On spacelike curve in nullcone 3-space (Applications of singularity theory to differential equations and differential geometry)

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On spacelike curve in nullcone 3-space

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1 Basic notions

The nullcone is one kind of pseudo-sphere of Minkowski space. Our aim in this article is to develop the study for spacelike curve in nullcone 3-space by Bruce and Giblin’s singularity theory. In order to study the spacelike curve of nullcone 3-space, we need to develop differential geometry of spacelike curve in nullcone 3-space similarly as it was done for curves in Euclidean space [2].

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)|x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the pseudo-scalar product of $x$ and $y$ is defined by $\langle x, y \rangle = -x_1y_1 + \sum_{i=2}^{4}x_iy_i$. $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ is called a Minkowski 4-space and denoted by $\mathbb{R}^4_1$. A vector $x$ in $\mathbb{R}^4_1 \setminus \{0\}$ is called spacelike, lightlike or timelike if $\langle x, x \rangle$ is positive, zero or negative respectively. The norm of a vector $x \in \mathbb{R}^4_1$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. For any $x, y \in \mathbb{R}^4_1$, we say $x$ pseudo-perpendicular to $y$ if $\langle x, y \rangle = 0$. For a vector $v \in \mathbb{R}^4_1$ and a real number $c$, we define a hyperplane with pseudo normal $v$ by $HP(v, c) = \{x \in \mathbb{R}^4_1|\langle x, v \rangle = c\}$. $HP(v, c)$ is called a timelike hyperplane, a spacelike hyperplane or a lightlike hyperplane if $v$ is timelike, spacelike or lightlike respectively. Now, define the nullcone 3-space by $NC^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_1 | x_1 \neq 0, \langle x, x \rangle = 0 \}$, the de Sitter 3-space by $S^3_1 = \{x \in \mathbb{R}^4_1 | \langle x, x \rangle = 1 \}$ and the hyperbolic 3-space by $H^3_1 = \{x \in \mathbb{R}^4_1 | \langle x, x \rangle = -1 \}$. If $x = (x_1, x_2, x_3, x_4)$ is a lightlike vector, then $x_1 \neq 0$. Therefore $\tilde{x} = \left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}\right) \in S^2_+ = \{x \in \mathbb{R}^4_1 | \langle x, x \rangle = 0, x_1 = 1 \}$. $S^2_+$ is called the nullcone unit 2-sphere.

For any $x_1, x_2, x_3 \in \mathbb{R}^4_1$, we define a vector $x_1 \wedge x_2 \wedge x_3$ by

$$x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix} -e_1, & e_2, & e_3, & e_4 \\ x_1^1, & x_2^1, & x_3^1, & x_4^1 \\ x_2^2, & x_2^3, & x_2^4, & x_4^2 \\ x_3^3, & x_3^4, & x_3^2, & x_4^3 \end{vmatrix},$$

where $e_1, e_2, e_3, e_4$ are the canonical basis of $\mathbb{R}^4_1$ and $x_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. It is easy to check that $\langle x_1 \wedge x_2 \wedge x_3 \rangle = \det(x_1, x_2, x_3)$, so that $x_1 \wedge x_2 \wedge x_3$ is pseudo orthogonal to $x_i (i = 1, 2, 3)$.

Let $\gamma : I \to NC^3$; $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a smooth regular curve in $NC^3$ (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where $I$ is an open interval. The curve $\gamma$ is called a spacelike curve if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$ is positive for any $t \in I$. The arc-length of a spacelike curve $\gamma$, measured from $\gamma(t_0), t_0 \in I$ is $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| dt$. Then a parameter $s$ is determined such that $\|\dot{\gamma}(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. We say that a spacelike curve $\gamma$ is parameterized by arc-length if it satisfies that $\|\dot{\gamma}(s)\| = 1$. Throughout the reminder in this article, $s$ will denote the arc-length parameter. Let $f(s) = \gamma'(s)$, we call $f(s)$ an unit tangent vector of $\gamma$ at $s$. The signature of $x$ is defined to be

$$\delta(x) = \text{sign}(x) = \begin{cases} 1 & x: \text{spacelike}; \\ 0 & x: \text{lightlike}; \\ -1 & x: \text{timelike}. \end{cases}$$

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For any nonlightlike curve $\gamma : I \to NC^3$, which is parameterized by arc-length and satisfies $k_1(s) \neq 0$. We can construct a pseudo-orthogonal frame \{${t(s), n_1(s), n_2(s), n_3(s)}$\} of $\mathbb{R}^4_1$ along $\gamma$ which satisfies the following Frenet-Serret type formulae:

$$\begin{align*}
  t(s) &= \gamma'(s); \\
  t'(s) &= k_1(s)n_1(s); \\
  n_1'(s) &= -\delta_1k_1(s)t(s) + k_2(s)n_2(s); \\
  n_2'(s) &= \delta_3k_2(s)n_1(s) + k_3(s)n_3(s); \\
  n_3'(s) &= \delta_1k_3(s)n_2(s),
\end{align*}$$

where $n_1 = \frac{\gamma''}{||\gamma''||}$, $n_i = n_{i-1}^{\pm 1} + \delta_1 \delta_3 \delta_{i-1} k_{i-2}^{\pm 1} n_{i-1}^{\pm 1}$, $\delta_0 = \delta(t)$ and $\delta_1 = \delta(n_i)$ ($i = 1, 2, 3$).

Let $n_2(s)$ be a timelike vector. Then $n_2(j \neq 2)$ is a spacelike vector.

Define maps

$$NG_{2,j} : I \to S^2_+$$

by $NG_{2,j}(s) = n_j \pm n_2(s)(j = 1, 3)$. Also define a map

$$\eta : S^2_+ \to S^2_+,$$

by $\eta(n_1 \pm n_2(s)) = n_2 \pm n_3(s)$, $\eta(n_2 \pm n_3(s)) = n_1 \pm n_2(s)$ and $\eta$ is identity on the other elements of $S^2_+$. Each one of $NG_{2,j}(j = 1, 3)$ is called the nullsphere Gauss map of $\gamma$.

2 Nullsphere height functions on spacelike curve in $NC^3$

Now the function

$$H_1 : I \times S^2_+ \to \mathbb{R}$$

is defined by $H_1(s, v) = \langle \gamma(s), v \rangle$ and the function

$$H_2 : I \times S^2_+ \to \mathbb{R}$$

is defined by $H_2(s, v) = \langle \gamma(s), \eta(v) \rangle$, $H_1$ and $H_2$ are called the nullsphere height function on the spacelike curve $\gamma$. For any fixed $v_0 \in S^2_+$, we denote that $h_{1,v_0}(s) = H_1(s, v_0)$ and $h_{2,v_0}(s) = H_2(s, v_0)$, then we have the following theorem.

**Theorem 2.1.** Let $\gamma : I \to NC^3$ be an unit speed space curve with $k_1(s) \neq 0$. Then we have the following assertions:

1. $h_{1,v_0}'(s_0) = 0$ (resp. $h_{2,v_0}'(s_0) = 0$) if and only if there exist $\lambda_1$ and $\lambda_2$ such that $\nu = \tilde{n}(s_0)$ (resp. $\eta(\nu) = \tilde{n}(s_0)$), $n(s_0) = (\lambda_1 n_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2} n_2 + \lambda_2 n_3)(s_0) \in NC^3$.
2. $h_{1,v_0}''(s_0) = 0$ (resp. $h_{2,v_0}''(s_0) = 0$) if and only if $\nu = n_3 \pm n_2(s_0)$ (resp. $\eta(\nu) = n_1 \pm n_2(s_0)$).
3. $h_{1,v_0}''(s_0) = h_{1,v_0}'''(s_0) = 0$ (resp. $h_{2,v_0}''(s_0) = h_{2,v_0}'''(s_0) = 0$) if and only if $\nu = n_3 \pm n_2(s_0)$ (resp. $\eta(\nu) = n_1 \pm n_2(s_0)$) and $k_2(s_0) = 0$.
4. $h_{1,v_0}''(s_0) = \cdots = h_{1,v_0}^{(4)}(s_0) = 0$ (resp. $h_{2,v_0}''(s_0) = \cdots = h_{2,v_0}^{(4)}(s_0) = 0$) if and only if $\nu = n_3 \pm n_2(s_0)$ (resp. $\eta(\nu) = n_1 \pm n_2(s_0)$) and $k_2(s_0) = k_2(s_0) = 0$.

**Theorem 2.2.** Let $\gamma(s)$ be a spacelike curve in nullcone 3-space. Then:

1. If $v_0 = \gamma(s_0)$, then $h_{1,v_0}''(s_0)$ never equal to zero.
2. If $\eta(v_0) = \gamma(s_0)$, then $h_{2,v_0}''(s_0)$ never equal to zero.

**Proposition 2.3.** If $\gamma(s)$ is an unit speed spacelike curve, $H_1$ and $H_2$ are nullsphere height functions, $B_{H_1} = \{v \in S^2_+ \mid h_{1,v}''(s) = h_{1,v}'''(s) = 0\}$ and $B_{H_2} = \{v \in S^2_+ \mid h_{2,v}''(s) = h_{2,v}'''(s) = 0\}$, then the following conditions are equivalent:

1. $h_{1,v}'''(s_0) = 0$ for $v_0 = (n_3 \pm n_2)(s_0)$ (resp. $h_{2,v}'''(s_0) = 0$ for $v_0 = (n_2 \pm n_2)(s_0)$);
2. $s_0$ is a singularity of nullsphere Gauss map $NG_{2,3}^{\pm}(v_0) = NG_{2,1}^{\pm}(v_0) = 0$ on $\gamma$.
3. $k_2(s_0) = 0$. 


Consider now the particular case of a curve $\gamma \subset NC^3$. Given a vector $v \in S^2_+(\text{resp. } S^2_-, H^3_\pm)$ and a number $c$, denote by $S(v, c)$ the null hyperhorosphere (resp. null hypersphere, null equidistant hyperplane) determined by the intersection of the hyperplane $HP(v, c)$ with $NC^3$.

**Proposition 2.4.** Suppose that $\overline{\gamma}(s) = NG_{2,j}^{\pm}(s)$. If $NG_{2,j}^{\pm}$ is constant, then $\gamma(s)$ is a straight line.

*Proof.* Since $\overline{\gamma}(s) = NG_{2,j}^{\pm}(s)$, $\gamma(s) = \gamma_1(s)NG_{2,j}^{\pm}(s)$. $NG_{2,j}^{\pm}(s)$ is constant, so $\gamma(s)$ is a straight line. $\square$

For an unit speed spacelike curve $\gamma : I \rightarrow NC^3$, we now define extended nullsphere height functions $\overline{H}_1 : I \times NC^3 \rightarrow \mathbb{R}$ by $\overline{H}_1(s, v) = H_1(s, \overline{v}) - v_1 = \langle \gamma(s), \overline{v} \rangle - v_1$ and $\overline{H}_2 : I \times NC^3 \rightarrow \mathbb{R}$ by $\overline{H}_2(s, v) = H_2(s, \overline{v}) - v_1 = \langle \gamma(s), \eta(v) \rangle - v_1$, where $H_1$ and $H_2$ are the nullsphere height function on $\gamma$. For any fixed $v_0 \in NC^3$, we denote $\overline{h}_{1,v_0}(s) = \overline{H}_1(s, v_0)$ and $\overline{h}_{2,v_0}(s) = \overline{H}_2(s, v_0)$.

Let $F : NC^3 \rightarrow \mathbb{R}$ be a submersion and $\gamma : I \rightarrow NC^3$ be a spacelike curve. We say that $\gamma$ and $F^{-1}(0)$ have $k$-point contact at $t_0$ if $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. Then we have the following corollary.

**Corollary 2.5.** Let $\gamma : I \rightarrow NC^3$ be an unit speed spacelike curve with $k_1(s) \neq 0$. Then $\gamma$ and the null hyperhorosphere $S(v_0^\pm, c_0^\pm)$ have 4-point contact at $s_0$ if and only if $k_2(s) = 0$ and $k'_2(s) \neq 0$, where $v_0^\pm = n_3 \pm n_2(s_0)$, $c_0^\pm = \langle \gamma(s_0), v_0^\pm \rangle$.

This work is only a preparation for further studying, in the following, we will give the classification of singularities of nullsphere Gausss map and discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

**References**


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