite sequence of ideal triangulations*

\[
\Delta = \Delta_0, \Delta_1, \cdots, \Delta_m = \Delta',
\]

*where each \(\Delta_i\) arises from \(\Delta_{i-1}\) by an elementary move.*
Using this theorem it can be shown that coordinate change between $\lambda$-length coordinates associated with two ideal triangulations is a bi-rational map:

**Theorem 4** If $\Delta$ and $\Delta'$ are ideal triangulations of $F$, then the coordinate change

\[
T \xrightarrow{\lambda_{\Delta}} \lambda_{\Delta}(T) \subset (\mathbb{R}_{+})^q
\]

extends to a rational transformation of $\mathbb{R}^q$

Let $\mathcal{MC} = \mathcal{MC}_{g,n}$ denote the mapping class group of $(S, P)$. Each $\varphi \in \mathcal{MC}$ acts on the Teichmüller space $T$. The theorem above yields

**Theorem 5** The correspondence

\[
\phi \mapsto \phi_* = \lambda_{\varphi^{-1}(\Delta)} \circ \lambda_{\Delta}^{-1}
\]

gives an isomorphism of $\mathcal{MC}$ to a group of rational transformations.

## 2 $SL(2, \mathbb{C})$-representation space of a punctured surface group

Let $\mathcal{R} = \mathcal{R}_{g,n}$ be the space of classes of faithful representations $[m]$ of the punctured surface group $G$ into $SL(2, \mathbb{C})$ such that $m(d_i)$ is parabolic with $\text{tr} m(d_i) = -2$ for $i = 1, 2, ..., n$. The Teichmüller space $T_{g,n}$ is a subspace of $\mathcal{R}_{g,n}$.

Our purpose is to give a coordinate-system for $\mathcal{R}_{g,n}$ whose restriction to $T_{g,n}$ coincides with Penner's $\lambda$-lengths coordinate-system.

### 2.1 Parabolic elements of $SL(2, \mathbb{C})$

Define

$\mathcal{P} = \{P \in SL(2, \mathbb{C}) : P$ is parabolic with $\text{tr} P = -2\}$.

If $P_1$ and $P_2 \in \mathcal{P}$ do not commute, then the square root of $-P_1 P_2$ in $SL(2, \mathbb{C})$

\[
Q = \pm \frac{1}{\sqrt{2 - \text{tr} P_1 P_2}} (I - P_1 P_2),
\]

is unique up to sign and satisfies

\[
P_2 = Q^{-1} P_1 Q.
\]

For the rest of this paper, the diagram

\[
P_1 \xrightarrow{Q} P_2
\]
will mean that $Q^2 = -P_1P_2$.

Cycles of parabolic elements

Let $P_1, \ldots, P_n, P_{n+1} = P_1 \in \mathcal{P}$. Suppose that no consecutive elements $P_i$ and $P_{i+1}$ commute. Let $Q_i$ be a square root of $-P_iP_{i+1}$, ($i = 1, 2, \ldots, n$). Then, since $P_{i+1} = Q_i^{-1}P_iQ_i$, $Q_1Q_2 \cdots Q_n$ commutes with $P_1$,

$$\text{tr}Q_1Q_2 \cdots Q_n = +2 \text{ or } -2.$$  \hfill (7)

Definition

$(Q_1, Q_2, \ldots, Q_n)$ is a $(+)$-system or a $(-)$-system according to if $\text{tr}Q_1Q_2 \cdots Q_n = +2$ or $-2$.

2.2 A trace identity of Ptolemy type

Let $P_1$, $P_2$, $P_3$ and $P_4$. Suppose that $P_i$ and $P_j$ do not commute unless $i = j$. Choose $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_5', Q_6' \in SL(2, \mathbb{C})$ so that

$$Q_1^2 = -P_1P_2, \quad Q_2^2 = -P_2P_3, \quad Q_3 = -P_3P_4,$$
$$Q_4^2 = -P_4P_1, \quad Q_5^2 = -P_5P_1, \quad Q_6 = -P_2P_4,$$
$$(Q_5')^2 = -P_1P_3, \quad (Q_6')^2 = -P_4P_2,$$

where

$$Q_5' = P_1Q_5P_1^{-1}, \quad Q_6' = P_4Q_6P_4^{-1}.$$

Theorem

If $(Q_1, Q_2, Q_5)$, $(Q_5', Q_3, Q_4)$ and $(Q_1, Q_6, Q_4)$ are $(-)$-systems, then

$$\text{tr}Q_5\text{tr}Q_6 = \text{tr}Q_1\text{tr}Q_3 + \text{tr}Q_2\text{tr}Q_4$$  \hfill (8)
3 Complexified $\lambda$-length

3.1 Definition of $\lambda$-length

A point of $\mathcal{R}$ is represented by a marked group $\Gamma_{m}$. Let $\mathcal{P}_{+}(\Gamma)$ be the set of parabolic elements in $[m(d_{1})] \cup \cdots \cup [m(d_{n})]$, where $[m(d_{i})]$ is the conjugacy class of $m(d_{i})$.

Let $c$ be an ideal arc in $(S, P)$. Then for each $\Gamma_{m} \in \mathcal{R}$, $c$ defines two parabolic elements $P_{1}, P_{2}$ of $\mathcal{P}_{+}(\Gamma)$, see the following figure. We define the $\lambda$-length of $c$ with respect to $\Gamma_{m}$ by

$$\lambda(c, \Gamma_{m}) = \text{tr}Q,$$

where $Q$ is a square root of $-P_{1}P_{2}$. The $\lambda$-length is defined up to sign.

3.2 $\lambda$-length coordinates for $\mathcal{R}_{g,n}$

Let $\Delta = (c_{1}, c_{2}, \ldots, c_{q})$ be an ideal triangulation of $(S, P)$. Let $T$ be a triangle in $\Delta$. $T$ inherits the orientation of the surface $S$. Label the sides of $T$ by $a$, $b$, $c$ in order. Then those sides determine matrices $Q_{a}$, $Q_{b}$, $Q_{c}$ whose traces give $\lambda$-lengths of $a$, $b$ and $c$ for $\Gamma_{m}$.

**Lemma 1** It is possible to choose branches of $\lambda$-length functions $\lambda(c_{1})$, $\lambda(c_{2})$, ..., $\lambda(c_{q})$ so that $(Q_{a}, Q_{b}, Q_{c})$ is a $(-)$-system for each triangle $T$ in $\Delta$.

With the choice of branches of $\lambda$-lengths as depicted in the lemma, we obtain

**Theorem 7** For each ideal triangulation $\Delta$,

$$\lambda_{\Delta} = \prod_{i=1}^{q} \lambda(c_{i}) : \mathcal{R}_{g,n} \rightarrow (\mathbb{C}^{*})^{q}$$

is an embedding. The image is contained in an algebraic variety.

3.3 Rational representation of the mapping class group

As in the case of $T$, the Ptolemy identity (8) yields

**Theorem 8** The mapping class group $\mathcal{MC}$ acts on $\mathcal{R}$ as a group of rational transformations.
4 Invariant holomorphic two-form

Let $T_1, \ldots, T_p$, $p = 4g - 2$, be triangles in an ideal triangulation of a once-punctured surface. Let the sequence of sides $a_i, b_i, c_i$ of $T_i$ agree with the positive orientation of $T_i$, then the 2-form

$$\sum_{i=1}^{p} (d\log \lambda(a_i) \wedge d\lambda(b_i) + d\log \lambda(b_i) \wedge d\log \lambda(c_i) + d\log \lambda(c_i) \wedge d\log \lambda(a_i))$$  (10)

is invariant under the mapping class group $\mathcal{MC}$. The proof is similar to the one of the corresponding result in [2].

5 A characterization of the rational map induced by a mapping class

5.1 Example: Once punctured torus

The Teichmüller space $T_{1,1}$ of once punctured tori is represented as the subspace of $(\mathbb{R}_+)^3$ defined by

$$x^2 + y^2 + z^2 = xyz,$$  (11)

where $x, y, z$ are $\lambda$-length functions related to an (essentially unique) triangulation of the once punctured torus (or $x, y, z$ are trace functions $\text{tr}_A, \text{tr}_B, \text{tr}_{AB}$, with $\{A, B\}$ the canonical generator-system of $G_{1,1}$.)

The mapping class group $\mathcal{MC}_{1,1}$ has generators

$$\sigma(x, y, z) = (x, z, \frac{x^2 + z^2}{y}) \quad \text{and} \quad \tau(x, y, z) = (\frac{x^2 + y^2}{z}, y, x),$$

with relations

$$\tau \circ \sigma)^3 = 1, \quad (\sigma \circ \tau \circ \sigma)^2 = 1.$$

Since $\mathcal{MC}_{1,1}$ acts on $T_{1,1}$, the group of rational transformations generated by $\sigma$ and $\tau$ preserves the equation (11) and $(x, y, z) = (3, 3, 3)$ gives integer solutions of (11).

**Theorem 9 (Markoff)** All positive integer solutions of (11) are in the orbit of $(3, 3, 3)$ under the action of $\mathcal{MC}_{1,1}$.

The viewpoint of understanding the Markoff transformations as mapping classes acting on $T_{1,1}$ is given in Penner's paper [1].

With $\lambda$-length coordinates, the Teichmüller space $T_{g,n}$ is determined by $n$ algebraic equations and the group of rational transformations induced by the mapping class group $\mathcal{MC}_{g,n}$ keep this space. So we can pursue analogies of the above result.
5.2 Example: twice punctured torus

Let $\Delta$ be the ideal triangulation of the twice punctured torus as depicted in the following figure.

![Twice punctured torus](image)

Consider the $\lambda$-lengths

$$\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e$$

associated with $\Delta$. Then it holds that $\lambda_e = \lambda_f$. The Teichmüller space $\mathcal{T}_{1,2}$ (or the space $\mathcal{R}_{1,2}$) is represented by the $\lambda$-lengths as the space

$$\frac{\lambda_e}{\lambda_a \lambda_b} + \frac{\lambda_a}{\lambda_b \lambda_e} + \frac{\lambda_b}{\lambda_a \lambda_e} + \frac{\lambda_c}{\lambda_d \lambda_e} + \frac{\lambda_d}{\lambda_c \lambda_e} = 1$$

or

$$\lambda_c \lambda_d (\lambda_a^2 + \lambda_b^2 + \lambda_e^2) + \lambda_a \lambda_b (\lambda_c^2 + \lambda_d^2 + \lambda_e^2) = \lambda_a \lambda_b \lambda_c \lambda_d \lambda_e. \quad (12)$$

The mapping class group $\mathcal{MC}_{1,2}$ (as a group of rational transformations) has generators

$$\omega_{1*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) = (\lambda_d, \lambda_b, \lambda_c, \frac{\lambda_a^2 + \lambda_b^2}{\lambda_c}, \lambda_e)$$

$$\omega_{2*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) = (\lambda_d, \lambda_a, \lambda_b, \lambda_c, \frac{\lambda_a \lambda_c + \lambda_b \lambda_c}{\lambda_e})$$

$$\omega_{3*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) = (\lambda_a, \frac{\lambda_b^2 + \lambda_e^2}{\lambda_c}, \lambda_b, \lambda_d, \lambda_e),$$

with relations

$$\omega_{2*}^2 \omega_{1*} \omega_{2*} = \omega_{3*} \quad \omega_{1*} \omega_{3*} = \omega_{3*} \omega_{1*}$$

$$(\omega_{1*}\omega_{2*})^3 = 1, \quad (\omega_{3*}\omega_{2*})^3 = 1$$

The point $p = (6, 6, 6, 6, 6)$ gives integer solutions of (12). An analogous result to the Markoff equation holds:

**Theorem 10** The orbit $\{\varphi_*(6, 6, 6, 6, 6) : \varphi \in \mathcal{MC}_{1,2}\}$, gives integer solutions of (12), but not all of its integer solutions.
5.3 Diophantine equations

We consider a once punctured surface.

**Lemma 2** Let \((\lambda_1, \lambda_2, \ldots, \lambda_q)\) be the \(\lambda\)-length coordinate-system for \(\mathcal{R}_{g,1}\) associated to an ideal triangulation \((c_1, c_2, \ldots, c_q)\), where \(q = 6g - 3\). Then the \(\lambda\)-length of a simple ideal arc \(c\) is expressed by a rational function of the form

\[
P(\lambda_1, \lambda_2, \ldots, \lambda_q) \quad \frac{1}{\lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_q^{m_q}}
\]

where \(P(\lambda_1, \lambda_2, \ldots, \lambda_q)\) is a homogeneous polynomial of degree

\[d = 1 + m_1 + m_2 + \cdots + m_q,
\]

with positive integer coefficients and \(m_i\) is the geometric intersection number of \(c\) and \(c_i\) in \(S - P\) for \(i = 1, 2, \ldots, q\).

For \(\varphi \in \mathcal{MC}_{g,1}\) let \(\varphi_*\) denote the rational transformation induced by \(\varphi\). Then entries of \(\varphi_*(\lambda_1, \lambda_2, \ldots, \lambda_q)\) are of the form as in (13). This fact leads us to the following observation.

Let \(D(\lambda_1, \ldots, \lambda_q) = 0\) be the algebraic equation which determines \(\mathcal{T}_{g,1}\) in the \(\lambda\)-length coordinates. Then the rational transformation \(\varphi_*\) induced by \(\varphi \in \mathcal{MC}_{g,1}\) preserves \(D(\lambda_1, \ldots, \lambda_q)\). Moreover, if

\[(\lambda, \lambda, \ldots, \lambda)\]

gives integer solutions of (14), then so does \(\varphi_*(\lambda, \lambda, \ldots, \lambda)\).

We remark that it is not true in general that all integer solutions are in the orbit of \((\lambda, \lambda, \ldots, \lambda)\) under \(\mathcal{MC}\).

6 3-manifolds which fiber over the circle

Let \(\varphi \in \mathcal{MC}_{g,n}\). Let \(M\) be a manifold which fibers over the circle and whose monodromy is \(\varphi\). If \(\varphi_*\) denotes the action of \(\varphi\) on the fundamental group \(G = G_{g,n}\) of the surface \(S\) of type \((g, n)\), then the fundamental group of \(M\) has the presentation

\[
\tilde{G} = \langle G, t : \varphi_*(g) = tgt^{-1} \text{ for all } g \in G \rangle
\]

If \(m : \tilde{G} \rightarrow SL(2, \mathbb{C})\) is a faithful representation of \(\tilde{G}\), then for all \(g \in G\)

\[
(\varphi_* \circ m)(g) = m(t)m(g)m(t)^{-1}.
\]

Hence the class \([m]\) is a fixed point of \(\varphi_*\) for its action on \(\mathcal{R}_{g,n}\).
The \( \lambda \)-length coordinates of \( \mathcal{R}_{g,n} \) represent \( \varphi_* \) as a rational function. Hence the fixed point \([m]\) corresponds to a solution of the algebraic equation

\[
\varphi_*(\lambda_1, \ldots, \lambda_q) = (\lambda_1, \ldots, \lambda_q).
\] (16)

If \( \varphi \) is reducible, then one of the solutions of (16) gives a faithful and discrete representation \( m \) of \( G \). We can find the Möbius transformation \( m(t) \) easily, because \( m(t) \) sends the fixed point of \( m(g) \) to that of \( m(\varphi_*(g)) \) for each parabolic element \( g \in G \). In this way hyperbolization of \( M_\varphi \) can be done. However, to carry this hyperbolization program into effect, we need efficient discreteness criteria.

**References**


