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VARIATION FORMULAS FOR PRINCIPAL FUNCTIONS AND SPANS OF RIEMANN SURFACES

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ABSTRACT. The purpose of this article is to give a summary of the seminar lecture with title Variational formulas for principal functions and for spans of Riemann surfaces by the author in the conference in RIMS, Kyoto, Japan, December 2008. The main theorems are in the manuscripts [2] and [6]. This note should be understood as a lectures summary version of these manuscripts.

1. INTRODUCTION

Let $\mathcal{R}=\bigcup_{t\in B}(t,R(t))$ and $\widetilde{\mathcal{R}}=\bigcup_{t\in B}(t,\widetilde{R}(t))$ be unramified sheeted domains over $B\times\mathbb{C}_z$, where $B=\{|t|<\rho\}$ is a disk in $\mathbb{C}_t$ and $R(t)\Subset\widetilde{R}(t)$ for $t\in B$. We set $\partial\mathcal{R}=\bigcup_{t\in B}(t,\partial R(t))$ in $\widetilde{\mathcal{R}}$. In this article, we assume that

$$\mathcal{R}:t\in B\to R(t)$$

is a C^ω smooth variation of domains $R(t)$ with C^ω smooth boundaries in $\widetilde{R}(t)$. Namely, we can choose a real analytic defining function $\varphi(t,z)$ of $\partial\mathcal{R}$ such that $\frac{\partial\varphi}{\partial z}\neq 0$ on $\partial\mathcal{R}$. We denote by $C_j(t)$ $(j=0,1,\ldots,\nu)$, where $\nu(\geq 0)$ is independent of $t\in B$, the boundary contours of $R(t)$ in $\widetilde{R}(t)$ with the orientation: $\partial R(t)=\sum_{j=0}^{\nu}C_j(t)$. We usually regard two-dimensional Riemann domain $\mathcal{R}$ over $B\times\mathbb{C}_z$ as a C^ω smooth variation of Riemann surface $R(t)$ over $\mathbb{C}_z$ with C^ω smooth boundary $\partial R(t)$ with one complex parameter $t\in B$.

Assume that the total space $\mathcal{R}$ of complex 2-dimension contains $B\times\{0\}$. Precisely, there exists at least one constant section $\mathbf{O}$ of $\mathcal{R}$ over $B\times\{0\}$. For each $t\in B$, we conventionally write 0 for the point $\mathbf{O}\cap R(t)$.

Let $t\in B$ be fixed. In the theory of one complex variable, it is known that there uniquely exists a real-valued function $u(t,z)$ on $R(t)\setminus\{0\}$ with the following four conditions:

(1) $u(t,z)$ is harmonic on $R(t)\setminus\{0\}$ and is continuous on $\overline{R(t)}$;
(2) $u(t,z)-\log\frac{1}{|z|}$ is harmonic at $z=0$;
(3) $u(t,z)=0$ on $C_0(t)$;

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(4) for each $i = 1, \dots, \nu$, we have

$$\text{(i) } u(t,z) = a_i(t) : \text{constant on } C_i(t);$$

$$\text{(ii) } \int_{C_i(t)} *du(t,z) = 0.$$

We note that $u(t,z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \widetilde{R}(t)$. By (2), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$u(t,z) = \log \frac{1}{|z|} + \gamma(t) + h(t,z) \qquad \text{on } U_0(t),$$

where $\gamma(t)$ is the constant term and $h(t,z)$ is harmonic for z on $U_0(t)$ such that

$$h(t,0) = 0, \qquad t \in B.$$

The function $u(t,z)$ is called the L_1-*principal function* on $R(t)$ with logarithmic pole at 0 with respect to $C_0(t)$, and $\gamma(t)$ is called the L_1-*constant* for $(R(t), 0)$ with respect to $C_0(t)$ (cf: [7]). In this paper, we simply call $u(t,z)$ the L_1-principal function for $(R(t), 0, C_0(t))$, and $\gamma(t)$ the L_1-constant for $(R(t), 0, C_0(t))$. We note that $u(t,z) > 0$ in $R(t) \setminus \{0\}$ and $a_i(t) > 0$ $(i = 1, \dots, \nu)$.

Then we have the following variation formula for the L_1-constant $\gamma(t)$ for $(R(t), 0, C_0(t))$.

Lemma 1. *It holds for $t \in B$ that*

$$\frac{\partial^2 \gamma(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial u(t,z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 u(t,z)}{\partial \bar{t} \partial z} \right|^2 dxdy.$$

Here

$$k_2(t,z) = \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2\mathrm{Re} \left\{ \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right) \Big/ \left| \frac{\partial \varphi}{\partial z} \right|^3$$

on $\partial \mathcal{R}$, which does not depend on the choice of defining functions $\varphi(t,z)$ of $\partial \mathcal{R}$, and ds_z is the arc length element of $\partial R(t)$ at z.

The function $k_2(t,z)$ on $\partial \mathcal{R}$ is due to Maitani-Yamaguchi in [5] which is based on [4] for the several complex variables [1]. This variation formula is formally the same as that for the Robin constant $\lambda(t)$ (induced by the Green function $g(t,z)$ on $R(t)$ with logarithmic pole at $z = 0$) in Theorem 3.1 in [5]. The essential difference of the proofs for $\gamma(t)$ and $\lambda(t)$ comes from the fact that $u(t,z)$ is not a defining function of $\partial \mathcal{R}$ contrary to the case of the Green function $g(t,z)$.

[1]The geometric meaning of $k_2(t,z)$ is studied in the language of Cartan's moving frame (see [3]).

Theorem 2. *Under the same conditions in Lemma 1, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $\gamma(t)$ is a C^ω superharmonic function on B.*

Remark 1. For Lemma 1, we assumed that $\mathcal{R}$ is unramified over $B \times \mathbb{C}_z$. However, even if each $R(t), t \in B$ has a finite number of branch points $\zeta_k(t)$ $(k = 1, \ldots, m)$ for $t \in B$ such that $\zeta_k(t)$ is a holomorphic function on B with $\zeta_k(t) \neq \zeta_l(t)$ $(k \neq l)$, $t \in B$, then Lemma 1 and hence Theorem 2 hold. For, this case can be reduced to Lemma 1 by the standard method by use of Y. Nishimura's theorem [8].

In the special case when $R(t)$ is a planar Riemann surface, the L_1-principal function $u(t,z)$ induces a circular slit mapping $f(t,z)$. That is, if we choose a branch $u^*(t,z)$ of harmonic conjugate function of $u(t,z)$ on $R(t), t \in B$ such that

$$f(t,z) = e^{\gamma(t)-(u(t,z)+iu^*(t,z))}$$

is of the form

$$w = f(t,z) = z + \sum_{j=2}^{\infty} b_j(t) z^j \quad \text{on } U_0(t),$$

then $f(t,z)$ conformally maps $R(t)$ onto a circular slit domain $\{|w| < e^{\gamma(t)}\} \setminus \left(\cup_{i=1}^{\nu} \ell_i\right)$, where $\ell_i(t) = f(t, C_i(t))$ (an arc of the circle $\{|w| = e^{\gamma(t)-a_i(t)}\}$). Since $e^{\gamma(t)}$ is logarithmic superharmonic on B, the total space $\bigcup_{t\in B}\{|w| < e^{\gamma(t)}\}$ is a Hartogs domain in $B \times \mathbb{C}_w$.

Remark 2. We note that the same formula for the radius $r_0(t)$ of radial slit mapping does not hold. We can give counterexamples of pseudoconvex domains $\mathcal{R}$ in $B \times \mathbb{C}_z$ such that $\log r_0(t)$ are not superharmonic or not subharmonic on B.

In the theory of conformal mappings in one complex variable, the L_1- and L_0-principal functions and the logarithmic span play the important role (cf: Ahlfors-Sario [1] and Nakai-Sario [7]). In section 2, we introduce the variation formula of the second order for L_1-principal function $p_1(t,z)$ on Riemann surface $R(t)$ moving smoothly with one complex parameter t in a disk B in $\mathbb{C}_t$. We apply it to show the simultaneous uniformization theorem for the Schottky covering $\widetilde{R}(t)$ of the (holomorphically) moving compact Riemann surface $R(t)$. In sections 3, we establish the variation formula of the second order for L_0-principal function $p_0(t,z)$ on the moving Riemann surface $R(t)$, and in section 4 we apply it to show the meaning of the logarithmic span $s(t)$ of the moving planar Riemann surface $R(t)$, in the theory of several complex variables.

2. Variation formula for L_1-principal functions $p(t,z)$ and application

Under the same conditions for the unramified domain $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ in $\widetilde{\mathcal{R}}$ over $B \times \mathbb{C}_z$ and $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$, we assume that there exist two holomorphic sections:

$$\Xi_0 : z = 0 \quad \text{and} \quad \Xi_1 : z = \xi(t)$$

of $\mathcal{R}$ over B such that $\Xi_0 \cap \Xi_1 = \emptyset$. Let $t \in B$ be fixed. Then it is known (cf: Ahlfors-Sario [1]) that $R(t)$ carries the following harmonic functions $p(t,z)$ and $q(t,z)$ with logarithmic poles at 0 and $\xi(t)$:

Definition. *L_1-principal function* $p(t,z)$ for $(R(t), 0, \xi(t))$ is a real valued-function $p(t,z)$ on $R(t) \setminus \{0, \xi(t)\}$ with the following four conditions:

(i) $p(t,z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and continuous on $\overline{R(t)}$;
(ii) $p(t,z) - \log 1/|z|$ is harmonic at $z = 0$ and

$$\lim_{z\to 0} (\, p(t,z) - \log 1/|z| \,) = 0;$$

(iii) $p(t,z) - \log|z - \xi(t)|$ is harmonic at $z = \xi(t)$;
(iv) for each $j = 1, \ldots, \nu$, we have

$$(1)\ p(t,z) = a_j(t) : \text{constant on } C_j(t);$$

$$(2)\ \int_{C_j(t)} *dp(t,z) = 0.$$

We note that $p(t,z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \widetilde{R}(t)$, $-\infty < p(t,z) < +\infty$, and $-\infty < a_j(t) < +\infty$. Moreover $p(t,z)$ is of class C^ω for (t,z) in $\mathcal{R} \setminus \{\Xi_0 \cup \Xi_1\}$. By (ii), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$p(t,z) = \log \frac{1}{|z|} + h_0(t,z) \qquad \text{on } U_0(t),$$

where $h_0(t,z)$ is harmonic for z on $U_0(t)$ and

$$h_0(t,0) = 0, \qquad t \in B.$$

By (iii), we find a neighborhood $U_\xi(t)$ of $z = \xi(t)$ such that

$$p(t,z) = \log|z - \xi(t)| + \alpha(t) + h_\xi(t,z) \qquad \text{on } U_\xi(t),$$

where $\alpha(t)$ is a real constant and $h_\xi(t,z)$ is harmonic for z on $U_\xi(t)$ and

$$h_\xi(t, \xi(t)) = 0, \qquad t \in B.$$

We call $\alpha(t)$ the *L_1-constant* for $(R(t), 0, \xi(t))$.

Under these situations, we showed in [2] the following variation formula of the second order for $\alpha(t)$:

Lemma 3 (H, [2]). *It holds for $t \in B$ that*

$$\frac{\partial^2 \alpha(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial p(t,z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 p(t,z)}{\partial \bar{t} \partial z} \right|^2 dxdy.$$

Since $k_2(t,z) \geq 0$ on $\partial \mathcal{R} = \cup_{t \in B}(t, \partial R(t))$ in case $\mathcal{R}$ is pseudoconvex, the lemma implies

Theorem 4. *Under the same conditions in Lemma 3, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $\alpha(t)$ is a C^ω subharmonic function on B. This is also true under the same condition for $\mathcal{R}$ as in Remark 1*

As an application of Theorem 4, we proved that the following fact. Let $\mathcal{S} : t \in B \to S(t)$ be a holomorphic family without singularity of a compact Riemann surface $S(t)$ over a simply connected domain B in $\mathbb{C}_t$, so that $S(t)$ varies holomorphically with respect to one complex parameter t in B. For a fixed $t \in B$, we consider the Schottky covering $\widetilde{S}(t)$ of each $S(t)$, and denote by $\widetilde{\mathcal{S}}$ the total space of the variation: $t \in B \to \widetilde{S}(t)$, namely, $\widetilde{\mathcal{S}} = \bigcup_{t \in B}(t, \widetilde{S}(t))$. Then we have:

Theorem 5. *The total space $\widetilde{\mathcal{S}}$ consisting of Schottky covering $\widetilde{S}(t)$ of a compact Riemann surface $S(t)$ with one complex parameter $t \in B$ is holomorphically uniformized to a univalent domain in $B \times \mathbb{P}$.*

Remark 3. In [9], Nishino showed that, if $\mathcal{R} = \cup_{t \in B}(t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ such that each $R(t)$, $t \in B$ is conformal equivalent to $\mathbb{C}^1$, then $\mathcal{R}$ is holomorphically equivalent to $B \times \mathbb{C}$. In [5], Maitani and Yamaguchi proved that, if $\mathcal{R} = \cup_{t \in B}(t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ such that each $R(t)$, $t \in B$ is planar and parabolic, then $\mathcal{R}$ is holomorphically uniformizable to a domain in $B \times \mathbb{P}^1$. Since the Schottky covering $\widetilde{S}(t)$ of a compact Riemann surface $S(t)$ of genus $g \geq 2$ is planar but not parabolic, their theorem and method cannot be applicable to our case.

Remark 4. In [10], Yamaguchi wrote a resumé about Theorem 5 with a rough sketch of the proof. However his sketch had a serious "gap". Then I bridge the gap by the variation formula for L_1-principal function, and obtain Theorem 5.

3. Variation formula for L_0-principal functions $q(t,z)$

Definition. *L_0-principal function $p(t,z)$ for $(R(t), 0, \xi(t))$* is a real valued-function $q(t,z)$ on $R(t) \setminus \{0, \xi(t)\}$ with the following four conditions:

(i) $q(t,z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and continuous on $\overline{R(t)}$;

(ii) $q(t,z) - \log 1/|z|$ is harmonic at $z = 0$ and

$$\lim_{z \to 0} (\, q(t,z) - \log 1/|z|\,) = 0;$$

(iii) $q(t,z) - \log|z - \xi(t)|$ is harmonic at $z = \xi(t)$;

(iv) $\frac{\partial q(t,z)}{\partial n_z} = 0$ on $\partial R(t)$.

We note that $q(t,z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \widetilde{R}(t)$, $-\infty < q(t,z) < +\infty$, and $q(t,z)$ is of class C^ω for (t,z) in $\mathcal{R} \setminus \{\Xi_0 \cup \Xi_1\}$.

By (ii), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$q(t,z) = \log \frac{1}{|z|} + k_0(t,z) \qquad \text{on } U_0(t),$$

where $k_0(t,z)$ is harmonic for z on $U_0(t)$ and

$$k_0(t,0) = 0, \qquad t \in B.$$

By (iii), we find a neighborhood $U_\xi(t)$ of $z = \xi(t)$ such that

$$q(t,z) = \log |z - \xi(t)| + \beta(t) + k_\xi(t,z) \qquad \text{on } U_\xi(t),$$

where $\beta(t)$ is a real constant and $k_\xi(t,z)$ is harmonic for z on $U_\xi(t)$ and

$$k_\xi(t,\xi(t)) = 0, \qquad t \in B.$$

We call $\beta(t)$ the L_0-*constant* for $(R(t), 0, \xi(t))$.

We showed in [6] the variation formula of the second order for L_0-constant $\beta(t)$. In order to prove the formula, we have to add a new idea to the proof of Lemma 3. To state the variation formula for L_0-constant $\beta(t)$ in case when $R(t)$ has positive genus, we need the following consideration (which was not necessary for the variation formula for L_1-constant $\alpha(t)$).

In case when $R(t)$ is of positive genus $g \geq 1$, we take $\{A_l(t), B_l(t)\}_{1 \leq l \leq g}$ be usual A, B cycles on $R(t)$ with intersection number condition: for $k, l = 1, \ldots, \nu$,

$$A_k(t) \times B_l(t) = \delta_{k,l}, A_k(t) \times A_l(t) = 0, B_k(t) \times B_l(t) = 0.$$

Here $\delta_{k,l}$ is Kronecker's delta; $A_k(t) \times B_l(t)$ means that $A_k(t)$ crosses $B_l(t)$ from the left-side to the right-side of the direction $B_l(t)$; and each $A_k(t)$ and $B_k(t)$ $(k = 1, \ldots, g)$ varies continuously with parameter $t \in B$ such that $A_k(t), B_k(t)$ do not pass through $\{0, \xi(t)\}$.

On each $R(t)$ for $t \in B$ we denote by $*dq(t,z)$ the conjugate differential of $dq(t,z)$. It follows that $*dq(t,z)$ has the following multi-valued poles at $z = 0$ and $z = \xi(t)$:

$$*dq(t,z) = \begin{cases} -d \arg z + dk_0^*(t,z) & \text{in } U_0(t), \\ d \arg(z - \xi(t)) + dk_\xi^*(t,z) & \text{in } U_\xi(t), \end{cases}$$

where $k_0^*(t,z)$ and $k_\xi^*(t,z)$ are the single-valued harmonic conjugate functions of $k_0(t,z)$ and $k_\xi(t,z)$ on $U_t(0)$ and $U_\xi(t)$, respectively. We may assume

$$k_0^*(t,0) = 0, \quad k_\xi^*(t,\xi(t)) = 0.$$

Under these notations, we have the following variation formula of the second order for $\beta(t)$:

Lemma 6 (M-Y-H, [6]). *It holds for $t \in B$ that*

$$\frac{\partial^2\beta(t)}{\partial t\partial\bar{t}} = -\frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial q(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 q(t,z)}{\partial\bar{t}\partial z}\right|^2 dxdy$$
$$-\frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{k=1}^{g}\frac{\partial}{\partial t}\left(\int_{A_k(t)} *dq(t,z)\right)\frac{\partial}{\partial\bar{t}}\left(\int_{B_k(t)} *dq(t,z)\right)\right\}$$

Since $k_2(t,z) \geq 0$ on $\partial\mathcal{R}$ in case $\mathcal{R}$ is pseudoconvex, the lemma immediately implies

Theorem 7 (M-Y-H,[6]). *Under the same conditions in Lemma 6, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$ and each $R(t)$, $t \in B$ is planar, then $\beta(t)$ is a C^ω superharmonic function on B.*

4. Variation for spans of Riemann surfaces

What do Theorems 4 and 7 mean in two conmplex variables? To answer it, we recall the study in one complex variable: Let R be a planar Riemann surfaces with smooth boundary $\partial R = \sum_{j=1}^{\nu} C_j$. We denote by $\mathcal{S}(R)$ the set of all univalent functions $f(z)$ on $R \cup \overline{R}$ such that

$$f(z) = u(z) + iu^*(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n \quad \text{near} \quad z = 0,$$
$$f(z) = C_1(z-1) + C_2(z-1)^2 + \dots \quad \text{near} \quad z = 1.$$

We draw a simple curve L in R starting at 0 and terminating at 1. Let $f \in \mathcal{S}$. Then $f(L)$ is a simple curve in $\mathbb{P}_w$ which starts at 0 and terminating at ∞, so that each branch $\log f(z)$ on $R\backslash L$ is single-valued and the Euclidean area of the complemant of $\log f(R \setminus L)$ is uniquely determind. Set

$$s(R) := \max\{E_{\log}(f) \mid f \in \mathcal{S}(R)\},$$

which is called the *logarithmic span* or *harmonic span* for $(R, 0, 1)$. Then the following fact is studied by G. Grunsky and M. Schiffer:

Fact. Let α and β be L_1-constant and L_0-constant for $(R, 0, 1)$, respectively. Then

$$s(R) = \alpha - \beta (> 0)$$

Therefore, by Theorem 7 we have:

Theorem 8 (M-Y-H,[6]). *Let $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ be a 2-dimensional pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary, where each $R(t), t \in B$ is planar and $R(t) \ni 0, \xi(t)$. Then the logarithmic span $\alpha(t) - \beta(t)$ for $(R, 0, \xi(t))$ varies subharmonically on B.*

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