Title: On spacelike curve in nullcone 3-space (Applications of singularity theory to differential equations and differential geometry)

Author(s): Kong, L.L.; Pei, D.H.

Citation: 数理解析研究所講究録 (2009), 1664: 68-70

Issue Date: 2009-09

URL: http://hdl.handle.net/2433/141034

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
On spacelike curve in nullcone 3-space

L.L. Kong and D.H. Pei

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China

1 Basic notions

The nullcone is one kind of pseudo-sphere of Minkowski space. Our aim in this article is to develop the study for spacelike curve in nullcone 3-space by Bruce and Giblin’s singularity theory. In order to study the spacelike curve of nullcone 3-space, we need to develop differential geometry of spacelike curve in nullcone 3-space similarly as it was done for curves in Euclidean space [2].

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the pseudo-scalar product of $x$ and $y$ is defined by $\langle x, y \rangle = -x_1y_1 + \sum_{i=2}^{4}x_iy_i$. $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ is called a Minkowski 4-space and denoted by $\mathbb{R}_1^4$. A vector $x$ in $\mathbb{R}_1^4 \setminus \{0\}$ is called \textit{spacelike}, \textit{lightlike} or \textit{timelike} if $\langle x, x \rangle$ is positive, zero or negative respectively. The norm of a vector $x \in \mathbb{R}_1^4$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. For any $x, y \in \mathbb{R}_1^4$, we say $x$ \textit{pseudo-perpendicular} to $y$ if $\langle x, y \rangle = 0$. For a vector $v \in \mathbb{R}_1^4$ and a real number $c$, we define a hyperplane with pseudo normal $v$ by $HP(v, c) = \{x \in \mathbb{R}_1^4 | \langle x, v \rangle = c\}$. $HP(v, c)$ is called a \textit{timelike hyperplane}, a \textit{spacelike hyperplane} or a \textit{lightlike hyperplane} if $v$ is timelike, spacelike or lightlike respectively. Now, define the \textit{nullcone 3-space} by $NC^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}_1^4 | x_1 \neq 0, \langle x, x \rangle = 0\}$, the \textit{de Sitter 3-space} by $S_1^3 = \{x \in \mathbb{R}_1^4 | \langle x, x \rangle = 1\}$ and the \textit{hyperbolic 3-space} by $H_1^3 = \{x \in \mathbb{R}_1^4 | \langle x, x \rangle = -1\}$. If $x = (x_1, x_2, x_3, x_4)$ is a lightlike vector, then $x_1 \neq 0$. Therefore $\bar{x} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}) \in S_1^2 = \{x \in \mathbb{R}_1^4 | \langle x, x \rangle = 0, x_1 = 1\}$. $S_1^2$ is called the \textit{nullcone unit 2-sphere}.

For any $x_1, x_2, x_3 \in \mathbb{R}_1^4$, we define a vector $x_1 \wedge x_2 \wedge x_3$ by

$$x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_2^2 & x_2^2 & x_3^2 & x_4^2 \\ x_3^3 & x_3^3 & x_3^3 & x_4^3 \\ x_4^4 & x_4^4 & x_4^4 & x_4^4 \end{vmatrix},$$

where $e_1, e_2, e_3, e_4$ are the canonical basis of $\mathbb{R}_1^4$ and $x_i = (x_{i1}^1, x_{i1}^2, x_{i1}^3, x_{i1}^4)$. It is easy to check that $\langle x, x_1 \wedge x_2 \wedge x_3 \rangle = \det(x_1, x_2, x_3)$, so that $x_1 \wedge x_2 \wedge x_3$ is pseudo orthogonal to $x_i (i = 1, 2, 3)$.

Let $\gamma : I \rightarrow NC^3$; $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$, a smooth regular curve in $NC^3$ (i.e., $\gamma(t) \neq 0$ for any $t \in I$), where $I$ is an open interval. The curve $\gamma$ is called a \textit{spacelike curve} if $\langle \gamma'(t), \gamma'(t) \rangle$ is positive for any $t \in I$. The \textit{arc-length} of a spacelike curve $\gamma$, measured from $\gamma(t_0)$, $t_0 \in I$ is $s(t) = \int_{t_0}^{t} \|\gamma'(t)\| \, dt$. Then a parameter $s$ is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. We say that a spacelike curve $\gamma$ is \textit{parameterized by arc-length} if it satisfies that $\|\gamma'(s)\| = 1$. Throughout the reminder in this article, $s$ will denote the arc-length parameter. Let $t(s) = \gamma'(s)$. We call $t(s)$ an \textit{unit tangent vector} of $\gamma$ at $s$. The \textit{signature} of $x$ is defined to be

$$\delta(x) = \text{sign}(x) = \begin{cases} 1 & : \text{spacelike}; \\ 0 & : \text{lightlike}; \\ -1 & : \text{timelike}. \end{cases}$$
For any nonlightlike curve $\gamma : I \rightarrow NC^3$, which is parameterized by arc-length and satisfies $k_1(s) \neq 0$.

We can construct a pseudo-orthogonal frame $\{t(s), n_1(s), n_2(s), n_3(s)\}$ of $\mathbb{R}^3_1$ along $\gamma$ which satisfies the following Frenet-Serret type formulæ:

$$
\begin{align*}
\mathbf{t}(s) &= \gamma'(s); \\
\mathbf{t}'(s) &= k_1(s)n_1(s); \\
n_1'(s) &= -\delta_1k_1(s)t(s) + k_2(s)n_2(s); \\
n_2'(s) &= \delta_3k_3(s)n_1(s) + k_3(s)n_3(s); \\
n_3'(s) &= \delta_1k_3(s)n_2(s),
\end{align*}
$$

where $n_1 = \gamma''/||\gamma''||$, $n_i = n_{i-1} + \delta_{s,k_i} ... \delta_{s,k_{i-1}}n_{i-2} - \delta_0$, $\delta_0 = \delta(t)$ and $\delta_i = \delta(n_i)$ ($i = 1, 2, 3$).

Let $n_2(s)$ be a timelike vector. Then $n_2(j \neq 2)$ is a spacelike vector.

Define maps

$$NG_{2,j} : I \rightarrow S_{+}^2$$

by $NG_{2,j}(s) = n_j \pm n_2(s) (j = 1, 3)$. Also define a map

$$\eta : S_{+}^2 \rightarrow S_{+}^2,$$

by $\eta(n_1 \pm n_2(s)) = n_2 \pm n_3(s)$, $\eta(n_2 \pm n_3(s)) = n_1 \pm n_2(s)$ and $\eta$ is identity on the other elements of $S_{+}^2$. Each one of $NG_{2,j}(j = 1, 3)$ is called the nullsphere Gauss map of $\gamma$.

2 Nullsphere height functions on spacelike curve in $NC^3$

Now the function

$$H_1 : I \times S_{+}^2 \rightarrow \mathbb{R}$$

is defined by $H_1(s, v) = \langle \gamma(s), v \rangle$ and the function

$$H_2 : I \times S_{+}^2 \rightarrow \mathbb{R}$$

is defined by $H_2(s, v) = \langle \gamma(s), \eta(v) \rangle$, $H_1$ and $H_2$ are called the nullsphere height function on the spacelike curve $\gamma$. For any fixed $v_0 \in S_{+}^2$, we denote that $h_{1,v_0}(s) = H_1(s, v_0)$ and $h_{2,v_0}(s) = H_2(s, v_0)$, then we have the following theorem.

**Theorem 2.1.** Let $\gamma : I \rightarrow NC^3$ be an unit speed spacelike curve with $k_1(s) \neq 0$. Then we have the following assertions:

1. $h_{1,v_0'}(s_0) = 0$ (resp. $h_{2,v_0'}(s_0) = 0$) if and only if there exist $\lambda_1$ and $\lambda_2$ such that $v = \tilde{n}(s_0)$ (resp. $\eta(v) = \tilde{n}(s_0)$), $n(s_0) = (\lambda_1n_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2}n_2 + \lambda_3n_3)(s_0) \in NC^3$.

2. $h_{1,v_0''}(s_0) = h_{1,v_0}''(s_0) = 0$ (resp. $h_{2,v_0''}(s_0) = h_{2,v_0}''(s_0) = 0$) if and only if $v = n_3 \pm n_2(s_0)$ (resp. $v = n_1 \pm n_2(s_0)$).

3. $h_{1,v_0'}(s_0) = h_{1,v_0''}(s_0) = h_{1,v_0}'(s_0) = 0$ (resp. $h_{2,v_0'}(s_0) = h_{2,v_0}''(s_0) = h_{2,v_0}'(s_0) = 0$) if and only if $v = n_3 \pm n_2(s_0)$ (resp. $v = n_1 \pm n_2(s_0)$) and $k_2(s_0) = 0$.

4. $h_{1,v_0}(s_0) = \cdots = h_{1,v_0}^{(4)}(s_0) = 0$ (resp. $h_{2,v_0}(s_0) = \cdots = h_{2,v_0}^{(4)}(s_0) = 0$) if and only if $v = n_3 \pm n_2(s_0)$ (resp. $v = n_1 \pm n_2(s_0)$) and $k_2(s_0) = k_2'(s_0) = 0$.

**Theorem 2.2.** Let $\gamma(s)$ be a spacelike curve in nullcone 3-space. Then:

1. If $v_0 \ni \gamma(s_0)$, then $h_{1,v_0}''(s_0)$ never equal to zero.

2. If $\eta(v_0) = \gamma(s_0)$, then $h_{2,v_0}''(s_0)$ never equal to zero.

**Proposition 2.3.** If $\gamma(s)$ is an unit speed spacelike curve, $H_1$ and $H_2$ are nullsphere height functions, $B_{H_1} = \{v \in S_{+}^2 \mid h_{1,v'}(s) = h_{1,v''}(s) = 0\}$ and $B_{H_2} = \{v \in S_{+}^2 \mid h_{2,v'}(s) = h_{2,v''}(s) = 0\}$, then the following conditions are equivalent:

1. $h_{1,v_0''}(s_0) = 0$ for $v_0 = (n_3 \pm n_2)(s_0)$ (resp. $h_{2,v_0''}(s_0) = 0$ for $v_0 = (n_1 \pm n_2)(s_0)$);

2. $s_0$ is a singularity of nullsphere Gauss map $NG_{2,3,1}^\pm$ (resp. $NG_{2,1}^\pm$) on $\gamma$;

3. $k_2(s_0) = 0$. 
Consider now the particular case of a curve \( \gamma \subset NC^3 \). Given a vector \( \mathbf{v} \in S_2^0(\text{resp. } S_1^0, H_1^3) \) and a number \( c \), denote by \( S(\mathbf{v}, c) \) the null hyperhorosphere (resp. null hypersphere, null equidistant hyperplane) determined by the intersection of the hyperplane \( HP(\mathbf{v}, c) \) with \( NC^3 \).

**Proposition 2.4.** Suppose that \( \tilde{\gamma}(s) = NG_{2,j}^{\pm}(s) \). If \( NG_{2,j}^{\pm} \) is constant, then \( \gamma(s) \) is a straight line.

**Proof.** Since \( \tilde{\gamma}(s) = NG_{2,j}^{\pm}(s) \), \( \gamma(s) = \gamma_1(s)NG_{2,j}^{\pm}(s) \). \( NG_{2,j}^{\pm}(s) \) is constant, so \( \gamma(s) \) is a straight line. \( \square \)

For an unit speed spacelike curve \( \gamma : I \rightarrow NC^3 \), we now define extended nullsphere height functions \( \tilde{H}_1 : I \times NC^3 \rightarrow \mathbb{R} \) by \( \tilde{H}_1(s, \mathbf{v}) = H_1(s, \overline{\mathbf{v}}) - \mathbf{v}_1 = \langle \gamma(s), \overline{\mathbf{v}} \rangle - \mathbf{v}_1 \) and \( \tilde{H}_2 : I \times NC^3 \rightarrow \mathbb{R} \) by \( \tilde{H}_2(s, \mathbf{v}) = H_2(s, \overline{\mathbf{v}}) - \mathbf{v}_1 = \langle \gamma(s), \eta(\overline{\mathbf{v}}) \rangle - \mathbf{v}_1 \), where \( H_1 \) and \( H_2 \) are the nullsphere height function on \( \gamma \). For any fixed \( \mathbf{v}_0 \in NC^3 \), we denote \( \tilde{h}_{1,v_0}(s) = \tilde{H}_1(s, \mathbf{v}_0) \) and \( \tilde{h}_{2,v_0}(s) = \tilde{H}_2(s, \mathbf{v}_0) \).

Let \( F : NC^3 \rightarrow \mathbb{R} \) be a submersion and \( \gamma : I \rightarrow NC^3 \) be a spacelike curve. We say that \( \gamma \) and \( F^{-1}(0) \) have \( k \)-point contact at \( t_0 \) if \( g(t) = F \circ \gamma(t) \) satisfies \( g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0, \ g^{(k)}(t_0) \neq 0 \). Then we have the following corollary.

**Corollary 2.5.** Let \( \gamma : I \rightarrow NC^3 \) be an unit speed spacelike curve with \( k_1(s) \neq 0 \). Then \( \gamma \) and the null hyperhorosphere \( S(\mathbf{v}_0^\pm, c_0^\pm) \) have 4-point contact at \( s_0 \) if and only if \( k_2(s) = 0 \) and \( k_2'(s) \neq 0 \), where \( \mathbf{v}_0^\pm = n_3 \pm n_2(s_0), c_0^\pm = \langle \gamma(s_0), \mathbf{v}_0^\pm \rangle \).

This work is only a preparation for further studying, in the following, we will give the classification of singularities of nullsphere Gauss map and discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

**References**


Lingling Kong, School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China e-mail:kongl1111@nenu.edu.cn

Donghe Pei, School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China e-mail:peidh340@nenu.edu.cn