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Singularities of nullsphere Gauss map for spacelike surface in nullcone 3-space

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1 Introduction

The nullcone in Minkowski 4-space is one kind of Minkowski pseudo-sphere, which is similar with the sphere in Euclidean 4-space. In [6], Izumiya has studied the details of spacelike hypersurface in the nullcone by Legendrian dualities. Our aim in this article is to study spacelike surfaces in nullcone 3-space by the method similar to that in [5].

We shall assume throughout the whole article that all maps and manifolds are C^{∞} unless the contrary is explicitly stated.

Let $\mathbb{R}^4=\{(x_1,x_2,x_3,x_4)|x_1,x_2,x_3,x_4\in\mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $\mathbf{x}=(x_1,x_2,x_3,x_4)$ and $\mathbf{y}=(y_1,y_2,y_3,y_4)$ in $\mathbb{R}^4$, the *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle\mathbf{x},\mathbf{y}\rangle=-x_1y_1+\sum_{i=2}^{4}x_iy_i$. $(\mathbb{R}^4,\langle,\rangle)$ is called a *Minkowski* 4-space and written by $\mathbb{R}^4_1$. A vector $\mathbf{x}$ in $\mathbb{R}^4_1\setminus\{\mathbf{0}\}$ is called *spacelike*, *lightlike* or *timelike* if $\langle\mathbf{x},\mathbf{x}\rangle$ is positive, zero or negative respectively. The norm of a vector $\mathbf{x}\in\mathbb{R}^4_1$ is defined by $\|\mathbf{x}\|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}$. For any $\mathbf{x},\mathbf{y}\in\mathbb{R}^4_1$, we say $\mathbf{x}$ *pseudo-perpendicular* to $\mathbf{y}$ if $\langle\mathbf{x},\mathbf{y}\rangle=0$. For a vector $\mathbf{v}\in\mathbb{R}^4_1$ and a real number c, a hyperplane with pseudo normal $\mathbf{v}$ is defined by $HP(\mathbf{v},c)=\{\mathbf{x}\in\mathbb{R}^4_1\mid\langle\mathbf{x},\mathbf{v}\rangle=c\}$. $HP(\mathbf{v},c)$ is called *a timelike hyperplane, a spacelike hyperplane or a lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively. Now, *hyperbolic 3-space* is defined by $H^3_1=\{\mathbf{x}\in\mathbb{R}^4_1\mid\langle\mathbf{x},\mathbf{x}\rangle=-1\}$, *de Sitter 3-space* is defined by $S^3_1=\{\mathbf{x}\in\mathbb{R}^4_1\mid\langle\mathbf{x},\mathbf{x}\rangle=1\}$ and the *nullcone 3-space* is defined by $NC^3=\{\mathbf{x}=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4_1\mid x_1\neq0,\langle\mathbf{x},\mathbf{x}\rangle=0\}$. The *3-dimension nullcone with vertex* λ in $\mathbb{R}^4_1$ is defined by $NC^3_\lambda=\{\mathbf{x}\in\mathbb{R}^4_1\mid\langle\mathbf{x}-\lambda,\mathbf{x}-\lambda\rangle=0\}$. If $\mathbf{x}=(x_1,x_2,x_3,x_4)$ is a lightlike vector, then $x_1\neq0$. Therefore we have $\widetilde{\mathbf{x}}=\left(1,\frac{x_2}{x_1},\frac{x_3}{x_1},\frac{x_4}{x_1}\right)\in S^2_+=\{\mathbf{x}\in\mathbb{R}^4_1\mid\langle\mathbf{x},\mathbf{x}\rangle=0,x_1=1\}$. S^2_+ is called the *nullcone unit 2-sphere*.

For any $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\mathbb{R}^4_1$, we define a vector $\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3$ by

$$\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3=\begin{vmatrix}-e_1, & e_2, & e_3, & e_4\\ x_1^1, & x_1^2, & x_1^3, & x_1^4\\ x_2^1, & x_2^2, & x_2^3, & x_2^4\\ x_3^1, & x_3^2, & x_3^3, & x_3^4\end{vmatrix},$$

where e_1,e_2,e_3,e_4 is the canonical basis of $\mathbb{R}^4_1$ and $\mathbf{x}_i=(x_i^1,x_i^2,x_i^3,x_i^4)$. It can easily check that $\langle\mathbf{x},\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3\rangle=\det(\mathbf{x},\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$, so that $\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3$ is pseudo orthogonal to any $\mathbf{x}_i(i=1,2,3)$.

We fix an orientation and timelike orientation of $\mathbb{R}^4_1$ (i.e., a 4-volume form dV, and future time-like vector field, have been chosen). Let $X:U\to NC^3$ be an embedding, where U is an open subset of $\mathbb{R}^2$. Denote $M=X(U)$ and identify M with U by the embedding X. We say X a *spacelike surface* if X_{u_1} and X_{u_2} are spacelike vectors. Therefore, the tangent space T_pM of M is a spacelike subspace(i.e., consists of spacelike vectors) for any point $p\in M$. In this case, the pseudo-normal space N_pM is a timelike plane(i.e.,

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Lorentz plane). Denote by $N(M)$ the pseudo-normal bundle over M. Since this is a trivial bundle, we can arbitrarily choose a future directed unit timelike normal section $\boldsymbol{n}^T(u) \in N_pM \cap H^3_1$, where $p = X(u)$. Therefore we can define a spacelike unit normal section $\boldsymbol{n}^S(u)$ by

$$\boldsymbol{n}^S(u) = \frac{\boldsymbol{n}^T(u) \wedge X_{u_1}(u) \wedge X_{u_2}(u)}{\|\boldsymbol{n}^T(u) \wedge X_{u_1}(u) \wedge X_{u_2}(u)\|} \in S^3_1,$$

and we have $\langle \boldsymbol{n}^T, \boldsymbol{n}^S \rangle = 0$. Although we could also choose $-\boldsymbol{n}^S(u)$ as a spacelike unit normal section with the above properties, we fix the direction $\boldsymbol{n}^S(u)$ throughout this article. $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is called *a future directed normal frame* along $M = X(U)$. Clearly, the vector $\boldsymbol{n}^T \pm \boldsymbol{n}^S(u)$ is lightlike. Since $\{X_{u_1}, X_{u_2}\}$ is a basis of T_pM, the system $\{X_{u_1}, X_{u_2}, \boldsymbol{n}^T, \boldsymbol{n}^S\}$ provides a basis for $T_p\mathbb{R}^4_1$.

$X \in N_pM$, N_pM is a Lorentzian plane and $X(U)$ is a regular surface, so $\widetilde{X}(u) = \widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S}(u)$ for any $u \in U$ or $\widetilde{X}(u) = \widetilde{\boldsymbol{n}^T - \boldsymbol{n}^S}(u)$ for any $u \in U$.

Here, we only consider the case of $\widetilde{X}(u) = \widetilde{\boldsymbol{n}^T - \boldsymbol{n}^S}(u)$ for $u \in U$. The case of $\widetilde{X}(u) = \widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S}(u)$ can be discussed similarly. Define two maps of $M = X(U)$ as

$$NG^{\pm}_M : U \to S^2_+, \quad NG^{\pm}_M(u) = \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u),$$

each one of these maps is called *nullsphere Gauss map*. Under the identification of M with U through X, we have the linear mapping $d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}) : T_pM \to T_p\mathbb{R}^4_1 = T_pM \oplus N_pM$. Consider the orthogonal projections $\pi^t : T_pM \oplus N_pM \to T_pM$ and $\pi^n : T_pM \oplus N_pM \to N_pM$. Define $d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})^t = \pi^t \circ d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})$ and $d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})^n = \pi^n \circ d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})$. The linear transformations $S^{\pm}_p(\boldsymbol{n}^T, \boldsymbol{n}^S) = -d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})^t$ and $d_p(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})^n$ are respectively called the *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-shape operator and the normal connection with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ of $M = X(U)$ at $p = X(u)$.

The eigenvalues of $S^{\pm}_p(\boldsymbol{n}^T, \boldsymbol{n}^S)$ denoted by $\{\kappa^{\pm}_i(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)\}(i = 1, 2)$ are called the *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-nullsphere principal curvature* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p. Then the *nullsphere Gauss-Kronecker curvature* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at $p = X(u)$ is defined as

$$K^{\pm}_n(\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = \det S^{\pm}_p(\boldsymbol{n}^T, \boldsymbol{n}^S).$$

We say that a point $p = X(u)$ is a *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic point* if all the principal curvatures coincide at p and thus $S^{\pm}_p(\boldsymbol{n}^T, \boldsymbol{n}^S) = \kappa^{\pm}(\boldsymbol{n}^T, \boldsymbol{n}^S)I|_{T_pM}$ for some function $\kappa^{\pm}$. We say that $M = X(U)$ is totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic if all points on M are $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic.

We deduce now the nullcone Weingarten formula. Since X_{u_1} and X_{u_2} are spacelike vectors, we have a Riemannian metric (*the first fundamental form*) on M defined by $ds^2 = \sum_{i=1}^2 g_{ij}du_idu_j$, where $g_{ij}(u) = \langle X_{u_i}, X_{u_j} \rangle$ for any $u \in U$. We also have a *nullcone second fundamental invariant* with respect to the normal vector field $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ defined by $h^{\pm}_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle -(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_i}(u), X_{u_j}(u) \rangle$ for any $u \in U$.

Proposition 1.1. *Under the above notations, we have the following nullcone Weingarten formula with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$:

(a) $(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_i} = \frac{\mp(\boldsymbol{n}^T_1 \pm \boldsymbol{n}^S_1)\langle \boldsymbol{n}^S_{u_i}, \boldsymbol{n}^T \rangle - (\boldsymbol{n}^T_1 \pm \boldsymbol{n}^S_1)_{u_i}}{(\boldsymbol{n}^T_1 \pm \boldsymbol{n}^S_1)^2}(\boldsymbol{n}^T \pm \boldsymbol{n}^S) - \sum_{j=1}^2 h^{j\pm}_i(\boldsymbol{n}^T, \boldsymbol{n}^S)X_{u_i}$;

(b) $\pi^t \circ (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_i} = -\sum_{j=1}^2 h^{j\pm}_i(\boldsymbol{n}^T, \boldsymbol{n}^S)X_{u_i}$.

Here, $h^{j\pm}_i(\boldsymbol{n}^T, \boldsymbol{n}^S) = h^{\pm}_{ik}(\boldsymbol{n}^T, \boldsymbol{n}^S)g^{kj}$, $g^{kj} = (g_{kj})^{-1}$ *and* $\boldsymbol{n}^i = (\boldsymbol{n}^i_1, \boldsymbol{n}^i_2, \boldsymbol{n}^i_3, \boldsymbol{n}^i_4)(i = T, S)$.

As a corollary of the above proposition, we have an explicit expression of the nullsphere Gauss-Kronecker curvature by Riemannian metric and the nullcone second fundamental invariant.

Corollary 1.2. *Under the same notations as in the above proposition, the nullsphere Gauss-Kronecker curvature is given by*

$$K^{\pm}_n(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \frac{\det(h^{\pm}_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u))}{\det(g_{\alpha\beta})}.$$

If $K_n^{\pm}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0) = 0$, the point $p_0 = X(u_0)$ is called a *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-nullcone parabolic point* of $X : U \to NC^3$. And we say that a point p_0 is a *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-nullcone flat point* if it is a $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-nullcone umbilical point and $K_n^{\pm}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0) = 0$.

Theorem 1.3. $K_n^-(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) \not\equiv 0$.

2 Nullsphere height function

The *nullsphere height function* family on $M = X(U)$ is defined by

$$H : U \times S_+^2 \to \mathbb{R},\ H(u, v) = \langle X(u), v \rangle.$$

The Hessian matrix of the nullsphere height function $h_{v_0} = H(u, v_0)$ at u_0 is denoted by $\mathrm{Hess}(h_{v_0})(u_0)$.

Proposition 2.1. *Let H be a nullsphere height function on M. Then*

(1)$\partial h_{v_0}/\partial u_i(u_0) = 0 (i = 1, 2)$ *if and only if* $v_0 = \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0)$.

(2)$\partial h_{v_0}/\partial u_i(u_0) = \mathrm{detHess}(h_{v_0}(u_0)) = 0 (i = 1, 2)$ *if and only if* $v_0 = \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0)$ *and* $K_n^{\pm}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0) = 0$.

(3) *p_0 is a nullcone flat point if and only if rankHess$(h_{v_0})(u_0) = 0$.*

Corollary 2.2. *For a point $p_0 = X(u_0) \in M$, the following conditions are equivalent:*

(1) *The point $p_0 \in M$ is a $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-nullcone parabolic point.*

(2) *The point $p_0 \in M$ is a singular point of the nullsphere Gauss map $NG_M^{\pm}$.*

(3) $K_n^{\pm}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0) = 0$.

(4) $\mathrm{detHess}(h_{v_0})(u_0) = 0$ *for* $\mathbf{v}_0 = \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0)$.

Corollary 2.3. *NG_M^- is a regular nullsphere Gauss map.*

Consider now the particular case of a surface $M \subset NC^3$. Given a vector $\mathbf{v} \in S_+^2$(resp. S_1^3, H_1^3) and a number c, denoted by $S(\mathbf{v}, c)$ the *null hyperhorosphere*(resp. *null equidistant hyperplane, null hypersphere*) determined by the intersection of the hyperplane $HP(\mathbf{v}, c)$ with NC^3.

Proposition 2.4. *Let M be a spacelike surface in NC^3. If NG_M^- is constant, then M degenerate to a straight line.*

We now define a family of functions

$$\widetilde{H} : U \times NC^3 \to \mathbb{R},\ \widetilde{H}(u, v) = \langle X(u), \widetilde{\mathbf{v}} \rangle - v_1,$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)$. $\widetilde{H}$ is called the *extended nullsphere height function* of $M = X(U)$. The Hessian matrix of the extended nullsphere height function $\widetilde{h}_{v_0} = \widetilde{H}(u, v_0)$ at u_0 is denoted by $\mathrm{Hess}(\widetilde{h}_{v_0})(u_0)$.

Proposition 2.5. *Let M be a spacelike surface in NC^3. $\widetilde{H}$ is the extended nullsphere height function of M. For $\mathbf{v}_0 \in NC^3$, we have the following:*

(1) $\widetilde{h}_{v_0}(p_0) = \frac{\partial \widetilde{h}_{v_0}}{\partial u_i}(p_0) = 0$ *if and only if* $\widetilde{\mathbf{v}}_0 = \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0)$ and $v_1 = \langle X(u_0), \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0) \rangle$.

(2) $\widetilde{h}_{v_0}(p_0) = \frac{\partial \widetilde{h}_{v_0}}{\partial u_i}(p_0) = \mathrm{detHess}\widetilde{h}_{v_0}(p_0) = 0$ *if and only if* $\widetilde{\mathbf{v}}_0 = \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0)$, $v_1 = \langle X(u_0), \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u_0) \rangle$ *and* $K_n^{\pm}(\boldsymbol{n}^T, \boldsymbol{n}^S)(p_0) = 0$.

The assertions of proposition 2.5 means that the discriminant set of the extended nullsphere height function $\widetilde{H}$ is given by $\mathcal{D}_{\widetilde{H}} = \{\mathbf{v} \mid \mathbf{v} = \langle X(u), \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u) \rangle (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u)\}$. Therefore we now define a pair of singular surfaces in NC^3 by $NP_M^{\pm}(u) = \langle X(u), \widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}(u) \rangle (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u)$, each one of $NP_M^{\pm}$ is called *the nullcone pedal surface* of $X(U) = M$. A singularity of the nullcone pedal surface exactly corresponds to a singularity of the nullsphere Gauss map.

Corollary 2.6. *NP_M^- is a zero map.*

This work is only a preparation for further studying, in the following, we will discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

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