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On spacelike curve in nullcone 3-space

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1 Basic notions

The nullcone is one kind of pseudo-sphere of Minkowski space. Our aim in this article is to develop the study for spacelike curve in nullcone 3-space by Bruce and Giblin's singularity theory. In order to study the spacelike curve of nullcone 3-space, we need to develop differential geometry of spacelike curve in nullcone 3-space similarly as it was done for curves in Euclidean space [2].

Let $\mathbb{R}^4=\{(x_1,x_2,x_3,x_4)|x_1,x_2,x_3,x_4\in\mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $\mathbf{x}=(x_1,x_2,x_3,x_4)$ and $\mathbf{y}=(y_1,y_2,y_3,y_4)$ in $\mathbb{R}^4$, the *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle\mathbf{x},\mathbf{y}\rangle=-x_1y_1+\sum_{i=2}^{4}x_iy_i$. $(\mathbb{R}^4,\langle,\rangle)$ is called a *Minkowski* 4-space and denoted by $\mathbb{R}_1^4$. A vector $\mathbf{x}$ in $\mathbb{R}_1^4\setminus\{\mathbf{0}\}$ is called *spacelike*, *lightlike* or *timelike* if $\langle\mathbf{x},\mathbf{x}\rangle$ is positive, zero or negative respectively. The norm of a vector $\mathbf{x}\in\mathbb{R}_1^4$ is defined by $\|\mathbf{x}\|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}$. For any $\mathbf{x},\mathbf{y}\in\mathbb{R}_1^4$, we say $\mathbf{x}$ *pseudo-perpendicular* to $\mathbf{y}$ if $\langle\mathbf{x},\mathbf{y}\rangle=0$. For a vector $\mathbf{v}\in\mathbb{R}_1^4$ and a real number c, we define a hyperplane with pseudo normal $\mathbf{v}$ by $HP(\mathbf{v},c)=\{\mathbf{x}\in\mathbb{R}_1^4\mid\langle\mathbf{x},\mathbf{v}\rangle=c\}$. $HP(\mathbf{v},c)$ is called *a timelike hyperplane, a spacelike hyperplane or a lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively. Now, define the *nullcone 3-space* by $NC^3=\{\mathbf{x}=(x_1,x_2,x_3,x_4)\in\mathbb{R}_1^4\mid x_1\neq 0,\langle\mathbf{x},\mathbf{x}\rangle=0\}$, the *de Sitter 3-space* by $S_1^3=\{\mathbf{x}\in\mathbb{R}_1^4\mid\langle\mathbf{x},\mathbf{x}\rangle=1\}$ and the *hyperbolic 3-space* by $H_1^3=\{\mathbf{x}\in\mathbb{R}_1^4\mid\langle\mathbf{x},\mathbf{x}\rangle=-1\}$. If $\mathbf{x}=(x_1,x_2,x_3,x_4)$ is a lightlike vector, then $x_1\neq 0$. Therefore $\widetilde{\mathbf{x}}=\left(1,\frac{x_2}{x_1},\frac{x_3}{x_1},\frac{x_4}{x_1}\right)\in S_+^2=\{\mathbf{x}\in\mathbb{R}_1^4\mid\langle\mathbf{x},\mathbf{x}\rangle=0,x_1=1\}$. S_+^2 is called the *nullcone unit 2-sphere*.

For any $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\mathbb{R}_1^4$, we define a vector $\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3$ by

$$\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3=\begin{vmatrix}-e_1, & e_2, & e_3, & e_4\\ x_1^1, & x_1^2, & x_1^3, & x_1^4\\ x_2^1, & x_2^2, & x_2^3, & x_2^4\\ x_3^1, & x_3^2, & x_3^3, & x_3^4\end{vmatrix},$$

where e_1,e_2,e_3,e_4 are the canonical basis of $\mathbb{R}_1^4$ and $\mathbf{x}_i=(x_i^1,x_i^2,x_i^3,x_i^4)$. It is easy to check that $\langle\mathbf{x},\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3\rangle=\det(\mathbf{x},\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$, so that $\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3$ is pseudo orthogonal to $\mathbf{x}_i (i=1,2,3)$.

Let $\gamma:I\to NC^3$; $\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t),\gamma_4(t))$ be a smooth regular curve in NC^3 (i.e.,$\dot\gamma(t)\neq 0$ for any $t\in I$), where I is an open interval. The curve γ is called a *spacelike curve* if $\langle\dot\gamma(t),\dot\gamma(t)\rangle$ is positive for any $t\in I$. The *arc-length* of a spacelike curve γ, measured from $\gamma(t_0)$, $t_0\in I$ is $s(t)=\int_{t_0}^{t}\|\dot\gamma(t)\|\,dt$. Then a parameter s is determined such that $\|\gamma'(s)\|=1$, where $\gamma'(s)=d\gamma/ds(s)$. We say that a spacelike curve γ is *parameterized by arc-length* if it satisfies that $\|\gamma'(s)\|=1$. Throughout the reminder in this article, s will denote the arc-length parameter. Let $\boldsymbol{t}(s)=\gamma'(s)$. we call $\boldsymbol{t}(s)$ an *unit tangent vector* of γ at s. The *signature* of $\mathbf{x}$ is defined to be

$$\delta(\mathbf{x})=\mathrm{sign}(\mathbf{x})=\begin{cases}1 & \mathbf{x}:\text{spacelike};\\ 0 & \mathbf{x}:\text{lightlike};\\ -1 & \mathbf{x}:\text{timelike}.\end{cases}$$

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For any nonlightlike curve $\gamma : I \to NC^3$, which is parameterized by arc-length and satisfies $k_1(s) \neq 0$. We can construct a pseudo-orthogonal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s), \boldsymbol{n}_3(s)\}$ of $\mathbb{R}^4_1$ along γ which satisfies the following Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{t}(s) & = & \gamma'(s); \\ \boldsymbol{t}'(s) & = & k_1(s)\boldsymbol{n}_1(s); \\ \boldsymbol{n}_1'(s) & = & -\delta_1 k_1(s)\boldsymbol{t}(s) + k_2(s)\boldsymbol{n}_2(s); \\ \boldsymbol{n}_2'(s) & = & \delta_3 k_2(s)\boldsymbol{n}_1(s) + k_3(s)\boldsymbol{n}_3(s); \\ \boldsymbol{n}_3'(s) & = & \delta_1 k_3(s)\boldsymbol{n}_2(s), \end{cases}$$

where $\boldsymbol{n}_1 = \frac{\gamma''}{\|\gamma''\|} = \frac{\gamma''}{k_1}$, $\boldsymbol{n}_i = \frac{\boldsymbol{n}'_{i-1} + \delta_0\delta_1\ldots\delta_{i-1}k_{i-1}\boldsymbol{n}_{i-2}}{\delta_0 k_i}$, $\delta_0 = \delta(\boldsymbol{t})$ and $\delta_i = \delta(\boldsymbol{n}_i)$ $(i = 1, 2, 3)$.

Let $\boldsymbol{n}_2(s)$ be a timelike vector. Then $\boldsymbol{n}_j (j \neq 2)$ is a spacelike vector.

Define maps

$$NG^{\pm}_{2,j} : I \to S^2_+$$

by $NG^{\pm}_{2,j}(s) = \widetilde{\boldsymbol{n}_j \pm \boldsymbol{n}_2}(s) (j = 1, 3)$. Also define a map

$$\eta : S^2_+ \to S^2_+,$$

by $\eta(\widetilde{\boldsymbol{n}_1 \pm \boldsymbol{n}_2}(s)) = \widetilde{\boldsymbol{n}_2 \pm \boldsymbol{n}_3}(s)$, $\eta(\widetilde{\boldsymbol{n}_2 \pm \boldsymbol{n}_3}(s)) = \widetilde{\boldsymbol{n}_1 \pm \boldsymbol{n}_2}(s)$ and η is identity on the other elements of S^2_+. Each one of $NG^{\pm}_{2,j}(j = 1, 3)$ is called *the nullsphere Gauss map* of γ.

2 Nullsphere height functions on spacelike curve in NC^3

Now the function

$$H_1 : I \times S^2_+ \to \mathbb{R}$$

is defined by $H_1(s, \boldsymbol{v}) = \langle \gamma(s), \boldsymbol{v} \rangle$ and the function

$$H_2 : I \times S^2_+ \to \mathbb{R}$$

is defined by $H_2(s, \boldsymbol{v}) = \langle \gamma(s), \eta(\boldsymbol{v}) \rangle$, H_1 and H_2 are called the *nullsphere height function* on the spacelike curve γ. For any fixed $\boldsymbol{v}_0 \in S^2_+$, we denote that $h_{1,v_0}(s) = H_1(s, \boldsymbol{v}_0)$ and $h_{2,v_0}(s) = H_2(s, \boldsymbol{v}_0)$, then we have the following theorem.

Theorem 2.1. *Let $\gamma : I \to NC^3$ be an unit speed spacelike curve with $k_1(s) \neq 0$. Then we have the following assertions:*

(1) $h_{1,v_0}{}'(s_0) = 0$ *(resp.* $h_{2,v_0}{}'(s_0) = 0$*) if and only if there exist λ_1 and λ_2 such that* $\boldsymbol{v} = \tilde{\boldsymbol{n}}(s_0)$ *(resp.* $\eta(\boldsymbol{v}) = \tilde{\boldsymbol{n}}(s_0)$*),* $\boldsymbol{n}(s_0) = (\lambda_1\boldsymbol{n}_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2}\boldsymbol{n}_2 + \lambda_2\boldsymbol{n}_3)(s_0) \in NC^3$.

(2) $h_{1,v_0}{}'(s_0) = h_{1,v_0}{}''(s_0) = 0$ *(resp.* $h_{2,v_0}{}'(s_0) = h_{2,v_0}{}''(s_0) = 0$*) if and only if* $\boldsymbol{v} = \widetilde{\boldsymbol{n}_3 \pm \boldsymbol{n}_2}(s_0)$ *(resp.* $\boldsymbol{v} = \widetilde{\boldsymbol{n}_1 \pm \boldsymbol{n}_2}(s_0)$*).*

(3) $h_{1,v_0}{}'(s_0) = h_{1,v_0}{}''(s_0) = h_{1,v_0}{}'''(s_0) = 0$ *(resp.* $h_{2,v_0}{}'(s_0) = h_{2,v_0}{}''(s_0) = h_{2,v_0}{}'''(s_0) = 0$*) if and only if* $\boldsymbol{v} = \widetilde{\boldsymbol{n}_3 \pm \boldsymbol{n}_2}(s_0)$ *(resp.* $\boldsymbol{v} = \widetilde{\boldsymbol{n}_1 \pm \boldsymbol{n}_2}(s_0)$*) and* $k_2(s_0) = 0$.

(4) $h_{1,v_0}{}'(s_0) = \cdots = h^{(4)}_{1,v_0}(s_0) = 0$ *(resp.* $h_{2,v_0}{}'(s_0) = \cdots = h^{(4)}_{2,v_0}(s_0) = 0$*) if and only if* $\boldsymbol{v} = \widetilde{\boldsymbol{n}_3 \pm \boldsymbol{n}_2}(s_0)$ *(resp.* $\boldsymbol{v} = \widetilde{\boldsymbol{n}_1 \pm \boldsymbol{n}_2}(s_0)$*) and* $k_2(s_0) = k_2'(s_0) = 0$.

Theorem 2.2. *Let $\gamma(s)$ be a spacelike curve in nullcone 3-space. Then:*

(1)*If* $\mathbf{v}_0 = \tilde{\gamma}(s_0)$, *then* $h_{1,v_0}{}''(s_0)$ *never equal to zero.*

(2)*If* $\eta(\mathbf{v}_0) = \tilde{\gamma}(s_0)$, *then* $h_{2,v_0}{}''(s_0)$ *never equal to zero.*

Proposition 2.3. *If $\gamma(s)$ is an unit speed spacelike curve, H_1 and H_2 are nullsphere height functions,* $B_{H_1} = \{\boldsymbol{v} \in S^2_+ \mid h_{1,v}{}'(s) = h_{1,v}{}''(s) = 0\}$ *and* $B_{H_2} = \{\boldsymbol{v} \in S^2_+ \mid h_{2,v}{}'(s) = h_{2,v}{}''(s) = 0\}$, *then the following conditions are equivalent:*

(1) $h_{1,v_0}{}'''(s_0) = 0$ *for* $\boldsymbol{v}_0 = (\widetilde{\boldsymbol{n}_3 \pm \boldsymbol{n}_2})(s_0)$ *(resp.* $h_{2,v_0}{}'''(s_0) = 0$ *for* $\boldsymbol{v}_0 = (\widetilde{\boldsymbol{n}_1 \pm \boldsymbol{n}_2})(s_0)$*);*

(2) s_0 *is a singularity of nullsphere Gauss map* $NG^{\pm}_{2,3}$ *(resp.* $NG^{\pm}_{2,1}$*) on* γ*;*

(3) $k_2(s_0) = 0$.

Consider now the particular case of a curve $\gamma \subset NC^3$. Given a vector $\mathbf{v} \in S^2_+$(resp. S^3_1, H^3_1) and a number c, denote by $S(\mathbf{v}, c)$ the *null hyperhorosphere*(resp. *null hypersphere, null equidistant hyperplane*) determined by the intersection of the hyperplane $HP(\mathbf{v}, c)$ with NC^3.

Proposition 2.4. *Suppose that* $\widetilde{\gamma}(s) = NG^{\pm}_{2,j}(s)$. *If* $NG^{\pm}_{2,j}$ *is constant, then* $\gamma(s)$ *is a straight line.*

Proof. Since $\widetilde{\gamma}(s) = NG^{\pm}_{2,j}(s)$, $\gamma(s) = \gamma_1(s)NG^{\pm}_{2,j}(s)$. $NG^{\pm}_{2,j}(s)$ is constant, so $\gamma(s)$ is a straight line. $\square$

For an unit speed spacelike curve $\gamma : I \longrightarrow NC^3$, we now define *extended nullsphere height functions* $\widetilde{H}_1 : I \times NC^3 \longrightarrow \mathbb{R}$ by $\widetilde{H}_1(s, \boldsymbol{v}) = H_1(s, \tilde{\boldsymbol{v}}) - \boldsymbol{v}_1 = \langle \gamma(s), \tilde{\boldsymbol{v}} \rangle - \boldsymbol{v}_1$ and $\widetilde{H}_2 : I \times NC^3 \longrightarrow \mathbb{R}$ by $\widetilde{H}_2(s, \boldsymbol{v}) = H_2(s, \tilde{\boldsymbol{v}}) - \boldsymbol{v}_1 = \langle \gamma(s), \eta(\tilde{\boldsymbol{v}}) \rangle - \boldsymbol{v}_1$, where H_1 and H_2 are the nullsphere height function on γ. For any fixed $\mathbf{v}_0 \in NC^3$, we denote $\tilde{h}_{1,v_0}(s) = \widetilde{H}_1(s, v_0)$ and $\tilde{h}_{2,v_0}(s) = \widetilde{H}_2(s, v_0)$.

Let $F : NC^3 \longrightarrow \mathbb{R}$ be a submersion and $\gamma : I \longrightarrow NC^3$ be a spacelike curve. We say that γ and $F^{-1}(0)$ have *k-point contact at* t_0 if $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0,\ g^{(k)}(t_0) \neq 0$. Then we have the following corollary.

Corollary 2.5. *Let* $\gamma : I \longrightarrow NC^3$ *be an unit speed spacelike curve with* $k_1(s) \neq 0$. *Then* γ *and the null hyperhorosphere* $S(\boldsymbol{v}^{\pm}_0, c^{\pm}_0)$ *have 4-point contact at* s_0 *if and only if* $k_2(s) = 0$ *and* $k_2'(s) \neq 0$, *where* $\boldsymbol{v}^{\pm}_0 = \widetilde{\boldsymbol{n}_3 \pm \boldsymbol{n}_2}(s_0), c^{\pm}_0 = \langle \gamma(s_0), \boldsymbol{v}^{\pm}_0 \rangle$.

This work is only a preparation for further studying, in the following, we will give the classification of singularities of nullsphere Gausss map and discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

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