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FINDING IDEAL POINTS OF THE PSL(2, $\mathbb{C}$)-CHARACTER VARIETIES OF 3-MANIFOLDS FROM IDEAL TRIANGULATIONS

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Culler and Shalen developed the theory which relates ideal points of the character variety of a 3-manifold and incompressible surfaces. The theory has a great influence on the study of 3-manifolds. Although their theory is powerful and beautiful, it is difficult to find ideal points of the character variety of a 3-manifold. In this article, we show a computable method for finding ideal points from an ideal triangulation of a 3-manifold.

This article is organized as follows. In section 2, we review the basic notions. In section 3, we explain ideal triangulations and a parametrization of PSL(2, $\mathbb{C}$)-representations. In section 4, we give an exposition of logarithmic limit set. This material is not essential to understand main theorem, but it might give some insight. In section 5, we state the main theorem. In section 6 we give an example of 9_{32} knot complement case.

In this article we assume that all 3-manifolds are compact orientable with torus boundary.

2. Ideal points and incompressible surfaces

In this section we review the notions of character varieties and ideal points and the relationship between ideal points and incompressible surfaces. The original work was done by Culler and Shalen [Cu-Sh]. For PSL(2, $\mathbb{C}$) case, see [Bo-Zh], [He-Po].

2.1. Ideal points of a character variety. Let M be a 3-manifold with torus boundary $T=\partial M$. Let $R(M)$ be the affine algebraic set consisting of PSL(2, $\mathbb{C}$) representations of the fundamental group of M i.e. $R(M)=\mathrm{Hom}(\pi_1(M),\mathrm{PSL}(2,\mathbb{C}))$. PSL(2, $\mathbb{C}$) acts on $R(X)$ by conjugation: $\rho\mapsto g\rho g^{-1}$. The *character variety* of M is the algebraic geometric quotient of $R(M)$ by the action of PSL(2, $\mathbb{C}$). It is known that $X(M)$ also has a structure of an affine algebraic set.

Let C be a complex affine algebraic curve. Let $\widetilde{C}$ be the smooth projective model of C. Roughly speaking an ideal point of C is a point of $\widetilde{C}-C$. For precise definition, see [Cu-Sh]. Let $\mathbb{C}[C]$ be the coordinate ring of C and $\mathbb{C}(C)$ be the function field of C. There is a correspondence between points on C and valuations of $\mathbb{C}(C)$. A valuation of $\mathbb{C}(C)$ is a function $\mathbb{C}(C)\to\mathbb{Z}$ satisfying the following conditions:

(i) $v(fg)=v(f)+v(g)$,
(ii) $v(f+g)\geq\min\{v(f),v(g)\}$.

The valuation v corresponding to an ideal point satisfies $v(f) < 0$ for some $f \in \mathbb{C}[C]$. An ideal point represents a point at infinity of C. So an ideal point of a character variety represents a degeneration of $\mathrm{PSL}(2,\mathbb{C})$-representations. Culler and Shalen showed that for each ideal point of the character variety, there is a corresponding essential surface.

Let S be an properly embedded orientable surface in M. S is called *incompressible* if $\pi_1(S) \to \pi_1(M)$ is injective. An incompressible surface is called *essential* if S is not boundary parallel. The boundary of an essential surface is a slope on the boundary torus T. It is called the *boundary slope.* Boundary slopes are extensively studied. Hatcher showed that the set of all boundary slopes is finite [Ha]. For knot complement case, there are a lot of works of essential surfaces and their boundary slopes. For example there is an algorithm to compute boundary slopes of Montesinos knot complements [Ha-Oe]. But there are few works on boundary slopes for non-Montesinos knot complements.

2.2. **A-polynomial.** In general it is difficult to find ideal points and corresponding boundary slopes from the definition. But if we know the A-polynomial of M, it is easy.

The inclusion $\partial M \to M$ induces the algebraic $r : X(M) \to X(\partial M)$. Let $(\mathcal{M}, \mathcal{L})$ be a set of generators of $\pi_1(\partial M)$. Let $\Delta \subset R(\partial M)$ be the set of all diagonal representations on the boundary. Define M and L by

$$\rho(\mathcal{M}) = \pm \begin{pmatrix} \sqrt{M} & 0 \\ 0 & \sqrt{M}^{-1} \end{pmatrix}, \quad \rho(\mathcal{L}) = \pm \begin{pmatrix} \sqrt{L} & 0 \\ 0 & \sqrt{L}^{-1} \end{pmatrix}$$

where $\rho \in \Delta$. Then Δ can be regarded as $\mathbb{C}^* \times \mathbb{C}^*$. Let $t_\Delta : \Delta \to X(\partial M)$ be the quotient map. For each curve $Y \subset X(M)$, $t^{-1}(r(Y)) \subset \mathbb{C}^* \times \mathbb{C}^*$ defines a plane curve D_M. The defining equation of D_M is denoted by $A(M, L)$ and called the *A-polynomial* [CCGLS].

Let $A(M, L) = \sum c_{OJ} M^i L^j$ be the A-polynomial of M. The *Newton polygon* of A is the convex hull of the set $\{(i,j) \in \mathbb{Z}^2 | c_{OJ} \neq 0\}$. Let p/q be the slope of an edge of the Newton polygon. Then there is a corresponding valuation v of D_M satisfying $-v(M)/v(L) = p/q$.

3. Ideal triangulation and parametrization of PSL(2, ℂ)-representations

3.1. **Ideal tetrahedron.** Let $\mathbb{H}^3$ be the upper half space model of the hyperbolic 3-space. $\mathbb{C}P^1$ can be regarded as the ideal boundary of $\mathbb{H}^3$. $\mathrm{PSL}(2,\mathbb{C})$ acts on the $\mathbb{H}^3$ and also its ideal boundary $\mathbb{C}P^2$. An ideal tetrahedron is a convex hull of distinct 4 points of $\mathbb{C}P^1$ in $\mathbb{H}^3$. We assume that every ideal tetrahedron has an orientation. Let (z_0, z_1, z_2, z_3) be distinct points of $\mathbb{C}P^1$. For an edge (z_0, z_1), we define the complex parameter by cross ratio:

$$z = [z_0 : z_1 : z_2 : z_3] = \frac{(z_2 - z_1)(z_3 - z_0)}{(z_2 - z_0)(z_3 - z_1)},$$

where (z_0, z_1, z_2, z_3) forms the orientation of the ideal tetrahedron. Because $z_0, \ldots, z_3$ are distinct, this complex number is not equal to 0 or 1. We can easily show that edge (z_2, z_3) has same complex parameter. The edges (z_1, z_2) and (z_0, z_3) have complex parameter $\frac{1}{1-z}$ and the edges (z_1, z_3) and (z_0, z_2) have complex parameter $1 - \frac{1}{z}$.

3.2. **Ideal triangulation and developing map.** Let M be a 3-manifold with torus boundary. A (topological) *ideal triangulation* of M is a cell complex K formed by gluing tetrahedra along their faces so that $K - N(K^{(0)})$ is homeomorphic to M. Let K be an ideal triangulation of M with n ideal tetrahedra. Give a complex parameter for each ideal tetrahedron of K. We denote these complex parameters by $z_\nu(\nu = 1, \ldots, n)$. For each 1-simplex e_k of K, there are the edges of ideal tetrahedra which are adjacent to e_k. There are complex parameters corresponding to these edges. They are z_ν, $\frac{1}{1-z_\nu}$ or $1 - 1/z_\nu$. Let R_k be the multiplication of these complex parameters. $R_k = 1$ is called the *gluing equation* of e_k. There are n 1-simplices of K but there is one relation among $R_1, \ldots, R_n$. So we only have to consider $n - 1$ gluing equations. We define integers $(p_{k,\nu}, p'_{k,\nu}, p''_{k,\nu})$ by

$$\begin{aligned} R_k &= \prod_{\nu=1}^{n} z_\nu^{p_{k,\nu}} \left(\frac{1}{1-z_\nu}\right)^{p'_{k,\nu}} \left(1 - \frac{1}{z_\nu}\right)^{p''_{k,\nu}} \\ &= \prod_{\nu=1}^{n} (-1)^{p''_{i,\nu}} z_\nu^{r'_{i,\nu}} (1 - z_\nu)^{r''_{i,\nu}} \quad (k = 1, \ldots n-1). \end{aligned}$$

We put $r'_{k,\nu} = p_{k,\nu} - p''_{k,\nu}$ and $r''_{k,\nu} = p''_{i,\nu} - p'_{k,\nu}$.

Let $\widetilde{M}$ be the universal covering of M. If $z_1, \ldots, z_n$ satisfies the gluing equations, we can construct a $\pi_1(M)$-equivariant map $\widetilde{M} \to \mathbb{H}^3$ as follows. Put one ideal tetrahedron parametrized by z_1 on $\mathbb{H}^3$. Then develop adjacent ideal tetrahedron in $\widetilde{M}$ to $\mathbb{H}^3$. By continuing this process, we obtain the $\pi_1(M)$-equivariant map $\widetilde{M} \to \mathbb{H}^3$. Let

$$\mathcal{D}(M, K) = \{(z_1, \ldots, z_n) \in (\mathbb{C} - \{0, 1\})^n | R_i = 1, i = 1, \ldots, n-1\}$$

We denote $\mathcal{D}(M, K)$ by $\mathcal{D}(M)$ for short. Each point of $\mathcal{D}(M)$ gives a developing map $\widetilde{M} \to \mathbb{H}^3$. The holonomy map of the developing map gives a $\mathrm{PSL}(2, \mathbb{C})$-representation. This defines a representation $\pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$. So we obtain the algebraic map $\mathcal{D}(M) \to X(M)$. By construction we can show that this map is algebraic. So we can study ideal points of $X(M)$ from ideal points of $\mathcal{D}(M)$. We remark that the defining equation of $\mathcal{D}(M)$ is very simple. In fact each equation is only a product of z_ν and $1 - z_\nu$. In general it is much more complicated to describe defining equations of $X(M)$ in terms of relations and generators of the fundamental group.

On the boundary torus ∂M, we choose a set of generators $\mathcal{M}, \mathcal{L}$ of $H_1(\partial M; \mathbb{Z})$. We can choose a pair of integers (m'_i, m''_i) and (l'_i, l''_i) so that

$$M = \pm \prod_{j=1}^{n} z_j^{m'_j} (1 - z_j)^{m''_j}, \quad L = \pm \prod_{j=1}^{n} z_j^{l'_j} (1 - z_j)^{l''_j}.$$

represent the squares of eigenvalues of $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$, where ρ is a holonomy representation associated to $(z_1, \ldots, z_n) \in \mathcal{D}(M)$. We denote $m = (m'_1, m''_1 \ldots, m_n,' m''_n)$ and $l = (l'_1, l''_1, \ldots, l'_n, l''_n)$.

Let $x = (x'_1, \ldots, x'_n, x''_1, \ldots x''_n)$ and $y = (y'_1, \ldots, y'_n, y''_1, \ldots y''_n)$. We define the symplectic form of $\mathbb{R}^{2n}$ by

$$x \wedge y = \sum_{j=1}^{n} x'_j y''_j - x''_j y'_j.$$

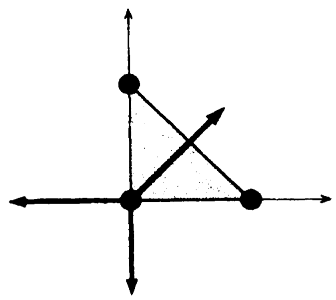

FIGURE 1. Newton polygon of $z + w - 1$ and its spherical dual.

Let $r_k = (r'_{k,1}, r''_{k,1}, \ldots, r'_{k,n}, r''_{k,n})$ and $[R] = \mathrm{span}_{\mathbb{R}}\langle r_1, \ldots, r_{n-1}\rangle$. We denote the orthogonal complement of $[R]$ with respect to $\wedge$ by $[R]^{\perp}$. The wedge product is useful and natural to describe some combinatorial formula (see [Ne-Za] and [Ne]).

4. LOGARITHMIC LIMIT SET AND REAL VALUATIONS

In this section we explain the logarithmic limit of a subvariety of $(\mathbb{C}^*)^n$. Tillman's paper [Ti1] is a good reference for these materials. [Ti2], [Mo-Sh] and [Yo] relate the logarithmic limit set to degenerations of PSL$(2, \mathbb{C})$-representations. We denote $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ by $\mathbb{C}[X]$. We use the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ and denote $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ by X^{α}.

4.1. **Logarithmic limit set.** Let V be a subvariety of $(\mathbb{C}^*)^n$. The *logarithmic limit set* of V is the set of limit points on S^{n-1} of the following set:

$$\left\{ \frac{(\log|x_1|, \ldots, \log|x_n|)}{\sqrt{1 + \sum(\log|x_i|)^2}} \middle| x \in V \right\} \subset B^n$$

where $B^n = \{x \in \mathbb{R}^n \,||x| \leq 1\} \subset \mathbb{R}^n$. We denote the logarithmic limit set by $V_{\infty}^{(a)}$.

4.2. **Real valuation.** Let J be the ideal corresponding to V. A (real) valuation on $\mathbb{C}[X]/J$ is a function $v : \mathbb{C}[X]/J \to \mathbb{R}$ satisfying the conditions of 2.1. We define the subset $V_{\infty}^{(b)}$ of S^{n-1} as the set of $(-v(x_1), \ldots, -v(x_n))$ where v runs over all real valuations of $\mathbb{C}[X]/J$. If V is an algebraic curve, we only have to consider discrete valuations.

4.3. **Newton polytope and spherical dual.** Let $f = \sum_{\alpha} a_{\alpha} X^{\alpha} \in \mathbb{C}[X^{\pm}]$. The *Newton polytope* of the polynomial f is the convex hull of $s(f) = \{\alpha | a_{\alpha} \neq= 0\}$ in $\mathbb{R}^n$ i.e. $Newt(f) = Conv(s(f))$. The set of $\xi \in S^{n-1}$ such that the maximum value of the dot product $\xi \cdot x$ is achieved for more than one as x runs over $New(f)$. For example Newton polygon and its spherical dual of $z + w - 1$ are shown in Figure 1. We define

$$V_{\infty}^{(c)} = \bigcap_{f \neq 0 \in J} Sph(f).$$

Then we have the following theorem:

Theorem 4.1 (Bergman [Be], Bieri-Groves [Bi-Gr]).

$$V_{\infty}^{(a)} = V_{\infty}^{(b)} = V_{\infty}^{(c)}.$$

4.4. **For $\mathcal{D}(M)$ case.** Let $w_\nu = 1 - z_\nu$ then $\mathcal{D}(M)$ can be regarded as a subvariety of $(\mathbb{C}^*)^{2n}$:

$$\mathcal{D}(M) = \{(w_1^{-1}, z_1, \dots, w_n^{-1}, z_n) \in (\mathbb{C}^*)^{2n} | z_\nu + w_\nu = 1 \quad (\nu = 1, \dots n),$$
$$R_k = \pm \prod_{\nu=1}^{n} z_\nu^{r'_{k,\nu}} w_\nu^{r''_{k,\nu}} = 1 \quad (k = 1, \dots, n-1)\}$$

(We take coordinate $(w_1^{-1}, z_1, \dots,)$ because it is suitable for the wedge product.) In this case, we have

(4.1)

$$\mathcal{D}(M)_\infty^{(c)} \subset \bigcap_{\nu=1}^{n} Sph(z_\nu + w_\nu - 1) \cap \bigcap_{\nu=1}^{n-1} Sph(R_k - 1)$$
$$= \left(\prod_{\nu=1}^{n} \left(\mathbb{R}_{\geq 0} \cdot \mathbf{e}_{2\nu-1} \cup \mathbb{R}_{\geq 0} \cdot (-\mathbf{e}_{2\nu}) \cup \mathbb{R}_{\geq 0} \cdot (-\mathbf{e}_{2\nu-1} + \mathbf{e}_{2\nu})\right) \cap [R]^\perp\right) \cap S^{n-1}$$

where $\mathbf{e}_i$ is the i-th unit vector. We remark that the right hand side can be computed by only using linear algebra.

5. Main theorem

In the previous section, we observed that there is a necessary condition that valuations of $\mathbb{C}(\mathcal{D}(M))$ satisfy. In this section we give a criterion to ensure that they are really valuations of $\mathbb{C}(\mathcal{D}(M))$ (so there exist corresponding ideal points).

Let $I = (i_1, \dots, i_n) \in \{1, 0, \infty\}^n$ and call it a *degeneration index*. A degeneration index I describes how each ideal tetrahedron degenerates ($z_\nu \to 1, 0$ or ∞). For a degeneration index I, we define

$$r(I)_{k,\nu} = \begin{cases} r''_{k,\nu} & \text{if } i_\nu = 1 \\ r'_{k,\nu} & \text{if } i_\nu = 0 \\ -r'_{k,\nu} - r''_{k,\nu} & \text{if } i_\nu = \infty. \end{cases}$$

$r(I)_{k,\nu}$ represents the main contribution from ν-th simplex on gluing equation at 1-simplex e_k. Then let

$$d(I)_\nu = (-1)^{\nu+1} \det \begin{pmatrix} r(I)_{1,1} & \dots & \widehat{r(I)_{1,\nu}} & \dots & r(I)_{1,n} \\ \vdots & & \vdots & & \vdots \\ r(I)_{n-1,1} & \dots & \widehat{r(I)_{n-1,\nu}} & \dots & r(I)_{n-1,n} \end{pmatrix}.$$

Then we define a *degeneration vector* by

$$d(I) = (d(I)_1, d(I)_2, \dots, d(I)_n) \in \mathbb{Z}^n \subset \mathbb{R}^n.$$

Let $\rho_1 = (1,0)$, $\rho_0 = (0,-1)$ and $\rho_\infty = (-1,1)$. A simple calculation shows that if all the coefficient of d_ν are non-negative, normalized $(d_1\rho_{i_1}, \dots, d_n\rho_{i_n})$ is in the rig hand side of the inclusion (4.1). The following is our main theorem:

Theorem 5.1 ([Ka]). *Let $I = (i_1, \dots, i_n)$ be an element of $\{1, 0, \infty\}^n$. If $d(I) > 0$ or $d(I) < 0$ then there are ideal points of $\mathcal{D}(M)$ corresponding to I. The number of the ideal points is* $\gcd(d(I)_1, \dots, d(I)_n)$.

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Edge equations:
 1 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0
 0 1 0 0 0 1 0 1 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0
 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0
 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0
 0 0 0 1 0 0 1 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 1
 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0
 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 0 0 0
 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0
 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 0 0 0
 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 1 0 0 1 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1 0 0 1 0 1 0 0 0 0 0 0 1 0 0 0
Cusp equations:
 0 0 0 0 0 0 1 0 0 -1 0 0 0 1 0 0 -1 0 0 0 -1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 1 1 0 -1 1 0 -3 0 2 6 0 0 0 -3 -1 0 2 -1 0 -1 4 0 -1 -3 0 -1 0 0 0 -1 1 -1 0 1 0 0 -1 0 0 1 0 -1
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FIGURE 2. A system of gluing equations of 9_{32}. The k-th row of Edge equations represents $(p_{k,1}, p'_{k,1}, p''_{k,1}, p_{k,2}, \dots)$ of subsection 3.2.

6. Computations

In this section we give an example for knot complement case. As mentioned before, Hatcher and Oertel [Ha-Oe] gave an algorithm to compute boundary slopes of Montesinos knots. So we give an example to compute boundary slopes of non-Montesinos knots. In general, computation of the degeneration vector for a degeneration index is very easy. But for finding ideal points, we try to compute all degeneration vectors to find degeneration vectors which satisfy the condition of our theorem. If the number of the ideal tetrahedra is n, we have to compute degeneration vectors 3^n times. So when the number of ideal tetrahedra increases, we have to compute much more number of degeneration vectors.

6.1. **The knot 9_{32}.** The knot 9_{32} is a non-Montesinos knot [Du]. By using SnapPea [We], we can find an ideal triangulation of the complement of 9_{32} with 14 ideal tetrahedra. By using Snap [Go] we can obtain the gluing equations for this ideal triangulation (Figure 2). In this case we have to compute $3^{14} = 4782969$ number of degeneration vectors! But a modern computer calculates these about 3 hours!! The degeneration indices which satisfy our theorem are

$$(0, \infty, 0, 1, 1, 0, 1, \infty, 1, 1, 1, \infty, 0, 0)$$
$$(0, \infty, \infty, 1, \infty, 0, 1, 1, 1, 1, 1, 1, 1, 0)$$
$$(1, \infty, \infty, 1, \infty, 0, 1, 1, \infty, 1, 1, 1, 1, 0)$$
$$(\infty, 1, 0, \infty, \infty, 1, 1, 0, 1, 0, \infty, \infty, 1, \infty)$$

and corresponding degeneration vectors and $(v(M), v(L))$ are

$$-(1, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \quad (1, -8)$$
$$-(1, 3, 3, 3, 1, 2, 1, 2, 1, 4, 3, 2, 1, 1) \quad (1, -18)$$
$$(1, 3, 3, 3, 1, 3, 1, 2, 1, 4, 2, 2, 1, 1) \quad (1, -14)$$
$$(3, 5, 1, 5, 4, 1, 5, 5, 2, 1, 3, 4, 1, 2) \quad (-1, 24).$$

v is the valuation corresponding to the ideal point and M and L are the elements of $\mathcal{D}(M)$ defined as the square of the eigenvalues of $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$. By a similar argument to [CCGLS], we can show that the corresponding boundary slopes are 8, 18, 14 and 24.

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