Singularities ofnullsphere Gauss map for spacelike
surface in nullcone 3-space

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1 Introduction

The nullcone in Minkowski 4-space is one kind of Minkowski pseudo-sphere, which is similar with the
sphere in Euclidean 4-space. In [6], Izumiya has studied the details of spacelike hypersurface in
the nullcone by Legendrian dualities. Our aim in this article is to study spacelike surfaces in nullcone 3-space
by the method similar to that in [5].

We shall assume throughout the whole article that all maps and manifolds are $C^\infty$ unless the contrary
is explicitly stated.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors
x = $(x_1, x_2, x_3, x_4)$ and y = $(y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the pseudo-scalar product of x and y is defined by
$(x, y) = -x_1y_1 + \sum_{i=2}^{4}x_iy_i$ $(\mathbb{R}^4, \langle \rangle)$ is called a Minkowski 4-space and written by $\mathbb{R}^4_1$. A vector x in
$\mathbb{R}^4_1 \setminus \{0\}$ is called spacelike, lightlike or timelike if $(x, x)$ is positive, zero or negative respectively.
The norm of a vector $x \in \mathbb{R}^4_1$ is defined by $||x|| = \sqrt{\langle x, x \rangle}$. For any $x, y \in \mathbb{R}^4_1$, we say x pseudo-perpendicular
to y if $(x, y) = 0$. For a vector $v \in \mathbb{R}^4_1$ and a real number c, a hyperplane with pseudo normal $v$ is defined by
$HP(v, c) = \{x \in \mathbb{R}^4_1 \mid \langle x, v \rangle = c\}$. $HP(v, c)$ is called a timelike hyperplane, a spacelike hyperplane or a
lightlike hyperplane if $v$ is timelike, spacelike or lightlike respectively. Now, hyperbolic 3-space is defined by
$H^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = -1\}$, de Sitter 3-space is defined by $S^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = 1\}$ and
the nullcone 3-space is defined by $NC^3_1 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_1 \mid x_1 \neq 0, (x, x) = 0\}$. The 3-dimension
nullcone with vertex $\lambda$ in $\mathbb{R}^4_1$ is defined by $NC^3_\lambda = \{x \in \mathbb{R}^4_1 \mid \langle x - \lambda, x - \lambda \rangle = 0\}$. If $x = (x_1, x_2, x_3, x_4)$ is a
lightlike vector, then $x_1 \neq 0$. Therefore we have $\tilde{x} = \left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}\right) \in S^2_+ = \{x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = 0, x_1 = 1\}$.
$S^2_+$ is called the nullcone unit 2-sphere.

For any $x_1, x_2, x_3 \in \mathbb{R}^4_1$, we define a vector $x_1 \wedge x_2 \wedge x_3$ by

\[
x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix}
-e_1 & e_2 & e_3 & e_4 \\
x_1^1 & x_2^1 & x_3^1 & x_4^1 \\
x_2^1 & x_3^1 & x_1^1 & x_4^1 \\
x_3^1 & x_1^1 & x_2^1 & x_4^1 \\
x_4^1 & x_1^1 & x_2^1 & x_3^1
\end{vmatrix},
\]

where $e_1, e_2, e_3, e_4$ is the canonical basis of $\mathbb{R}^4_1$ and $x_i = (x_1^i, x_2^i, x_3^i, x_4^i)$. It can easily check that
$(x, x_1 \wedge x_2 \wedge x_3) = \det(x_1, x_1, x_2, x_3)$, so that $x_1 \wedge x_2 \wedge x_3$ is pseudo orthogonal to any $x_i (i = 1, 2, 3)$.

We fix an orientation and timelike orientation of $\mathbb{R}^4_1$ (i.e., a 4-volume form $dV$, and future time-like
vector field, have been chosen). Let $X : U \rightarrow NC^3_1$ be an embedding, where $U$ is an open subset of $\mathbb{R}^2$.
Denote $M = X(U)$ and identify $M$ with $U$ by the embedding $X$. We say $X$ a spacelike surface if $X_{u_1}$ and
$X_{u_2}$ are spacelike vectors. Therefore, the tangent space $T_p M$ of $M$ is a spacelike subspace (i.e., consists of
spacelike vectors) for any point $p \in M$. In this case, the pseudo-normal space $N_p M$ is a timelike plane (i.e.,
Lorentz plane. Denote by $N(M)$ the pseudo-normal bundle over $M$. Since this is a trivial bundle, we can arbitrarily choose a future directed unit timelike normal section $n^T(u) \in N_pM \cap H^3_1$, where $p = X(u)$. Therefore we can define a spacelike unit normal section $n^S(u)$ by

$$n^S(u) = \frac{n^T(u) \wedge X_{u_1}(u) \wedge X_{u_2}(u)}{||n^T(u) \wedge X_{u_1}(u) \wedge X_{u_2}(u)||} \in S^1,$$

and we have $(n^T, n^S) = 0$. Although we could also choose $-n^S(u)$ as a spacelike unit normal section with the above properties, we fix the direction $n^S(u)$ throughout this article. $(n^T, n^S)$ is called a future directed normal frame along $M = X(U)$. Clearly, the vector $n^T \pm n^S(u)$ is lightlike. Since $\{X_{u_1}, X_{u_2}\}$ is a basis of $T_pM$, the system $\{X_{u_1}, X_{u_2}, n^T, n^S\}$ provides a basis for $T_p\mathbb{R}^4$.

$X \in N_pM$, $N_pM$ is a Lorentzian plane and $X(U)$ is a regular surface, so $\tilde{X}(u) = n^T + n^S(u)$ for any $u \in U$ or $\tilde{X}(u) = n^T - n^S(u)$ for any $u \in U$.

Here, we only consider the case of $\tilde{X}(u) = n^T - n^S(u)$ for $u \in U$. The case of $\tilde{X}(u) = n^T + n^S(u)$ can be discussed similarly. Define two maps of $M = X(U)$ as

$$NG_M^\pm : U \rightarrow S^2, \quad NG_M^\pm(u) = n^T \pm n^S(u),$$

each one of these maps is called nullsphere Gauss map. Under the identification of $M$ with $U$ through $X$, we have the linear mapping $d\pi(p)(n^T \pm n^S) : T_pM \rightarrow T_p\mathbb{R}^4 = T_pM \oplus N_pM$. Consider the orthogonal projections $\pi^T : T_pM \oplus N_pM \rightarrow T_pM$ and $\pi^S : T_pM \oplus N_pM \rightarrow N_pM$. Define $d\pi(p)(n^T \pm n^S)^t = \pi^T \circ d\pi(p)(n^T \pm n^S)$ and $d\pi(p)(n^T \pm n^S)^n = \pi^S \circ d\pi(p)(n^T \pm n^S)$. The linear transformations $S^\pm_M(n^T, n^S) = -d\pi(p)(n^T \pm n^S)^t$ and $d\pi(p)(n^T \pm n^S)^n$ are respectively called the $(n^T, n^S)$-shape operator and the normal connection with respect to $(n^T, n^S)$ of $M = X(U)$ at $p = X(u)$.

The eigenvalues of $S^\pm_M(n^T, n^S)$ denoted by $\{\kappa^\pm(n^T, n^S)(p)\}(i = 1, 2)$ are called the $(n^T, n^S)$-nullsphere principal curvature with respect to $(n^T, n^S)$ at $p$. Then the nullsphere Gauss-Kronecker curvature with respect to $(n^T, n^S)$ at $p = X(u)$ is defined as

$$K^\pm_M(n^T, n^S)(p) = \det S^\pm_M(n^T, n^S).$$

We say that a point $p = X(u)$ is a $(n^T, n^S)$-umbilic point if all the principal curvatures coincide at $p$ and thus $S^\pm_M(n^T, n^S) = \kappa^\pm(n^T, n^S)I_{T_pM}$ for some function $\kappa^\pm$. We say that $M = X(U)$ is totally $(n^T, n^S)$-umbilic if all points on $M$ are $(n^T, n^S)$-umbilic.

We deduce now the nullcone Weingarten formula. Since $X_{u_1}$ and $X_{u_2}$ are spacelike vectors, we have a Riemannian metric (the first fundamental form) on $M$ defined by $ds^2 = \sum_{i=1}^2 g_{ij}du_idu_j$, where $g_{ij}(u) = (X_{u_i}, X_{u_j})$ for any $u \in U$. We also have a nullcone second fundamental invariant with respect to the normal vector field $(n^T, n^S)$ defined by $h^\pm_M(n^T, n^S)(u) = -(n^T \pm n^S)_{u_1}(u), X_{u_1}(u))$ for any $u \in U$.

**Proposition 1.1.** Under the above notations, we have the following nullcone Weingarten formula with respect to $(n^T, n^S)$:

(a) $(n^T \pm n^S)_{u_1} = \sum_{i=1}^2 h^i_{u_1}(n^T \pm n^S)(n^T \pm n^S)_{u_1} - \sum_{i=1}^2 h^i_{u_2}(n^T \pm n^S)X_{u_2};$

(b) $\pi^t \circ (n^T \pm n^S)_{u_2} = - \sum_{i=1}^2 h^i_{u_1}(n^T \pm n^S)X_{u_2}.$

Here, $h^i_{u_1}(n^T, n^S) = h^i_{u_1}(n^T, n^S)g^{ij}$, $g^{ij} = (g_{ij})^{-1}$ and $n^i = (n^1, n^2, n^3, n^4)(i = T, S)$

As a corollary of the above proposition, we have an explicit expression of the nullsphere Gauss-Kronecker curvature by Riemannian metric and the nullcone second fundamental invariant.

**Corollary 2.1.** Under the same notations as in the above proposition, the nullsphere Gauss-Kronecker curvature is given by

$$K^\pm_M(n^T, n^S)(u) = \frac{\det(h^i_{u_1}(n^T, n^S)(u))}{\det(g_{ij})}.$$
If $K_{n}^{\pm}(n^{T}, n^{S})(u_{0}) = 0$, the point $p_{0} = X(u_{0})$ is called a $(n^{T}, n^{S})$-nullcone parabolic point of $X : U \to NC^{3}$. And we say that a point $p_{0}$ is a $(n^{T}, n^{S})$-nullcone flat point if it is a $(n^{T}, n^{S})$-nullcone umbilical point and $K_{n}^{\pm}(n^{T}, n^{S})(u_{0}) = 0$.

**Theorem 1.3.** $K_{n}^{-}(n^{T}, n^{S})(u) \neq 0$.

## 2 Nullsphere height function

The nullsphere height function family on $M = X(U)$ is defined by

$$H : U \times S_{n}^{4} \to \mathbb{R}, \quad H(u, v) = \langle X(u), v \rangle.$$  

The Hessian matrix of the nullsphere height function $h_{v_{0}} = H(u, v_{0})$ at $u_{0}$ is denoted by $\text{Hess}(h_{v_{0}})(u_{0})$.

**Proposition 2.1.** Let $H$ be a nullsphere height function on $M$. Then

(1) $\partial h_{v_{0}} / \partial u_{i}(u_{0}) = 0(i = 1, 2)$ if and only if $v_{0} = n^{T} \pm n^{S}(u_{0})$.

(2) $\partial h_{v_{0}} / \partial u_{i}(u_{0}) = \det \text{Hess}(h_{v_{0}}(u_{0})) = 0(i = 1, 2)$ if and only if $v_{0} = n^{T} \pm n^{S}(u_{0})$ and $K_{n}^{-}(n^{T}, n^{S})(u_{0}) = 0$.

(3) $p_{0}$ is a nullcone flat point if and only if $\text{rank Hess}(h_{v_{0}}(u_{0})) = 0$.

**Corollary 2.2.** For a point $p_{0} = X(u_{0}) \in M$, the following conditions are equivalent:

(1) The point $p_{0} \in M$ is a $(n^{T}, n^{S})$-nullcone parabolic point.

(2) The point $p_{0} \in M$ is a singular point of the nullsphere Gauss map $NG_{M}^{+}$.

(3) $K_{n}^{-}(n^{T}, n^{S})(u_{0}) = 0$.

(4) $\det \text{Hess}(h_{v_{0}})(u_{0}) = 0$ for $v_{0} = n^{T} \pm n^{S}(u_{0})$.

**Corollary 2.3.** $NG_{M}^{-}$ is a regular nullsphere Gauss map.

Consider now the particular case of a surface $M \subset NC^{3}$. Given a vector $v \in S_{n}^{4}$ (resp. $S_{n}^{1}$, $H_{n}^{2}$) and a number $c$, denoted by $S(v, c)$ the null hyperhorosphere (resp. null equidistant hyperplane, null hypersphere) determined by the intersection of the hyperplane $HP(v, c)$ with $NC^{3}$.

**Proposition 2.4.** Let $M$ be a spacelike surface in $NC^{3}$. If $NG_{M}^{-}$ is constant, then $M$ degenerate to a straight line.

We now define a family of functions

$$\tilde{H} : U \times NC^{3} \to \mathbb{R}, \quad \tilde{H}(u, v) = \langle X(u), \overline{v} \rangle - v_{1},$$

where $v = (v_{1}, v_{2}, v_{3}, v_{4})$. $\tilde{H}$ is called the extended nullsphere height function of $M = X(U)$. The Hessian matrix of the extended nullsphere height function $\tilde{h}_{v_{0}} = \tilde{H}(u, v_{0})$ at $u_{0}$ is denoted by $\text{Hess}(\tilde{h}_{v_{0}})(u_{0})$.

**Proposition 2.5.** Let $M$ be a spacelike surface in $NC^{3}$. $\tilde{H}$ is the extended nullsphere height function of $M$. For $v_{0} \in NC^{3}$, we have the following:

(1) $\tilde{h}_{v_{0}}(p_{0}) = \frac{\partial \tilde{h}_{v_{0}}}{\partial u_{i}}(p_{0}) = 0$ if and only if $\tilde{v}_{0} = n^{T} \pm n^{S}(u_{0})$ and $v_{1} = \langle X(u_{0}), n^{T} \pm n^{S}(u_{0}) \rangle$.

(2) $\tilde{h}_{v_{0}}(p_{0}) = \frac{\partial \tilde{h}_{v_{0}}}{\partial u_{i}}(p_{0}) = \det \text{Hess}(\tilde{h}_{v_{0}})(p_{0}) = 0$ if and only if $\tilde{v}_{0} = n^{T} \pm n^{S}(u_{0})$, $v_{1} = \langle X(u_{0}), n^{T} \pm n^{S}(u_{0}) \rangle$ and $K_{n}^{-}(n^{T}, n^{S})(p_{0}) = 0$.

The assertions of proposition 2.5 means that the discriminant set of the extended nullsphere height function $\tilde{H}$ is given by $D_{\tilde{H}} = \{ v \mid v = (X(u), n^{T} \pm n^{S}(u)) (n^{T} \pm n^{S}(u)) \}$. Therefore we now define a pair of singular surfaces in $NC^{3}$ by $NP_{M}^{+}(u) = (X(u), n^{T} \pm n^{S}(u)) (n^{T} \pm n^{S}(u))$, each one of $NP_{M}^{+}$ is called the nullcone pedal surface of $X(U) = M$. A singularity of the nullcone pedal surface exactly corresponds to a singularity of the nullsphere Gauss map.

**Corollary 2.6.** $NP_{M}^{-}$ is a zero map.

This work is only a preparation for further studying, in the following, we will discuss some geometrical properties of spacelike curve from singularity theory viewpoint.
References


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