数理解析研究所講究録
第 1664 巻 2009 年 84-104

A survey on horospherical geometry of submanifolds in hyperbolic space

泉屋 周一 (Shyuichi IZUMIYA)
北海道大学・大学院理学研究院
(Faculty of Science, Hokkaido University)

1 Introduction

Recently we discovered a new geometry on submanifolds in hyperbolic n-space which is called *horospherical geometry* ([5, 6, 13, 14, 15, 16, 17, 18, 19]). This is a survey article on horospherical geometry. This geometry is not invariant under the hyperbolic motions (it is invariant under the canonical action of $SO(n)$), but the flatness in this geometry is a hyperbolic invariant and the total curvatures are topological invariants. We also study horo-tight immersions of manifolds into hyperbolic spaces and give several characterizations of horo-tightness of spheres, answering a question proposed by T. Cecil and P. Ryan (1985) : **What are the horo-tight imersions of spheres?** It has been shown in [6] that a horo-tight immersion of sphere is hyperbolic tight in the sense of Cecil and Ryan [9] (cf., Theorem 5.2). Since the converse assertion has been shown in their paper [9], this is a complete answer to their question. According to this result, we have the following conjecture:

Conjecture A horo-tight immersion from any closed (orientable) manifold is hyperbolic tight.

Moreover, we consider a special class of surfaces in the hyperbolic space which are called horo-flat surfaces (i.e., flat surfaces in the sense of horospherical geometry).

2 Elementary horocyclic geometry

What is the horospherical geometry? We describe the basic idea of this geometry

2000 Mathematics Subject classification. 53A35, 57R45, 58K40
Keywords and Phrase. Hyperbolic 3-space, Horosphere, Horospherical geometry, Horo-flat surfaces, Singularities

in the hyperbolic plane which might be called the "horocyclic geometry". We consider the Poincaré disk model D^2 of the hyperbolic plane which is an open unit disk in the (x, y) plane with the Riemannian metric : $ds^2 = 4(dx^2 + dy^2)/(1 - x^2 - y^2)$. Therefore it is conformally equivalent to Euclidean plane, so that a circle in the Poincaré disk is also a circle in Euclidean plane. It is well-known that a geodesic in the Poincaré disk is the circle in Euclidean plane which is orthogonal to the ideal boundary (i.e., the unit circle). If we adopt geodesics as the lines in the Poincaré disk, we have a model of Hyperbolic geometry (the non-Euclidean geometry of Gauss-Bolyai-Lobachevski). However, we have another kind of curves in the Poincaré disk which have an analogous property of lines in Euclidean plane. A *horocycle* is a Euclidean circle which is tangent to the ideal boundary (cf., Fig. 1).

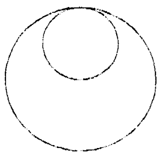


Fig. 1: Horocycle

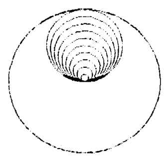


Fig. 2 : The limit of circles

We remind that a line in Euclidean plane can be considered as a limit of circles when the radii tend to infinity. A horocycle is also a curve as a limit of circles when the radii tend to infinity in the Poincaré disk (cf., Fig. 2). Therefore, horocycles are also an analogous notion of lines. If we adopt horocycles as lines, what kind of geometry we obtain? We say that two horocycles are *paralle* if they have the comon tangent point at the ideal boundary. Under this definition, the axiom of parallel is satisfied (cf., Fig. 3). However, for any two points in the disk, there are always two horocycles though the points, so that the axiom 1 of the Euclidean Geometry is not satisfied (cf., Fig. 4). We call this geometry a *horocyclic geometry*. Therefore, the horocyclic geometry is also a non-Euclidean geometry.

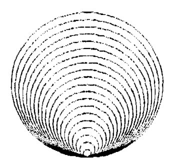


Fig. 3: The axiom of parallel

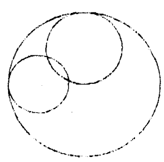
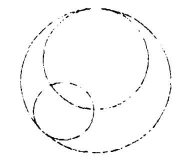


Fig. 4: The axiom 1

It might be said that horocycles have both the properties of lines and circles in Euclidean plane. We define the normal angle between two horocycles as follows: For a horocycle, we have a unit vector on Euclidean plane directed to the tangent points of the horocycle. We define that a *normal angle* between two horocycles is the Euclidean angle between corresponding two unit vectors (cf., Fig. 5, Fig. 6). It is clear that two horocycles are parallel if and only if the normal angle is zero. However, two horocycles are not paralle even if the normal angle is π.



⇒

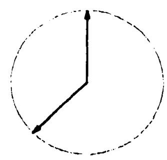
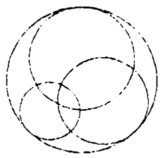


Fig. 5: Two horocycles **Fig. 6: Two unit horonormal vectors**

We now consider three horocycles in the disk (cf., Fig.7, Fig. 8). In this case, there are



⇒

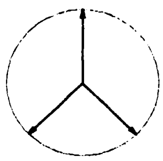


Fig. 7: Horo-triangles **Fig. 8: Three unit horo-normal vectors**

four horo-triangles on the disk. For the simplicity, we consider a horo-convex triangle. For any three horocycles, we say that a triangle is *horo-convex* if the horo-normal unit vector is directed to the inside of the triangle. If we have three horocycles sufficiently large radiuses parallel to given horocycles, there exists a horo-convex horo-triangle. By Fig. 7 and Fig. 8, we can recognize the following theorem:

Theorem 2.1 *The total sum of horo-normal angles of a horo-convex horo-triangle is 2π.*

If we consider the orientation of the horo-triangle, we have the similar theorem for other horo-triangles (under some careful considerations). Moeover, we can show that the total sum of the horo-normal angles of an oriented pieswise horo-cyclic curve is the winding number times 2π, so that it is a topological invariant. This suggests us a kind of the Gauss-Bonnet type theorem holds if we define a suitable curvature of a surface in hyperbolic space (cf., Theorem 4.1). However, the horo-normal angle is not a hyperbolic invariant. If we consider the absolute value of horo-normal angle we have the following inequality.

Theorem 2.2 *The total sum of absolute horo-normal angles of horo-triangle is greater than or equal to 2π. The equality holds if and only if the horo-triangle is horo-convex.*

This theorem suggests us a kindof Chern-Lashof type theorem (cf., Theorem 4.5).

On the other hand, the property that two horocycles are parallel is a hyperbolic invariant which corresponds to the flatness of the "horospherical curvature" in hyperbolic space.

3 Local Horospherical Geometry of submanifolds in Hyperbolic space

We outline in this section the local differential geometry of submanifolds in the hyperbolic n-space developed in the previous papers [13, 14, 15]. We adopt, for this purpose, the model of hyperbolic n-space in the Minkowski $(n+1)$-space. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\ \}$ be an $(n+1)$-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ *Minkowski* $(n+1)$-*space* and denote it by $\mathbb{R}^{n+1}_1$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *hyperbolic n-space* by $H^n_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1\}$ and *de Sitter n-space* by $S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}$. We have three kinds of totally umbilical hypersurfaces in $H^n_+(-1)$ which are given by the intersections of $H^n_+(-1)$ with hyperplanes. A hypersurface given by the intersection of $H^n_+(-1)$ with a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane is respectively called a *hypersphere*, an *equidistant hypersurface* or a *hyperhorosphere*. Especially, a hyperhorosphere is an important subject in this paper, so that we denote it by $HS(\boldsymbol{v}, c) = H^n_+(-1) \cap HP(\boldsymbol{v}, c)$, where $\boldsymbol{v}$ is a lightlike vector. We also define a set $LC^*_+ = \{\boldsymbol{x} = (x_0, \dots x_n) \in LC_0\ | x_0 > 0\ \}$, which is called the *future lightcone* at the origin.

In the first place, we review the results on hypersurfaces in $H^n_+(-1)$. Let $\boldsymbol{X} : U \longrightarrow H^n_+(-1)$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We shall identify $M = \boldsymbol{X}(U)$ and U through the embedding $\boldsymbol{X}$. Since $\langle \boldsymbol{X}, \boldsymbol{X} \rangle \equiv -1$, we have $\langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}(u) \rangle \equiv 0\ \ (i = 1, \dots, n-1)$, for any $u = (u_1, \dots u_{n-1}) \in U$. Therefore, we can define the spacelike unit normal $\mathcal{E}(u) \in S^n_1$. It follows that $\boldsymbol{X}(u) \pm \mathcal{E}(u) \in LC^*_+$ and hence we can define a map $\mathbb{L}^\pm : U \longrightarrow LC^*_+$ by $\mathbb{L}^\pm(u) = \boldsymbol{X}(u) \pm \mathcal{E}(u)$ which is called the *hyperbolic Gauss indicatrix* (or *the lightcone dual*) of $\boldsymbol{X}$. In order to define the hyperbolic Gauss-Kronecker curvature of the hypersurface $M = \boldsymbol{X}(U)$, we have shown in [13] $d\mathbb{L}^\pm(u_0)$ is a linear transformation on the tangent space T_pM. We call the linear transformation $S^\pm_p = -d\mathbb{L}^\pm(u_0) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. We denote the eigenvalues of $S^\pm_p$ by $\bar{\kappa}^\pm_p$ and the eigenvalues of A_p by $\kappa_i(p)$ $(i = 1, \dots n-1)$ which are called the *hyperbolic principal curvatures*. The *hyperbolic Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ is defined to be $K^\pm_h(u_0) = \det S^\pm_p = \kappa_1(p) \cdots \kappa_{n-1}(p)$. Since $\boldsymbol{X}_{u_i}$ $(i = 1, \dots n-1)$ are spacelike vectors, we have the Riemannian metric given by $ds^2 = \sum_{i=1}^{n-1} g_{ij} du_i du_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ and the *hyperbolic second fundamental invariant* defined by $\bar{h}^\pm_{ij}(u) = \langle -\mathbb{L}^\pm_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. In [13] the hyperbolic version of the Weingarten formula was shown and the formula $K^\pm_h = \det\left(\bar{h}^\pm_{ij}\right) / \det\left(g_{\alpha\beta}\right)$ was obtained.

In the previous paragraphs we reviewed the properties of hyperbolic Gauss indicatrices and hyperbolic Gauss-Kronecker curvatures. The original definition of the hyperbolic Gauss map introduced by Bryant [4] and Epstein [7] is given in the Poincaré ball model. Here, we introduce the corresponding definition in Minkowski model as follows: If $\boldsymbol{x} =$

$(x_0, x_1, \ldots, x_n)$ is a lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in S_+^{n-1} = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\ \}.$$

We call S_+^{n-1} the *lightcone* $(n-1)$*-sphere*. We define a map

$$\widetilde{\mathbb{L}}^{\pm} : U \longrightarrow S_+^{n-1}$$

by $\widetilde{\mathbb{L}}^{\pm}(u) = \widetilde{\mathbb{L}^{\pm}(u)}$ and call it the *hyperbolic Gauss map* of $\boldsymbol{X}$. Let N_pM be the pseudo-normal space of T_pM in $T_p\mathbb{R}_1^{n+1}$. We have the decomposition $T_p\mathbb{R}_1^{n+1} = T_pM \oplus N_pM$, so that we also have the Whitney sum $T\mathbb{R}^{n+1} = TM \oplus NM$. Therefore we have the canonical projection $\Pi : T\mathbb{R}^{n+1} \longrightarrow TM$. It follows that we have a linear transformation $\Pi_p \circ d\widetilde{\mathbb{L}}^{\pm}(u) : T_pM \longrightarrow T_pM$ for $p = \boldsymbol{X}(u)$ by the identification of U and $\boldsymbol{X}(U) = M$ via $\boldsymbol{X}$. In [18] the following formula was shown:

Proposition 3.1 *Under the above notation we have the following horospherical Weingarten formula:*

$$\Pi_p \circ \widetilde{\mathbb{L}}^{\pm}_{u_i} = -\sum_{j=1}^{n-1} \frac{1}{\ell_0^{\pm}(u)} \left(\bar{h}^{\pm}\right)_i^j \boldsymbol{X}_{u_j},$$

where $\mathbb{L}^{\pm}(u) = (\ell_0^{\pm}(u), \ell_1^{\pm}(u), \ldots, \ell_n^{\pm}(u))$.

We call the linear transformation $\widetilde{S}_p^{\pm} = -\Pi_p \circ d\widetilde{\mathbb{L}}^{\pm}$ the *horospherical shape operator* of $M = \boldsymbol{X}(U)$. The *horospherical Gauss-Kronecker curvature* of $\boldsymbol{X}(U) = M$ is defined to be $\widetilde{K}_h^{\pm}(u) = \det \widetilde{S}_p^{\pm}$. It follows that we have the following relation between the horospherical Gauss-Kronecker curvature and the hyperbolic Gauss-Kronecker curvature:

$$\widetilde{K}_h^{\pm}(u) = \left(\frac{1}{\ell_0^{\pm}(u)}\right)^{n-1} K_h^{\pm}(u).$$

We remark that $\widetilde{K}_h^{\pm}(u)$ is not invariant under hyperbolic motions but it is an $SO(n)$-invariant. We also remark that the notion of horospherical curvatures is independent of the choice of the model of hyperbolic space. For the purpose, we introduce a smooth function on the unit tangent sphere bundle of hyperbolic space which plays the principal role of the horospherical geometry. Let $SO_0(n,1)$ be the identity component of the matrix group

$$SO(n,1) = \{g \in GL(n+1, \mathbb{R}) \mid gI_{n,1}{}^t g = I_{n,1}\ \},$$

where

$$I_{n,1} = \left(\begin{array}{c|c} -1 & \mathbf{0} \\ \hline {}^t\mathbf{0} & I_n \end{array}\right) \in GL(n+1, \mathbb{R}).$$

It is well-known that $SO_0(n,1)$ acts transitively on $H_+^n(-1)$ and the isotropic group at $p = (1, 0, \ldots, 0)$ is $SO(n)$ which is naturally embedded in $SO_0(n,1)$. Moreover the action induces isometries on $H_+^n(-1)$. On the other hand, we consider a submanifold

$\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \}$ of $H^n_+(-1) \times S^n_1$ and the canonical projection $\bar{\pi} : \Delta \longrightarrow H^n_+(-1)$. Let $\pi : S(TH^n_+(-1)) \longrightarrow H^n(-1)$ be the unit tangent sphere bundle over $H^n_+(-1)$. For any $\boldsymbol{v} \in H^n_+(-1)$, we have the coordinates $(v_1, \dots, v_n)$ of $H^n_+(-1)$ such that $\boldsymbol{v} = (\sqrt{v_1^2 + \cdots + v_n^2 + 1}, v_1, \dots, v_n)$. We can canonically identify $\pi : S(TH^n_+(-1)) \longrightarrow H^n_+(-1)$ with $\bar{\pi} : \Delta \longrightarrow H^n_+(-1)$. Moreover, the linear action of $SO_0(n,1)$ on $\mathbb{R}^{n+1}_1$ induces the canonical action on Δ (i.e., $g(\boldsymbol{v}, \boldsymbol{w}) = (g\boldsymbol{v}, g\boldsymbol{w})$ for any $g \in SO_0(n,1)$). For any $(\boldsymbol{v}, \boldsymbol{w}) \in \Delta$, the first component of $\boldsymbol{v} \pm \boldsymbol{w}$ is given by

$$v_0 \pm w_0 = \sqrt{v_1^2 + \cdots + v_n^2 + 1} \pm \frac{1}{\sqrt{v_1^2 + \cdots + v_n^2 + 1}} \sum_{i=1}^{n} v_i w_i,$$

so that it can be considered as a function on the unit tangent bundle $S(TH^n_+(-1))$.

We now define a function

$$\mathcal{N}_h : \Delta \longrightarrow \mathbb{R} \ ; \ \mathcal{N}_h(\boldsymbol{v}, \boldsymbol{w}) = \frac{1}{v_0 + w_0}.$$

We call $\mathcal{N}_h^{\pm}$ a *horospherical normalization function* on $H^n_+(-1)$. Since $v_1^2 + \cdots + v_n^2 + 1$ and $\sum_{i=1}^{n} v_i w_i$ are $SO(n)$-invariant functions, $\mathcal{N}_h$ is an $SO(n)$-invariant function. Therefore, $\mathcal{N}_h^{\pm}$ can be considered as a function on the unit tangent sphere bundle over the hyperbolic space $SO_0(n,1)/SO(n)$ which is independent of the choice of the model space. For any embedding $\boldsymbol{X} : U \longrightarrow H^n_+(-1)$ $(U \subset \mathbb{R}^{n-1})$, we have the unit normal vector field $\mathcal{E} : U \longrightarrow S^n_1$, so that $(\boldsymbol{X}(u), \mathcal{E}(u)) \in \Delta$ for any $u \in U$. It follows that

$$\widetilde{K}_h^{\pm}(u) = \mathcal{N}_h(\boldsymbol{X}(u), \pm\mathcal{E}(u))^{n-1} K_h^{\pm}(u).$$

The right hand side of the above equality is independent of the choice of the model space.

We now consider general submanifolds in $H^n_+(-1)$ (cf., [17]). Let $\boldsymbol{X} : U \to H^n_+(-1)$ be an embedding of codimension $(s+1)$, where $U \subset \mathbb{R}^r$ is an open subset $(r + s + 1 = n)$. We also write that $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. Let $N_p(M)$ be the normal space of M at $p = \boldsymbol{X}(u)$ in $\mathbb{R}^{n+1}_1$ and we define $N^h_p(M) = N_p(M) \cap T_pH^n_+(-1)$. Since the normal bundle $N(M)$ is trivial, we can arbitrarily choose a unit normal section $\boldsymbol{N}(u) \in S^s(N^h_p(M))$. We consider the orthogonal projections $\pi^T : T_pM \oplus N^h_p(M) \longrightarrow T_pM$ and $\pi^N : T_pM \oplus N^h_p(M) \longrightarrow N^h_p(M)$. Let $d\boldsymbol{N}_u : T_uU \longrightarrow T_pM \oplus N^h_p(M)$ be the derivative of $\boldsymbol{N}$. We define that $d\boldsymbol{N}^T_u = \pi^T \circ d\boldsymbol{N}_u$ and $d\boldsymbol{N}^N_u = \pi^N \circ d\boldsymbol{N}_u$. Under the identification of U and M, the derivative $d\boldsymbol{X}_u$ can be identified with the identity mapping id_{T_pM}. We call the linear transformation $S_{p_0}(\boldsymbol{N}) = -(\mathrm{id}_{T_{p_0}M} + d\boldsymbol{N}^T_{u_0}) : T_{p_0}M \longrightarrow T_{p_0}M$ the *hyperbolic $\boldsymbol{N}$-shape operator* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$. The *hyperbolic curvature* with respect to $\boldsymbol{N}$ at $p_0 = \boldsymbol{X}(u_0)$ is defined to be

$$K_h(\boldsymbol{N})(\boldsymbol{X}(u_0)) = K_h(\boldsymbol{N})_{p_0} = \det S_{p_0}(\boldsymbol{N}).$$

We give the following generalized hyperbolic Weingarten formula. Since $\boldsymbol{X}_{u_i}$ $(i = 1, \dots r)$ are spacelike vectors, we induce the Riemannian metric (the *hyperbolic first fundamental* *form*) $ds^2 = \sum_{i=1}^r g_{ij}du_idu_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We also define the *hyperbolic second fundamental invariant with respect to the unit normal vector field* $\boldsymbol{N}$ by $\bar{h}_{ij}(\boldsymbol{N})(u) = \langle -(\boldsymbol{X}+\boldsymbol{N})_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. If we define the *second fundamental invariant with respect to the normal vector field* $\boldsymbol{N}$ by $h_{ij}(\boldsymbol{N})(u) = -\langle \boldsymbol{N}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$, then we have the following relation:

$$\bar{h}_{ij}(\boldsymbol{N})(u) = -g_{ij}(u) + h_{ij}(\boldsymbol{N})(u), \ (i,j = 1,\dots,r).$$

Proposition 3.2 *Under the above notations, we have the following horospherical (or, hyperbolic) Weingarten formula with respect to* $\boldsymbol{N}$:

$$\pi^T \circ (\boldsymbol{X}+\boldsymbol{N})_{u_i} = -\sum_{j=1}^r \bar{h}_i^j(\boldsymbol{N})\boldsymbol{X}_{u_j},$$

where $\left(\bar{h}_i^j(\boldsymbol{N})\right) = \left(\bar{h}_{ik}(\boldsymbol{N})\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = (g_{kj})^{-1}$. *It follows that the hyperbolic curvature with respect to* $\boldsymbol{N}$ *is given by*

$$K_h(\boldsymbol{N})(\boldsymbol{X}(u)) = \frac{\det\left(\bar{h}_{ij}(\boldsymbol{N})(u)\right)}{\det\left(g_{\alpha\beta}(u)\right)}.$$

Since $\langle -(\boldsymbol{X}+\boldsymbol{N})(u), \boldsymbol{X}_{u_j}(u)\rangle = 0$, we have $\bar{h}_{ij}(\boldsymbol{N})(u) = \langle \boldsymbol{X}(u)+\boldsymbol{N}(u), \boldsymbol{X}_{u_iu_j}(u)\rangle$. Therefore the hyperbolic second fundamental invariant at a point $p_0 = \boldsymbol{X}(u_0)$ depends only on $\boldsymbol{X}(u_0)+\boldsymbol{N}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$. By the above corollary, the hyperbolic curvature also depends only on $\boldsymbol{X}(u_0)+\boldsymbol{N}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$. It is independent on the choice of the normal vector field $\boldsymbol{N}$. We write $K_h(\boldsymbol{n})(\boldsymbol{X}(u_0))$ as the hyperbolic curvature at $p_0 = \boldsymbol{X}(u_0)$ with respect to $\boldsymbol{n} = \boldsymbol{N}(u_0)$ (i.e., $K_h(\boldsymbol{n})(\boldsymbol{X}(u_0)) = K_h(\boldsymbol{N})(\boldsymbol{X}(u_0))$).

4 Total horospherical curvatures

We now consider the global properties of curvatures. We first consider the hypersurface case. Let M be a closed orientable $(n-1)$-dimensional manifold and $f : M \longrightarrow H_+^n(-1)$ an embedding. We consider the canonical projection $\pi : \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (0, x_1, \dots, x_n)$. Then we have orientation preserving diffeomorphisms $\pi|H_+^n(-1) : H_+^n(-1) \longrightarrow \mathbb{R}^n$ and $\pi|S_+^{n-1} : S_+^{n-1} \longrightarrow S^{n-1}$. Consider the outward unit normal $\mathbb{E}$ of $f(M)$ in $H_+^n(-1)$, then we define the hyperbolic Gauss indicatrix in the global

$$\mathbb{L}^{\pm} : M \longrightarrow LC_+^*$$

by

$$\mathbb{L}^{\pm}(p) = f(p) \pm \mathbb{E}(p).$$

The *global hyperbolic Gauss-Kronecker curvature function* $\mathcal{K}_h : M \longrightarrow \mathbb{R}$ is then defined in the usual way in terms of the global hyperbolic Gauss indicatrix $\mathbb{L}$. We also define the hyperbolic Gauss map in the global

$$\widetilde{\mathbb{L}}^{\pm} : M \longrightarrow S_+^{n-1}$$

by

$$\widetilde{\mathbb{L}^{\pm}}(p) = \widetilde{\mathbb{L}^{\pm}(p)}.$$

We now define a *global horospherical Gauss-Kronecker curvature function* $\widetilde{\mathcal{K}}_h^{\pm} : M \longrightarrow \mathbb{R}$ by

$$\widetilde{\mathcal{K}}_h^{\pm}(p) = \mathcal{N}_h(f(p), \pm\mathbb{E}(p))^{n-1}\mathcal{K}_h^{\pm}(p).$$

In [19] the following Gauss-Bonnet type theorem for the horospherical Gauss-Kronecker curvature was shown.

Theorem 4.1 *If M is a closed orientable even-dimensional hypersurface in hyperbolic n-space, then*

$$\int_M \widetilde{\mathcal{K}}_h^{\pm} d\mathfrak{v}_M = \frac{1}{2}\gamma_{n-1}\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M, $d\mathfrak{v}_M$ is the volume form of M and the constant γ_{n-1} is the volume of the unit $(n-1)$-sphere S^{n-1}.

In order to prove the above theorem, it has been shown that $\widetilde{\mathcal{K}}_h^{\pm} d\mathfrak{v}_M = (\widetilde{\mathbb{L}}^{\pm})^* d\mathfrak{v}_{S_+^{n-1}}$, where $d\mathfrak{v}_{S_+^{n-1}}$ is the canonical volume form of S_+^{n-1} [19]. Let $D \subset S_+^{n-1}$ denote the set of regular values of $\widetilde{\mathbb{L}}^{\pm}$. Since M is compact, D is open and, by Sard's theorem, the complement of D in S_+^{n-1} has null measure. We define the integer valued map $\eta^{\pm} : D \longrightarrow \mathcal{E}$ by setting

$$\eta^{\pm}(\boldsymbol{v}) = \text{the number of elements of } (\widetilde{\mathbb{L}}^{\pm})^{-1}(\boldsymbol{v}),$$

which turns out to be continuous. We have the following theorem.

Theorem 4.2 *Let $f : M^{n-1} \longrightarrow H_+^n(-1)$ be an immersion of the compact manifold M^{n-1}. Then*

$$\int_M \left|\widetilde{\mathcal{K}}_h^{\pm}\right| d\mathfrak{v}_M = \int_D \eta^{\pm}(\boldsymbol{v})\, d\mathfrak{v}_{S_+^{n-1}}.$$

For the surface $M \subset H_+^3(-1)$, we have shown the following theorem as an application of Theorem 4.2 ([5])

Theorem 4.3 *Let M^2 be an embedded closed surface in $H_+^3(-1)$, then*

$$\int_M \left|\widetilde{\mathcal{K}}_h^{\pm}\right| d\mathfrak{a}_M \geq 2\pi(4 - \chi(M)).$$

We remark that the right hand side of the inequality will be much more complicated if we consider a hypersurface $M \subset H_+^n(-1)$. Actually we need some information on the Betti numbers of M and the volume of the unit sphere S^{n-1}. However, we have the following rough estimate:

Theorem 4.4 *Let $f : M \longrightarrow H^n_+(-1)$ be an embedding from a closed orientable manifold with dimension $n-1$. Then we have*

$$\int_M \left|\widetilde{\mathcal{K}}_h^{\pm}\right| d\mathfrak{v}_M \geq \gamma_{n-1},$$

where γ_{n-1} is the volume of the unit sphere S^{n-1}_+. The equality holds if and only if $\widetilde{\mathbb{L}}^{\pm}$ is bijective on the regular values.

We now define the *total absolute horospherical curvature* for an embedding $f : M \longrightarrow H^n_+(-1)$ from a closed orientable manifold with dimension $n-1$ by

$$\tau_h^{\pm}(f; M) = \frac{1}{\gamma_{n-1}} \int_M \left|\widetilde{\mathcal{K}}_h^{\pm}\right| d\mathfrak{v}_M.$$

On the other hand, we consider general submanifolds in $H^n_+(-1)$. Let M be a compact r-dimensional manifold and $f : M^r \to H^n_+(-1)$ denotes an immersion of codimension $(s+1)$. Let $\nu^1(M)$ denote the unitary normal bundle of the immersion f, i.e.:

$$\nu^1(M) = \{(p,\xi); \xi \in N^h_p(M) \text{ and } \langle \xi, \xi \rangle = 1\}.$$

The *horospherical Gauss map* $\widetilde{\mathbb{L}} : \nu^1(M) \to S^{n-1}_+$ of the immersion $f : M^s \to H^n_+(-1)$ is defined by the following commutative diagram

$$\begin{tikzcd} \nu^1(M) \arrow[r, "\mathbb{L}"] \arrow[dr, "\widetilde{\mathbb{L}}"'] & LC^*_+ \arrow[d, "\Pi"] \\ & S^{n-1}_+ \end{tikzcd}$$

where $\mathbb{L} : \nu^1(M) \to LC^*_+; \mathbb{L}(p,\xi) = f(p) + \xi$ is called the *hyperbolic Gauss indicatrix* of the immersion f and $\Pi(\boldsymbol{v}) = \widetilde{\boldsymbol{v}}$. The horospherical Gauss map lead us to a curvature in the framework of horospherical geometry. Let $T_{(\boldsymbol{x},\boldsymbol{n})}\nu^1(M)$ be the tangent space of $\nu^1(M)$ at $(\boldsymbol{x},\boldsymbol{n})$. We have the canonical identification $T_{(\boldsymbol{x},\boldsymbol{n})}\nu^1(M) = T_{\boldsymbol{x}}M \oplus T_{\boldsymbol{n}}S^s \subset T_{\boldsymbol{x}}M \oplus N_{\boldsymbol{x}}M = T_{\boldsymbol{x}}\mathbb{R}^{n+1}_1$, where $N_{\boldsymbol{x}}M$ is the normal vector space of M at $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1$. Let $P : \widetilde{\mathbb{L}}^*T\mathbb{R}^{n+1}_1 = T\nu^1(M) \oplus \mathbb{R}^2 \to T\nu^1(M)$ be the canonical projection. It follows that we have a linear transformation

$$P_{\widetilde{\mathbb{L}}(\boldsymbol{x},\boldsymbol{n})} \circ d\widetilde{\mathbb{L}} : T_{(\boldsymbol{x},\boldsymbol{n})}\nu^1(M) \to T_{(\boldsymbol{x},\boldsymbol{n})}\nu^1(M).$$

The *horospherical curvature* with respect to $\boldsymbol{n}$ at $\boldsymbol{x}$ is defined to be

$$\widetilde{K}_h(\boldsymbol{x},\boldsymbol{n}) = \det\left(P_{\widetilde{\mathbb{L}}(\boldsymbol{x},\boldsymbol{n})} \circ (-d\widetilde{\mathbb{L}})\right).$$

In [5] we have shown that

$$\widetilde{K}_h(\boldsymbol{x},\boldsymbol{n})(p) = \mathcal{N}_h^{n-1}(\boldsymbol{x},\boldsymbol{n}) K_h(\boldsymbol{n})(f(p)).$$

and

$$(\widetilde{\mathbb{L}}^* d\mathfrak{v}_{S_+^{n-1}})_{(\boldsymbol{x},\boldsymbol{n})} = |\widetilde{K}_h(\boldsymbol{x},\boldsymbol{n})|\, d\mathfrak{v}_{\nu^1(M)}.$$

The *total absolute horospherical curvature* of the immersion f is defined by

$$\tau_h(f;M) = \frac{1}{\gamma_{n-1}} \int_{\nu^1(M)} \widetilde{\mathbb{L}}^* \sigma.$$

It follows from the above formula that we have

$$\tau_h(f;M) = \frac{1}{\gamma_{n-1}} \int_{\nu^1(M)} |\widetilde{K}_h(\boldsymbol{x},\boldsymbol{n})|\, d\mathfrak{v}_{\nu^1(M)},$$

In [5] we have shown the following Chern-Lashof type theorem.

Theorem 4.5 *Let $f : M^r \longrightarrow H_+^n(-1)$ be an immersion of the compact manifold M. Then*

1. $\tau_h(f;M) \geqslant \gamma(M) \geqslant 2$;
2. *if $\tau_h(f;M) < 3$ then M is homeomorphic to the sphere S^r.*

It has been posed the following question in [5]:

Question 4.6 *How is the geometry of $f(M) \subset H_+^n(-1)$ if $\tau_h(f;M) = 2$?*

We have also given an answer to this question in [6].

Remark 4.7 If $r = n-1$, then $\nu^1(M)$ is a double covering over M, so that $\widetilde{\mathbb{L}}(p, \pm\mathcal{E}(p)) = f(p) \pm \mathcal{E}(p) = \widetilde{\mathbb{L}}^{\pm}(p)$ (i.e., $\mathbb{L}(p, \pm\mathcal{E}(p)) = \mathbb{L}^{\pm}(p)$). Therefore, we have the following weaker inequality as a corollary of Theorem 3.5:

$$\tau_h^+(f;M) + \tau_h^-(f;M) = \frac{1}{\gamma_{n-1}} \left(\int_M \left|\widetilde{\mathcal{K}}_h^+\right| d\mathfrak{v}_M + \int_M \left|\widetilde{\mathcal{K}}_h^-\right| d\mathfrak{v}_M \right) = \tau_h(f;M) \geq 2.$$

In §6 we give one of the examples of curves in $H_+^2(-1)$ such that

$$\int_M \left|\widetilde{\mathcal{K}}_h^+\right| d\mathfrak{v}_M \neq \int_M \left|\widetilde{\mathcal{K}}_h^-\right| d\mathfrak{v}_M.$$

5 Horo-tight immersions of spheres

What are the horo-tight immersions of spheres? We address this section to this question proposed by Thomas E. Cecil and Patrick J. Ryan in ([10], pg 236). The notion of horo-tightness was introduced in [9], whose main subjects are tight and taut immersions into hyperbolic space. In [6] we have shown Theorems 5.2, 5.3, 5.5 and 5.7 which give several characterizations on horo-tight spheres in hyperbolic space. These results give a complete answer to the question of Cecil and Ryan.

We first define two families of functions

$$H^h : M \times S_+^{n-1} \longrightarrow \mathbb{R}$$

by $H^h(p, \boldsymbol{v}) = \langle f(p), \boldsymbol{v} \rangle$ and

$$H^d : M \times S_1^n \longrightarrow \mathbb{R}$$

by $H^d(p, \boldsymbol{v}) = \langle f(p), \boldsymbol{v} \rangle$. We call H^h a *horospherical height functions family* and H^d a *de Sitter height functions family* on $f : M \longrightarrow H_+^n(-1)$. Each $h_{v_0}^h(p) = H^h(p, \boldsymbol{v}_0)$ for fixed $\boldsymbol{v}_0 \in S_+^{n-1}$ (respectively, $h_{v_0}^d(p) = H^d(p, \boldsymbol{v}_0)$ for fixed $\boldsymbol{v}_0 \in S_1^n$) is called a *horospherical height function* (respectively, *de Sitter height function*). We denote the *Hessian matrix* of the horospherical height function $h_{v_0}^h$ at $p_0 \in M$ by $\operatorname{Hess}(h_{v_0}^h)(p_0)$. We say that the critical point $p \in M$ of $h_{v_0}^h$ is *non-degenerate* if $\det \operatorname{Hess}(h_{v_0}^h)(u_0) \neq 0$. We say that a function $f : M \longrightarrow \mathbb{R}$ is *non-degenerate* if f has only non-degenerate critical points. An immersion $f : M \longrightarrow H_+^n(-1)$ is said to be *hyperbolic tight* (*H-tight* for short) if every non-degenerate de Sitter height function h_v^d has the minimum number of critical points required by the Morse inequalities. We also say that $f : M \longrightarrow H_+^n(-1)$ is *horospherical tight* (*horo-tight* for short) if every non-degenerate horospherical height function h_v^h has the minimum number of critical points required by the Morse inequalities.

Remark 5.1 In [8] a function $L_h : H_+^n(-1) \longrightarrow \mathbb{R}$ has been defined to be $L_h(p) = \ln(-h_v^h(p))$ which is called the *distance function from p to the hyperhorosphere* $HS(\boldsymbol{v}, -1)$ for $\boldsymbol{v} \in S^{n-1}$. Therefore the minimum of L_h corresponds to the maximum of h_v^h (i.e., the minimum of $-h_v^h$).

The main results in this section are the following.

Theorem 5.2 *Let $f : S^r \to H_+^n(-1)$ be an immersion. Then f is horo-tight if and only if f is H-tight.*

We remark that the above theorem gives an answer to the question of Cecil and Ryan. For $n > r + 1$ this theorem is a corollary of the following characterization of horo-tight embeddings of spheres in higher codimension.

Theorem 5.3 *Let $f : S^r \to H_+^{r+k}(-1)$, $k > 1$ be an immersion. Then f is horo-tight if and only if f embeds S^r as an r-dimensional metric sphere.*

The following properties of horo-tight immersions of manifolds into hyperbolic space can be found in [3].

Theorem 5.4 [Bolton, Theorem 1] *Let $f : M \to H_+^n(-1)$ be an immersion of a compact manifold into the hyperbolic space. The following conditions are equivalent:*

(i) *M is homeomorphic to a sphere and $f(M)$ is horo-tight.*

(ii) *$f(M)$ lies in only one side of any tangent hyperhorosphere.*

(iii) *The horospherical Gauss map $\widetilde{\mathbb{L}} : \nu^1(M) \longrightarrow S_+^{n-1}$ takes every regular value exactly twice.*

We can give an answer to Question 4.6 as follows.

Theorem 5.5 *Let $f : M \to H^n_+(-1)$ be an immersion of a compact manifold into the hyperbolic space. Then M is homeomorphic to a sphere and $f(M)$ is horo-tight if and only if*

$$\tau_h(f; M) = 2.$$

Proposition 5.6 *Let $f : M^r \to H^n_+(-1)$, $n > r+1$ be an immersion of a compact manifold into the hyperbolic space. If one of the above conditions* (i) *to* (iii) (*and hence all of them*) *of Theorem 5.4 holds, then $f(M)$ lies in one hyperhorosphere.*

We now consider the characterization of hyperspheres in hyperbolic space which attend the minimum of the total absolute horospherical curvature. We first consider the case of hypersurfaces in hyperbolic space. Let $f : M \longrightarrow H^n_+(-1)$ be an embedding from an $(n-1)$-dimensional manifold. In the first place, we recall that the minimum for the *total absolute curvature* of a hypersphere in Euclidean space $\mathbb{R}^n$ is 1 and this minimum is attained precisely when the image is the convex hypersphere. Moreover, for codimension one embeddings of spheres in Euclidean spaces, the property of attending the minimum of the total absolute curvature is equivalent to the notion of tightness. We have obtained a similar result for the image of hyperspheres in hyperbolic space in [6].

A set $X \subset H^n_+(-1)$ is *convex* if for any pair of points in X the geodesic segment joining them is contained in X. Every hyperhorosphere $\mathcal{H}$ in $H^n_+(-1)$ is the boundary of a closed convex region of $H^n_+(-1)$. These convex subsets are called *h-convex*. We say that a submanifold (or, an immersion) $f : M \to H^n_+(-1)$ is *horospherical convex* (*horo-convex* for short) if for any $p \in M$, one of the h-convex sets determined by its tangent hyperhorosphere at $f(p)$ contains $f(M)$ entirely.

Theorem 5.7 *For an immersion $f : S^{n-1} \longrightarrow H^n_+(-1)$, the following conditions are equivalent:*

(1) *f is horo-convex.*

(2) $\tau_h^+(f; S^{n-1}) = \tau_h^-(f; S^{n-1}) = 1$.

(3) $\tau_h(f; S^{n-1}) = 2$.

(4) *Both mappings $\widetilde{\mathbb{L}}^+$ and $\widetilde{\mathbb{L}}^-$ are bijective on the regular values.*

(5) *f is horo-tight.*

(6) *f is H-tight.*

6 Horospherical flat surfaces

In this section we investigate a special class of surfaces in hyperbolic 3-space which are called *horospherical flat surfaces.* We say that a surface $M = \boldsymbol{X}(U)$ is *horospherical flat* (briefly, *horo-flat*) if $\widetilde{K}_h(p) = 0$ at any point $p \in M$. By a direct consequence of the relation in §3, $K_h(p) = 0$ if and only if $\widetilde{K}_h(p) = 0$, so that the horospherical flatness is a hyperbolic invariant. Moreover, there is an important class of surfaces called *linear Weingarten surfaces* which satisfy the relation $aK_I + b(2H-2) = 0$ $((a,b) \neq (0,0))$. In [11], the Weierstrass-Bryant type representation formula for such surfaces with $a+b \neq 0$ (called, a *linear Weingarten surface of Bryant type*) was shown. This class of surfaces contains flat surfaces (i.e., $a \neq 0, b = 0$) and CMC-1(constant mean curvature one) surfaces ($a = 0, b \neq 0$). In the celebrated paper [4], Bryant showed the Weierstrass type representation formula for CMC-1 surfaces in hyperbolic space. This is the reason why the class of the surface with $a + b \neq 0$ is called of Bryant type. By using such representation formula, there are a lot of results on such surfaces. We only refer [11, 22, 23, 25, 26] here. The horospherical flat surface is one of the linear Weingarten surfaces. It is, however, the exceptional case (a *linear Weingarten surface of non-Bryant type* : $a + b = 0$). There are no Weierstarass-Bryant type representation formula for such surfaces so far as we know. Therefore the horospherical flat surfaces are also very important subjects in the hyperbolic geometry. If we suppose that a surface is umbilically free, then we have the following expression: Let $\boldsymbol{X} : U \longrightarrow H^3_+(-1)$ be a horospherical surface without umbilical points, where $U \subset \mathbb{R}^2$ is a neighborhood around the origin. In this case, we have two lines of curvature at each point and one of which corresponds to the vanishing hyperbolic principal curvature. We may assume that both the u-curve and the v-curve are the lines of curvature for the coordinate system $(u,v) \in U$. Moreover, we assume that the u-curve corresponds to the vanishing hyperbolic principal curvature. By the hyperbolic Weingarten formula, we have

$$\mathbb{L}_u(u,v) = \mathbf{0} \quad \mathbb{L}_v(u,v) = -\bar{\kappa}(u,v)\boldsymbol{X}_v(u,v),$$

where $\bar{\kappa}(u,v) \neq 0$. It follows that $\mathbb{L}(0,v) = \mathbb{L}(u,v)$. We define a function $F : H^3_+(-1) \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ by $F(\boldsymbol{X}, v) = \langle \mathbb{L}(0,v), \boldsymbol{X} \rangle + 1$, for sufficiently small $\varepsilon > 0$. For any fixed $v \in (-\varepsilon, \varepsilon)$, we have a horosphere $HS^2(\mathbb{L}(0,v), -1)$, so that $F = 0$ define a one-parameter family of horospheres. In [20] we have shown that the surface $M = \boldsymbol{X}(U)$ is the envelope of the family of horospheres defined by $F = 0$.

On the other hand, we consider a surface $\widetilde{\boldsymbol{X}} : I \times J \longrightarrow H^3_+(-1)$ defined by

$$\widetilde{\boldsymbol{X}}(s,v) = \boldsymbol{X}(0,v) + s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + \frac{s^2}{2}\mathbb{L}(0,v),$$

where $I, J \subset \mathbb{R}$ are open intervals. We have also shown that the surface $\widetilde{M} = \widetilde{\boldsymbol{X}}(I \times J)$ is the envelope of the family of horospheres defined by $F = 0$. It follows that a horo-flat surface can be reparametrized (at least locally) by $\widetilde{\boldsymbol{X}}(s,v)$. If we fix $v = v_0$, we denote that $\boldsymbol{a}_0 = \boldsymbol{X}(0,v_0)$, $\boldsymbol{a}_1 = \boldsymbol{X}_u(0,v_0)/\|\boldsymbol{X}_u(0,v_0)\|$, $\boldsymbol{a}_2 = \boldsymbol{e}(0,v_0)$. Then we have a curve

$$\boldsymbol{\gamma}(s) = \boldsymbol{a}_0 + s\boldsymbol{a}_1 + \frac{s^2}{2}(\boldsymbol{a}_0 + \boldsymbol{a}_2).$$

We can show that $\boldsymbol{\gamma}(s)$ is a horocycle. Moreover, any horocyclic has the above parametrization. Therefore the horo-flat surface is given by the one-parameter family of horocycles.

We say that a surface is a *horocyclic surface* if it is (at least locally) parametrized by one-parameter families of horocycles around any point. Eventually we have the following theorem[20]:

Theorem 6.1 *If $M \subset H^3_+(-1)$ is an umbilically free horo-flat surface, it is a horocyclic surface. Moreover, each horocycle is the line of curvatures with the vanishing hyperbolic principal curvature.*

It follows that our main subjects are the horocyclic surfaces. Let $\gamma : I \longrightarrow H^3_+(-1)$ be a smooth map and $\boldsymbol{a}_i : I \longrightarrow S^3_1$ $(i = 1, 2)$ be smooth mappings from an open interval I with $\langle \gamma(t), \boldsymbol{a}_i(t)\rangle = \langle \boldsymbol{a}_1(t), \boldsymbol{a}_2(t)\rangle = 0$. We define a unit spacelike vector $\boldsymbol{a}_3(t) = \gamma(t) \wedge \boldsymbol{a}_1(t) \wedge \boldsymbol{a}_2(t)$, so that we have a pseudo-orthonormal frame $\{\gamma, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ of $\mathbb{R}^4_1$. We now define a mapping

$$F_{(\gamma,a_1,a_2)} : \mathbb{R} \times I \longrightarrow H^3_+(-1)\ ;\ F_{(\gamma,a_1,a_2)}(s,t) = \gamma(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t),$$

where $\boldsymbol{\ell}(t) = \gamma(t) + \boldsymbol{a}_2(t)$. We call $F_{(\gamma,a_1,a_2)}$ (or the image of it) a *horocyclic surface.* Each horocycle $F_{(\gamma,a_1,a_2)}(s, t_0)$ is called a *generating horocycle.* By using the above pseudo-orthonormal frame, we define the following fundamental invariants:

$$\begin{aligned}
c_1(t) &= \langle \gamma'(t), \boldsymbol{a}_1(t)\rangle = -\langle \gamma(t), \boldsymbol{a}_1'(t)\rangle, & c_4(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t)\rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_2'(t)\rangle,\\
c_2(t) &= \langle \gamma'(t), \boldsymbol{a}_2(t)\rangle = -\langle \gamma(t), \boldsymbol{a}_2'(t)\rangle, & c_5(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_3'(t)\rangle,\\
c_3(t) &= \langle \gamma'(t), \boldsymbol{a}_3(t)\rangle = -\langle \gamma(t), \boldsymbol{a}_3'(t)\rangle, & c_6(t) &= \langle \boldsymbol{a}_2'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_2(t), \boldsymbol{a}_3'(t)\rangle.
\end{aligned}$$

We can show that the following fundamental differential equations for the horocyclic surface:

$$\begin{pmatrix} \gamma'(t)\\ \boldsymbol{a}_1'(t)\\ \boldsymbol{a}_2'(t)\\ \boldsymbol{a}_3'(t)\end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t)\\ c_1(t) & 0 & c_4(t) & c_5(t)\\ c_2(t) & -c_4(t) & 0 & c_6(t)\\ c_3(t) & -c_5(t) & -c_6(t) & 0\end{pmatrix}\begin{pmatrix} \gamma(t)\\ \boldsymbol{a}_1(t)\\ \boldsymbol{a}_2(t)\\ \boldsymbol{a}_3(t)\end{pmatrix}.$$

We remark that

$$C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t)\\ c_1(t) & 0 & c_4(t) & c_5(t)\\ c_2(t) & -c_4(t) & 0 & c_6(t)\\ c_3(t) & -c_5(t) & -c_6(t) & 0\end{pmatrix} \in \mathfrak{so}(3,1),$$

where $\mathfrak{so}(3,1)$ is the Lie algebra of the Lorentzian group $SO_0(3,1)$. If $\{\gamma(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ is a pseudo-orthonormal frame field as the above, the 4×4-matrix determined by the frame defines a smooth curve $A : I \longrightarrow SO_0(3,1)$. Therefore we have the relation that $A'(t) = C(t)A(t)$. For the converse, let $A : I \longrightarrow SO_0(3,1)$ be a smooth curve, then we can show that $A'(t)A(t)^{-1} \in \mathfrak{so}(3,1)$. Moreover, for any smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$, we apply the existence theorem on the linear systems of ordinary differential equations, so that there exists a unique curve $A : I \longrightarrow SO_0(3,1)$ such that $C(t) = A'(t)A(t)^{-1}$ with an initial data $A(t_0) \in SO_0(3,1)$. Therefore, a smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$ might be identified with a horocyclic surface in $H^3_+(-1)$. Let $C : I \longrightarrow \mathfrak{so}(3,1)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO_0(3,1)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(3,1)$ is a hyperbolic invariant of the pseudo-orthonormal frame $\{\boldsymbol{\gamma}(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$, so that it is a hyperbolic invariant of the corresponding horocyclic surface. Let $C^\infty(I, \mathfrak{so}(3,1))$ be the space of smooth curves into $\mathfrak{so}(3,1)$ equipped with Whitney C^∞-topology. By the above arguments, we may regard $C^\infty(I, \mathfrak{so}(3,1))$ as the space of horocyclic surfaces, where I is an open interval or the unit circle.

On the other hand, we consider the singularities of horocyclic surfaces. By a straightforward calculation, (s,t) is a singular point of $F_{(\gamma,a_1,a_2)}(s,t)$ if and only if

$$c_2(t) + s(c_4(t) - c_1(t)) = 0, \quad \left(1 + \frac{s^2}{2}\right) c_3(t) + sc_5(t) + \frac{s^2}{2} c_6(t) = 0.$$

On the other hand, we have also shown in [20] that $F_{(\gamma,a_1,a_2)}(s,t)$ is horo-flat if and only if $c_2(t) = c_4(t) - c_1(t) = 0$. In this case each generating horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ is a line of curvature. Therefore, the first equation for the singularities is automatically satisfied for a horo-flat horocyclic surface. In this case, the singular set is given by a family if quadratic equations $\sigma_C(s,t) = (c_3(t) + c_6(t))s^2 + 2C_5(t)s + 2c_3(t) = 0$.

We now consider the space of horo-flat horocyclic surfaces. Remember that $C^\infty(I, \mathfrak{so}(3,1))$ is the space of horocyclic surfaces. We consider a linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{hf}(3,1) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & c_4 & c_5 \\ c_2 & -c_4 & 0 & c_6 \\ c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = 0 \right\}$$

By the previous arguments, the *space of horo-flat horocyclic surfaces* is defined to be the space $C^\infty(I, \mathfrak{hf}(3,1))$ with Whitney C^∞-topology. We expect the analogous properties of developable surfaces in $\mathbb{R}^3$ which are ruled surfaces with vanishing Gaussian curvature. However the situation is quite different. In Euclidean space, complete non-singular developable surfaces are cylindrical surfaces [12]. There are various kinds of horo-flat horocyclic surfaces even if these are regular surfaces. We only give some interesting examples of regular horo-flat horocyclic surfaces and which suggest that the situation is quite different form the developable surfaces in Euclidean space. Suppose that $\boldsymbol{\gamma}(t)$ is a unit speed curve with $\kappa_h(t) \neq 0$. Then we have the Frenet-type frame $\{\boldsymbol{\gamma}(t), \boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{e}(t)\}$. Define

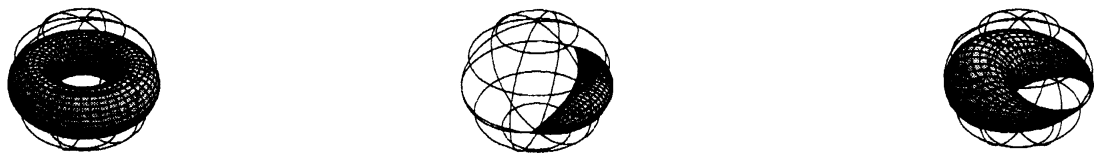
$$F_{(\gamma,e,\pm n)}(s,t) = \boldsymbol{\gamma}(t) + s\boldsymbol{e} + \frac{s^2}{2}(\boldsymbol{\gamma}(t) \pm \boldsymbol{n}(t))$$

which is called a *binormal horocyclic surface of a hyperbolic plane curve* $\boldsymbol{\gamma}$. By a straightforward calculation, the first fundamental form is given by $I_h = ds^2 + (1 + s^2(1 \mp \kappa_h(t))/2)^2 \, dt^2$.

Here, $\boldsymbol{\ell}(t) = \boldsymbol{\gamma}(t) \pm \boldsymbol{n}(t)$ is the lightlike normal vector field along the surface. Then we have

$$-\boldsymbol{\ell}'(t) == \frac{-2 \pm 2\kappa_h(t)}{2 + s^2(1 \mp \kappa_h(t))} \frac{\partial F_{(\gamma,e,\pm n)}}{\partial t}(s,t)$$

It follow that the de Sitter principal curvatures are 1 and $1-(2\mp 2\kappa_h(t))/(2+s^2(1\mp\kappa_h(t)))$. Since $\kappa_h(t) > 0$, $F_{(\gamma,e,-n)}$ is always umbilically free. We can draw the pictures of such surfaces in the Poincaré ball (cf., Fig. 9). However, $F_{(\gamma,e,n)}$ has umbilical points where



Horo-torus (γ : circle, a_1 =constant) Banana (γ : equidistant curve, a_1 =constant) Croissant (γ : horocycle, a_1 =constant)

Fig. 9.

$\kappa_h(t) = 1$. We can draw a horocylindrical surface which has umbilical points along the horocycle through $(0,0,0)$ in Fig. 10.

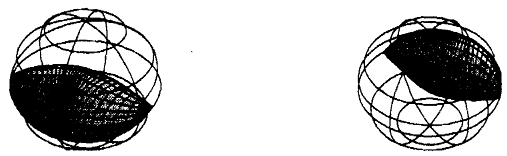


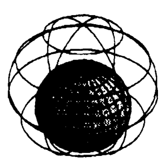
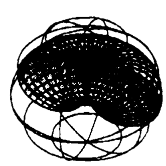
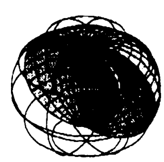
Fig. 10 : Hips ($\kappa_h(0) = 1$ of γ, a_1 =constant)

This gives a concrete example of the surface with a constant principal curvature which is not umbilically free ([1], Example 2.1) which is a counter example of the hyperbolic version of the Shiohama-Takagi theorem[24, 28]. If $\kappa_h \equiv 1$ (i.e., $\gamma(t)$ is a horocycle), then $F_{(\gamma,e,n)}$ is totally umbilical (i.e., a horosphere).

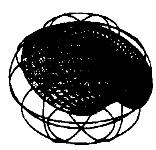
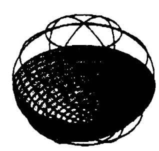
7 Singularities of horo-flat horocyclic surfaces

In this section we consider a horo-flat horocyclic suface $F_{(\gamma,a_1,a_2)}$ with singularities. Since the singularities satisfy the equation $\sigma_C(s,t) = 0$, $F_{(\gamma,a_1,a_2)}$ has at most two branches of singularities under the condition that $c_3(t)+c_4(t) \neq 0$. We suppose that one of the branches of the singularities is given by $\bar{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) + s(t)\boldsymbol{a}_1(t) + (s(t)^2/2)\boldsymbol{\ell}(t)$, where $s = s(t)$ is one of the real solutions of $\sigma_C(s,t)$ for any t. In this case we can reparametrize the horocyclic surface by $\bar{\boldsymbol{a}}_1(t), \bar{\boldsymbol{a}}_2(t)$ and $S = s - s(t), T = t$, then we have $F_{(\gamma,a_1,a_2)}(s,t) = F_{\bar{\gamma},\bar{a}_1,\bar{a}_2}(S,T)$, where$\bar{\boldsymbol{a}}_1(t) = \boldsymbol{a}_1(t) + s(t)\boldsymbol{\ell}(t)$ and $\bar{\boldsymbol{a}}_2(t) = \boldsymbol{\ell}(t) - \bar{\boldsymbol{\gamma}}(t)$. We can directly show that $c_2(t) = c_1(t) - c_4(t) = 0$ if and only if $\bar{c}_2(t) = \bar{c}_1(t) - \bar{c}_4(t) = 0$, so that one of the branch of the singularities is located on the curve $S = 0$. Therefore, we may always assume that one of the branch of singularities are located on $\boldsymbol{\gamma}(t)$. In this case, such singularities satisfy the condition $c_3(t) = 0$. Moreover, another branch of the singularities is given by the equation $2c_5(t) + sc_6(t) = 0$. If $c_6(t) \neq 0$, we denote that $\boldsymbol{\gamma}^\sharp(t) = \boldsymbol{\gamma}(t) + s(t)\boldsymbol{a}_1(t) + (s(t)^2/2)\boldsymbol{\ell}(t)$, where $s(t) = -2c_5(t)/c_6(t)$. We remark that the conditon $c_6(t) \neq 0$ is a generic condition for $C(t) \in C^\infty(I, \mathfrak{hl}(3,1))$.

A cone is one of the typical developable surfaces in Euclidean space which has very simple singularities (conical singularities). We have horo-flat horocyclic surfaces with analogous properties with cones, but the situation is complicated too. We call $F_{(\gamma,a_1,a_2)}$ is a *generalized horo-cone* if $\boldsymbol{\gamma}(t)$ is constant, $\boldsymbol{a}_1'(t) = c_5(t)\boldsymbol{a}_3(t)$ and $\boldsymbol{a}_2'(t) = c_6(t)\boldsymbol{a}_3(t)$. This condition is equivalent to the condition that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$. We say that a generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a *horo-cone with a single vertex* if $c_1(t) = c_2(t) = c_3(t) = c_4(t) = c_5(t) = 0$ and $c_6(t) \neq 0$. In this case, both of $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\gamma}^\sharp(t)$ are constant and $\boldsymbol{\gamma} = \boldsymbol{\gamma}^\sharp$. A generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is called a *horo-cone with two vertices* if both of $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\gamma}^\sharp(t)$ are constant and $\boldsymbol{\gamma} \neq \boldsymbol{\gamma}^\sharp$. By the calculation of the derivative of $\boldsymbol{\gamma}^\sharp(t)$, the above condition is equivalent to the condition that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$, $c_5(t) \neq 0$ and there exists a real number λ such that $c_5(t) = \lambda c_6(t)$. If the condition $c_1(t) = c_2(t) = c_3(t) = c_4(t) = c_6(t) = 0$, $c_5(t) \neq 0$ is satisfied, then $\boldsymbol{a}_2(t)$ is constant. It follows that the image of the generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a part of a horosphere (i.e., we call it a *conical horosphere*). We simply call $F_{(\gamma,a_1,a_2)}$ a *horo-cone* if it is one of the above three cases. We can draw the pictures of horo-cones in the Poincaré ball (Fig. 11).

Conical horosphere   Horo-cone with a single vertex   Horo-cone with two vertices

Half cut of horo-cone with a shifted single vertex   Half cut of horo-cone with shifted two vertices

Fig.11.

We also have the notion of *semi-horo-cones* which belongs to the class of generalized horo-cones. However, we omit the detail. Finally, we say that $F_{(\gamma,a_1,a_2)}$ is a *horo-flat tangent horocyclic surface* if both of γ and $\gamma^\sharp$ are not constant or γ is not constant and $c_6(t) = 0$. In the last case, the end is an isolated point and $F_{(\gamma,a_1,a_2)}$ is a subset of the horosphere (a one parameter family of horocycles which are tangent to γ on a horosphere).

By the above arguments, we also consider the linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{hf}_\sigma(3,1) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & c_4 & c_5 \\ c_2 & -c_4 & 0 & c_6 \\ c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = c_3 = 0 \right\}.$$

Therefore the *space of horo-flat horocyclic surfaces with curve singularities* can be regarded as the space $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$ with Whitney C^∞-topology. In this terminology, one of the branches of the singularities of the horo-flat surface is always located on the image of γ. In this space the condition $c_5(t) = 0$ is a codimension one condition (in the sufficiently higher order jet space $J^\ell(I, \mathfrak{hf}_\sigma(3,1))$. Therefore, we cannot generically avoid the points where $c_5(t) = 0$. Two branches of the singularities meet at such points. This fact suggests us the situation is quite different from the singularities of general wavefront sets or tangent developables in Euclidean space. In[20] we have shown the following theorem:

Theorem 7.1 *Let $F_{(\gamma,a_1,a_2)}$ be a horo-flat tangent horocyclic surface with singularities along γ.*

(A) *Suppose that $c_5(t_0) \neq 0$ and $c_6(t_0) \neq 0$, then both the points $(0, t_0)$ and $(-s(t_0), t_0)$ are singularities, where $s(t) = 2c_5(t)/c_6(t)$. In this case we have the following:*

(1) *The point $(0, t_0)$ is the cuspidal edge if and only if $c_1(t_0) \neq 0$.*

(2) *The point $(0, t_0)$ is the swallowtail if and only if $c_1(t_0) = 0$ and $c_1'(t_0) \neq 0$.*

(3) *The point $(-s(t_0), t_0)$ is the cuspidal edge if and only if $(c_1 - s')(t_0) \neq 0$.*

(4) *The point $(-s(t_0), t_0)$ is the swallowtail if and only if*

$$(c_1 - s')(t_0) = 0 \text{ and } (c_1 - s')'(t_0) \neq 0.$$

(B) *Suppose that $c_5(t_0) = 0$ and $c_6(t_0) \neq 0$, then $s(t_0) = 0$, so that $(0, t_0) = (-s(t_0), t_0)$ is a singular point. In this case, the point $(0, t_0)$ is the cuspidal beaks if and only if $c_5'(t_0) \neq 0$, $c_1(t_0) \neq 0$ and $(c_1 - s')(t_0) \neq 0$.*

(C) *Suppose that $c_5(t_0) \neq 0$ and $c_6(t_0) = 0$, then the point $(0, t_0)$ is the cuspidal cross cap if and only if $c_1(t_0) \neq 0$ and $c_6'(t_0) \neq 0$. In this case , $\gamma(t_0)$ is the only singular point on the generating horocycle $F_{(\gamma,a_1,a_2)}(s, t_0)$.*

Here, the cuspidal edge is a germ of surface diffeomorphic to $CE = \{(x_1, x_2, x_3) | {x_1}^2 = {x_2}^3\}$, the swallowtail is a germ of surface diffeomorphic to $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$, the cuspidal cross cap is a germ of surface diffeomorphic to $CCR = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = u, x_2 = uv^3, x_3 = v^2\}$ and the cuspidal beaks is a germ of surface diffeomorphic to $CBK = \{(x_1, x_2, x_3) | x_1 = v, x_2 = -2u^3 + v^2 u, x_3 = 3u^4 - v^2 u^2\}$.

By Thom's jet-transversality theorem, we can show that the above conditions on $C(t)$ is generic in the space $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$. This means that the conditions in the above theorem is generic in the space of horo-flat tangent horocyclic surfaces. Moreover, we emphasize that the above conditons on $C(t)$ are the exact conditions for the above singularities, so that we can easily recognize the singularities for given horo-flat horocyclic surfaces.

cuspidal edge swallowtail cuspidal cross cap cuspidal beaks

Fig. 12.

The singularities in the above theorem are depicted in Fig. 12. We remark that the cuspidal beaks appears as the center of one of the generic one-parameter bifurcations of wave front sets[27]. Usually it bifurcates into two swallowtails or two cuspidal edges. However, it never bifurcates under any small perturbations in the space of horo-flat horocyclic surfaces.

参考文献

[1] J. Aledo and J. A. Gálvez, *Complete surfaces in the hyperbolic space with a constant principal curvature*, Math. Nachr. **278** (2005), 1111–1116.

[2] T. F. Banchoff, Tightly embedded 2-dimensional polyhedral manifolds, Amer. J. Math. **87** (1965) 462-472.

[3] J. Bolton, Tight immersions into manifolds without conjugate points, Quart. J. Math. Oxford (2), **33** (1982), 159-167.

[4] R. L. Bryant, *Surfaces of mean curvature one in hyperbolic space* in Théorie des variétés minimales et applications (Palaiseau, 1983–1984), Astérisque No. 154–155, (1987),12, 321–347, 353 (1988).

[5] M. Buosi, S, Izumiya and M. A. Soares Ruas, *Total Absolute Horospherical curvature of Submanifolds in Hyperbolic Space*, to appear in Advances in Geometry.

[6] M. Buosi, S, Izumiya and M. A. Soares Ruas, *Horo-tight spheres in Hyperbolic space*, preprint (2009).

[7] C. L. Epstein, *The hyperbolic Gauss map and quasiconformal reflections*, J. Reine Angew. Math. **372** (1986), 96–135.

[8] T. E. Cecil and P. J. Ryan, *Distance functions and umbilic submanifolds of hyperbolic space*, Nagoya Math. J., **74** (1979), 67-75.

[9] T. E. Cecil and P. J. Ryan, *Tight ant taut immersions into hyperbolic space*, J. London Math. Soc., **19** (1979), 561-572.

[10] T. E. Cecil and P. J. Ryan, *Tight ant taut immersions of manifolds*, Research Notes in Mathematics, **107** (1985), Pitman.

[11] J. A. Gálvez, A. Martínez and F. Milán, *Complete linear Weingarten surfaces of Bryant type. A plateau problem at infinity*, Trans. A.M. S. **356** (2004), 3405–3428.

[12] P. Hartman and L. Nirenberg, *On spherical image whose jacobians do not change signs*, Amer. J. Math. **81**, (1959) 901–920.

[13] S. Izumiya, D. Pei and T. Sano, *Singularities of hyperbolic Gauss maps*, Proc. the London Math. Soc. **86** (2003), 485–512.

[14] S. Izumiya, D. Pei and T. Sano, *Horospherical surfaces of curves in Hyperbolic space*, Publ. Math. Debrecen, **64** (2004), 1–13.

[15] S. Izumiya, D-H. Pei and M. Takahashi, *Singularities of evolutes of hypersurfaces in hyperbolic space*, Proceedings of the Edinburgh Mathematical Society, **47** (2004), 131–153.

[16] S. Izumiya, D. Pei, M. C. Romero-Fuster and M. Takahashi, *On the horospherical ridges of submanifolds of codimension 2 in Hyperbolic n-space*, Bull. Braz. Math. Soc. **35** (2) (2004), 177–198.

[17] S. Izumiya, D. Pei, M. C. Romero-Fuster and M. Takahashi, *Horospherical geometry of submanifolds in hyperbolic n-space*, Journal of London Mathematical Society, **71** (2005), 779–800.

[18] S. Izumiya, D. Pei and M. C. Romero-Fuster, *The horospherical geometry of surfaces in Hyperbolic* 4*-space*, Israel Journal of Mathematics, **154** (2006), 361–379.

[19] S. Izumiya and M. C. Romero Fuster, *The horospherical Gauss-Bonnet type theorem in hyperbolic space*, Journal of Math. Soc. Japan, **58** (2006), 965–984.

[20] S. Izumiya , K. Saji and M. Takahashi, *Horospherical flat surfaces in Hyperbolic 3-space*, Preprint, Hokkaido University preprint series 838, (2007).

[21] T. Kobayashi, *Null varieties for convex domains* (*Japanese*), Reports on unitary representation seminar, **6** (1986), 1–18.

[22] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic* 3*-space*, Pacific J. Math. **221** (2005), no. 2, 303–351.

[23] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic* 3*-space and their caustics*, preprint.

[24] K. Shiohama and R. Takagi, *A characterization of a standard torus in* $\mathbb{E}^3$, J. Differential Geom. **4** (1970), 477–485.

[25] M. Umehara and K. Yamada, *Complete surfaces of constant mean curvature* 1 *in the hyperbolic 3-space*, Ann. of Math. **137** (1993), 611–638.

[26] M. Umehara and K. Yamada, *Surfaces of constant mean curvature c in $H^3(-c^2)$ with prescribed hyperbolic Gauss map*, Math. Ann. **304** (1996), 203–224.

[27] V. M. Zakalyukin, *Reconstructions of fronts and caustics depending one parameter and versality of mappings*, J. Sov. Math. **27** (1984), 2713-2735.

[28] M. Zhisheng, *Complete surfaces in $\mathbb{H}^3$ with a constant principal curvature*, in: Differential Geometry and Topology, Lecture Notes in Mathematics **1369** Springer-Verlag, Berlin, (1989), 176–182.

S. IZUMIYA,
Department of Mathematics,
Hokkaido University,
Sapporo 060-0810, Japan.
e-mail : `izumiya@math.sci.hokudai.ac.jp`