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<th>Title</th>
<th>On spacelike curve in nullcone 3-space (Applications of singularity theory to differential equations and differential geometry)</th>
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<tr>
<td>Author(s)</td>
<td>Kong, L.L.; Pei, D.H.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1664: 68-70</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141034">http://hdl.handle.net/2433/141034</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On spacelike curve in nullcone 3-space

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1 Basic notions

The nullcone is one kind of pseudo-sphere of Minkowski space. Our aim in this article is to develop the study for spacelike curve in nullcone 3-space by Bruce and Giblin’s singularity theory. In order to study the spacelike curve of nullcone 3-space, we need to develop differential geometry of spacelike curve in nullcone 3-space similarly as it was done for curves in Euclidean space [2].

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the pseudo-scalar product of $x$ and $y$ is defined by $\langle x, y \rangle = -x_1y_1 + \sum_{i=2}^{4} x_i y_i$. $\mathbb{R}_1^4 \backslash \{0\}$ is called spacelike, lightlike or timelike if $\langle x, x \rangle$ is positive, zero or negative respectively. The norm of a vector $x \in \mathbb{R}_1^4$ is defined by $|x| = \sqrt{\langle x, x \rangle}$. For any $x, y \in \mathbb{R}_1^4$, we say $x$ pseudo-perpendicular to $y$ if $\langle x, y \rangle = 0$. For a vector $v \in \mathbb{R}_1^4$ and a real number $c$, we define a hyperplane with pseudo normal $v$ by $HP(v, c) = \{x \in \mathbb{R}_1^4 | \langle x, v \rangle = c\}$. $HP(v, c)$ is called a timelike hyperplane, a spacelike hyperplane or a lightlike hyperplane if $v$ is timelike, spacelike or lightlike respectively. Now, define the nullcone 3-space by $NC^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}_1^4 | x_1 \neq 0, \langle x, x \rangle = 0\}$, the de Sitter 3-space by $S^3_+ = \{x \in \mathbb{R}_1^4 | \langle x, x \rangle = 1\}$ and the hyperbolic 3-space by $H^3_+ = \{x \in \mathbb{R}_1^4 | \langle x, x \rangle = -1\}$. If $x = (x_1, x_2, x_3, x_4)$ is a lightlike vector, then $x_1 \neq 0$. Therefore $\tilde{x} = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}\right) \in S^3_+ = \{x \in \mathbb{R}_1^4 | \langle x, x \rangle = 0, x_1 = 1\}$. $S^3_+$ is called the nullcone unit 3-sphere.

For any $x_1, x_2, x_3 \in \mathbb{R}_1^4$, we define a vector $x_1 \wedge x_2 \wedge x_3$ by

$$x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix} -e_1, & e_2, & e_3, & e_4 \\ e_1, & x_2, & x_3, & x_4 \\ e_2, & x_3, & x_1, & x_4 \\ e_3, & x_1, & x_2, & x_4 \\ e_4, & x_1, & x_2, & x_3 \end{vmatrix},$$

where $e_1, e_2, e_3, e_4$ are the canonical basis of $\mathbb{R}_1^4$ and $x_i = (x_1^i, x_2^i, x_3^i, x_4^i)$. It is easy to check that $\langle x_1 \wedge x_2 \wedge x_3 \rangle = \det(x_1, x_2, x_3)$, so that $x_1 \wedge x_2 \wedge x_3$ is pseudo orthogonal to $x_i (i = 1, 2, 3)$.

Let $\gamma : I \to NC^3$; $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a smooth regular curve in $NC^3$ (i.e., $\gamma(t) \neq 0$ for any $t \in I$, where $I$ is an open interval. The curve $\gamma$ is called a spacelike curve if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$ is positive for any $t \in I$. The arc-length of a spacelike curve $\gamma$, measured from $\gamma(t_0)$, $t_0 \in I$ is $s(t) = \int_{t_0}^{t} ||\dot{\gamma}(t)|| dt$. Then a parameter $s$ is determined such that $||\gamma'(s)|| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. We say that a spacelike curve $\gamma$ is parameterized by arc-length if it satisfies that $||\gamma'(s)|| = 1$. Throughout the reminder in this article, $s$ will denote the arc-length parameter. Let $t(s) = \gamma'(s)$, we call $t(s)$ an unit tangent vector of $\gamma$ at $s$. The signature of $x$ is defined to be

$$\delta(x) = \text{sign}(x) = \begin{cases} 1 & x : \text{spacelike}; \\ 0 & x : \text{lightlike}; \\ -1 & x : \text{timelike}. \end{cases}$$

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2000 Mathematics Subject classification. 53B30, 58K05, 57R70.
Key Words and Phrases. spacelike curve, nullsphere Gauss map, nullsphere height function
Work partially supported by NSF of China No.10871035 and NCET of China No.05-0319.
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For any nonlightlike curve $\gamma : I \to NC^3$, which is parameterized by arc-length and satisfies $k_1(s) \neq 0$. We can construct a pseudo-orthogonal frame $\{t(s), n_1(s), n_2(s), n_3(s)\}$ of $\mathbb{R}^3_1$ along $\gamma$ which satisfies the following Frenet-Serret type formulae:

$$
\begin{align*}
t(s) &= \gamma'(s); \\
t'(s) &= k_1(s)n_1(s); \\
n_1'(s) &= -\delta_1k_1(s)t(s) + k_2(s)n_2(s); \\
n_2'(s) &= \delta_2k_2(s)n_1(s) + k_3(s)n_3(s); \\
n_3'(s) &= \delta_3k_3(s)n_2(s),
\end{align*}
$$

where $n_1 = \frac{\gamma''}{||\gamma''||} = \frac{\gamma''}{k_1}$, $n_i = \frac{n_{i-1} + \delta_1 \delta_2 \cdots \delta_{i-1} k_{i-1} n_{i-2}}{\delta_0 k_i}$, $\delta_0 = \delta(t)$ and $\delta_i = \delta(n_i)$ ($i = 1, 2, 3$).

Let $n_2(s)$ be a timelike vector. Then $n_j(j \neq 2)$ is a spacelike vector.

Define maps

$$NG_{2,j}^\pm : I \to S^2_+$$

by $NG_{2,j}^\pm(s) = n_j \pm n_2(s)(j = 1, 3)$. Also define a map

$$\eta : S^2_+ \to S^2_+,$$

by $\eta(n_1 \pm n_2(s)) = n_2 \pm n_3(s)$, $\eta(n_2 \pm n_3(s)) = n_1 \pm n_2(s)$ and $\eta$ is identity on the other elements of $S^2_+$. Each one of $NG_{2,j}^\pm(j = 1, 3)$ is called the nullsphere Gauss map of $\gamma$.

## 2 Nullsphere height functions on spacelike curve in $NC^3$

Now the function

$$H_1 : I \times S^2_+ \to \mathbb{R}$$

is defined by $H_1(s, v) = (\gamma(s), v)$ and the function

$$H_2 : I \times S^2_+ \to \mathbb{R}$$

is defined by $H_2(s, v) = (\gamma(s), \eta(v))$, $H_1$ and $H_2$ are called the nullsphere height function on the spacelike curve $\gamma$. For any fixed $v_0 \in S^2_+$, we denote that $h_{1,v_0}(s) = H_1(s, v_0)$ and $h_{2,v_0}(s) = H_2(s, v_0)$, then we have the following theorem.

**Theorem 2.1.** Let $\gamma : I \to NC^3$ be an unit speed nonlightlike curve with $k_1(s) \neq 0$. Then we have the following assertions:

1. $h_{1,v_0}'(s_0) = 0$ (resp. $h_{2,v_0}'(s_0) = 0$) if and only if there exist $\lambda_1$ and $\lambda_2$ such that $v = n_1(s_0)$ (resp. $\eta(v) = n_2(s_0)$), $n_1(s_0) = (\lambda_1 n_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2} n_2 + \lambda_2 n_3)(s_0) \in NC^3$.

2. $h_{1,v_0}''(s_0) = 0$ (resp. $h_{2,v_0}''(s_0) = 0$) if and only if $v = n_3 \pm n_2(s_0)$ (resp. $v = n_1 \pm n_2(s_0)$).

3. $h_{1,v_0}'''(s_0) = 0$ (resp. $h_{2,v_0}'''(s_0) = 0$) if and only if $v = n_3 \pm n_2(s_0)$ (resp. $v = n_1 \pm n_2(s_0)$) and $k_2(s_0) = 0$.

4. $h_{1,v_0}''''(s_0) = \cdots = h_{2,v_0}''''(s_0) = 0$ if and only if $v = n_3 \pm n_2(s_0)$ (resp. $v = n_1 \pm n_2(s_0)$) and $k_2(s_0) = k_2'(s_0) = 0$.

**Theorem 2.2.** Let $\gamma(s)$ be a spacelike curve in nullcone 3-space. Then:

1. If $v_0 = \gamma(s_0)$, then $h_{1,v_0}''(s_0)$ never equal to zero.

2. If $\eta(v_0) = \gamma(s_0)$, then $h_{2,v_0}''(s_0)$ never equal to zero.

**Proposition 2.3.** If $\gamma(s)$ is an unit speed spacelike curve, $H_1$ and $H_2$ are nullsphere height functions, $B_{H_1} = \{v \in S^2_+ \mid h_{1,v}''(s) = h_{1,v''}'(s) = 0\}$ and $B_{H_2} = \{v \in S^2_+ \mid h_{2,v}''(s) = h_{2,v''}'(s) = 0\}$, then the following conditions are equivalent:

1. $h_{1,v_0}''(s_0) = 0$ for $v_0 = (n_3 \pm n_2)(s_0)$ (resp. $h_{2,v_0}''(s_0) = 0$ for $v_0 = (n_1 \pm n_2)(s_0)$); $s_0$ is a singularity of nullsphere Gauss map $NG_{2,3}^\pm$ (resp. $NG_{1,1}^\pm$) on $\gamma$.

2. $k_2(s_0) = 0$. 

(3) $k_2(s_0) = 0$. 

Consider now the particular case of a curve $\gamma \subset NC^3$. Given a vector $v \in S^2_+(\text{resp. } S^3_+, H^2_+)$ and a number $c$, denote by $S(v, c)$ the null hyperhorosphere (resp. null hypersphere, null equidistant hyperplane) determined by the intersection of the hyperplane $HP(v, c)$ with $NC^3$.

**Proposition 2.4.** Suppose that $\tilde{\gamma}(s) = NG^\pm_{2,j}(s)$. If $NG^\pm_{2,j}$ is constant, then $\gamma(s)$ is a straight line.

**Proof.** Since $\tilde{\gamma}(s) = NG^\pm_{2,j}(s)$, $\gamma(s) = \gamma_1(s)NG^\pm_{2,j}(s)$. $NG^\pm_{2,j}(s)$ is constant, so $\gamma(s)$ is a straight line. □

For an unit speed spacelike curve $\gamma : I \rightarrow NC^3$, we now define extended nullsphere height functions $\bar{H}_1 : I \times NC^3 \rightarrow \mathbb{R}$ by $\bar{H}_1(s, v) = H_1(s, \bar{v}) - v_1 = \langle \gamma(s), \bar{v} \rangle - v_1$ and $\bar{H}_2 : I \times NC^3 \rightarrow \mathbb{R}$ by $\bar{H}_2(s, v) = H_2(s, \bar{v}) - v_1 = \langle \gamma(s), \eta(\bar{v}) \rangle - v_1$, where $H_1$ and $H_2$ are the nullsphere height function on $\gamma$. For any fixed $v_0 \in NC^3$, we denote $h_{1,v_0}(s) = \bar{H}_1(s, v_0)$ and $\tilde{h}_{2,v_0}(s) = \bar{H}_2(s, v_0)$.

Let $F : NC^3 \rightarrow \mathbb{R}$ be a submersion and $\gamma : I \rightarrow NC^3$ be a spacelike curve. We say that $\gamma$ and $F^{-1}(0)$ have $k$-point contact at $t_0$ if $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. Then we have the following corollary.

**Corollary 2.5.** Let $\gamma : I \rightarrow NC^3$ be an unit speed spacelike curve with $k_1(s) \neq 0$. Then $\gamma$ and the null hyperhorosphere $S(v_0^\pm, c_0^\mp)$ have 4-point contact at $s_0$ if and only if $k_2(s) = 0$ and $k_2'(s) \neq 0$, where $v_0^\pm = n_3 \pm n_2(s_0), c_0^\mp = \langle \gamma(s_0), v_0^\pm \rangle$.

This work is only a preparation for further studying, in the following, we will give the classification of singularities of nullsphere Gausss map and discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

**References**


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