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A set of integral elements of higher order jet spaces

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1. Introduction

This is a survey of [S1] and [S2]. The theory of exterior differential systems originates form the geometric theory of partial differential equations studied by Cartan, Darboux, Goursat Lie, Monge and others. Roughly speaking, the geometry of subbundles of the tangent bundle of smooth manifolds. In this theory, we consider partial differential equations to be varieties in jet spaces endowed with the canonical contact system. If the varieties induced by partial differential equations are submanifolds in jet spaces, then we consider that integral manifolds of the submanifolds are solutions of the original partial differential equations. Hence, the jet spaces endowed with the canonical contact system play important role in this theory and the characterization of the jet spaces was an important problem.

By a differential system (R, D), we mean a distribution D on a manifold R, i.e., D is a subbundle of the tangent bundle $T(R)$. The *derived system* ∂D of D is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D. In general ∂D is obtained as a subsheaf of the tangent sheaf of R (for the precise argument, see e.g.[Y1], [BCG3]). Moreover higher derived systems $\partial^i D$ are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put $\partial^0 D = D$ by convention. A differential system (R, D) is called regular if $\partial^i D$ are subbundles of $T(M)$ for every $i \geq 1$.

It is known that a Goursat flag of length k is locally isomorphic, at a generic point, to the canonical system on the k-jet spaces of 1 independent and 1 dependent variable by Engel, Goursat and Cartan. Here, a Goursat flag is a differential system D on a manifold R such that the differences of ranks between the i-th derived system $\partial^i D$ of D and the (i+1)-th derived system $\partial^{i+1} D$ of D are 1 for any i, and the k-th derived system $\partial^k D$ is the whole tangent bundle TR.

The characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [B] for the first order systems and was given by K. Yamaguchi in [Y1] and [Y2] for higher order systems for n independent and m dependent variables.

However, it was first explicitly exhibited by A.Giaro, A. Kumpera and C. Ruiz in [GKR] that a Goursat flag of length 3 has singularities and the research of singularities of Goursat flags of length k ($k \geq 3$) began as in [Morm].

To this situation, R. Montgomery and M. Zhitomirskii constructed the "Monster Goursat manifold" by successive applications of the "Cartan prolongation of rank 2 distributions" to a surface and showed that every germ of a Goursat flag of length k appears in this "Monster Goursat manifold" in [MZ].

After [MZ], P.Mormul defined, for the case of 1 independent and m dependent variables, the "generalized Monster Goursat manifold" which are obtained by successive applications of the "generalized Cartan prolongation" to the space of 1-jets of 1 independent and m dependent variables.

We gave a characterization of the "generalized Monster Goursat manifold" for the case of 1 independent and m dependent variables using the Cartan rank condition ($m \geq 3$), the Engel rank conditions ($m \geq 4$) and the existence of completely integrable subbundle ($m \geq 2$) in [SY]. This is a generalization of the result of R. Montgomery and M. Zhitomirskii in [MZ].

Moreover, I considered the construction of the "generalized Monster Goursat manifold" for the multi independent variables case which should be obtained by successive applications of the "prolongation" to the 2-jets of 1 dependent variables and 1-jets of m dependent variables for $m \geq 2$. These cases are completely different from the case of 1 independent variables. I showed that we can not define the "prolongation" except for the "prolongation" of 2-jet space of 2 independent and 1 dependent variables in [S1] and [S2]. In other wards, the "prolongation" of 2-jet space of 2 independent and 1 dependent variables is a manifold endowed with a canonical differential system, the others are not. Furthermore, in [S2], I classified the singularities in the "prolongation" of 2-jet space of 2 independent and 1 dependent variables.

2. Review of geometric construction of jet spaces

Let M be a manifold of dimension $m+n$. Fixing the number n, we form the space of n-dimensional *contact elements* to M, i.e., the Grassmann bundle $J(M,n) = \mathrm{Gr}(TM,n)$ over M consisting of n-dimensional subspaces of tangent spaces to M. Namely, $J(M,n)$ is defined by

$$J(M,n) = \bigcup_{x \in M} J_x, \qquad J_x = \mathrm{Gr}(T_x(M),n),$$

where $\mathrm{Gr}(T_x(M),n)$ denotes the Grassmann manifold of n-dimensional subspaces in $T_x(M)$. Let $\pi : J(M,n) \to M$ be the bundle projection. The *canonical system* C on $J(M,n)$ is, by definition, the differential system of codimension m on $J(M,n)$ defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M,n)) \mid \pi_*(v) \in u\} \subset T_u(J(M,n)) \xrightarrow{\pi_*} T_x(M),$$

where $\pi(u) = x$ for $u \in J(M,n)$.

Let us describe C in terms of a canonical coordinate system in $J(M,n)$. Let $u_o \in J(M,n)$. Let $(x_1,\dots,x_n,z^1,\dots,z^m)$ be a coordinate system on a neighborhood U' of $x_o = \pi(u_o)$ such that $dx_1,\dots,dx_n$ are linearly independent when restricted to $u_o \subset T_{x_o}(M)$. We put $U = \{ u \in \pi^{-1}(U') | dx_1|_u,\dots,dx_n|_u \text{ are linearly independent } \}$. Then U is a neighborhood of u_o in $J(M,n)$. Here $dz^\alpha|_u$ is a linear combination of $dx_i|_u$'s,i.e., $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$. Thus, there exist unique functions p_i^α on U

such that C is defined on U by the following 1-forms;

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^{n} p_i^\alpha dx_i \qquad (\alpha = 1, \ldots, m),$$

where we identify z^α and x_i on U' with their lifts on U. The system of functions $(x_i, z^\alpha, p_i^\alpha)$ $(\alpha = 1, \ldots, m, i = 1, \ldots, n)$ on U is called a *canonical coordinate system* of $J(M, n)$ subordinate to (x_i, z^α).

The space $(J(M, n), C)$ is called the (geometric) 1-jet space and especially, in case $m = 1$, is the so- called contact manifold. Let M, $\hat{M}$ be manifolds of dimension $m + n$ and $\varphi : M \longrightarrow \hat{M}$ be a diffeomorphism. Then φ induces the isomorphism $\varphi_* : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$, i.e., the differential map $\varphi_* : J(M, n) \longrightarrow J(\hat{M}, n)$ is a diffeomorphism sending C onto $\hat{C}$. The reason why the case $m = 1$ is special is explained by the following theorem of Bäcklund.

Theorem(Bäcklund) *Let M and $\hat{M}$ be manifolds of dimension $m + n$. Assume $m \geq 2$. Then, for an isomorphism $\Phi : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$, there exists a diffeomorphism $\varphi : M \longrightarrow \hat{M}$ such that $\Phi = \varphi_*$.*

The essential part of this theorem is to show that $F = \operatorname{Ker} \pi_*$ is the covariant system of $(J(M, n), C)$ for $m \geq 2$. Namely an isomorphism Φ sends F onto $\hat{F} = \operatorname{Ker} \hat{\pi}_*$ for $m \geq 2$. (For the proof, see [Y2] Theorem 1.4.)

In case $m = 1$, it is a well known fact that the group of isomorphisms of $(J(M, n), C)$, i.e., the group of contact transformations, is larger than the group of diffeomorphisms of M. Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number m of dependent variables is 1 or greater.

(1) Case $m = 1$. We should start from a contact manifold (J, C) of dimension $2n + 1$, which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space $(L(J), E)$ as follows: We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J, C);

$$L(J) = \bigcup_{u \in J} L_u \subset J(J, n),$$

where L_u is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Namely, $v \in J(J, n)$ is an integral element if and only if $v \subset C(u)$ and $d\varpi|_v = 0$, where $u = \pi(v)$. Let $\pi : L(J) \longrightarrow J$ be the projection. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where $\pi(v) = u$ for $v \in L(J)$.

We denote by $\operatorname{Ch}(C)$ the Cauchy characteristic system of C.

Then we have $\partial E = \pi_*^{-1}(C)$ and $\mathrm{Ch}\,(C) = \{0\}$ (cf.[Y1]). Hence we get $\mathrm{Ch}\,(\partial E) = \mathrm{Ker}\,\pi_*$, which implies the Bäcklund theorem for $(L(J), E)$ (cf. [Y1]).

Now we put

$$(J^2(M,n), C^2) = (L(J(M,n)), E),$$

where M is a manifold of dimension $n+1$.

Here recall that the Cauchy characteristic system of a differential system (R, D) are generally defined as follows;

The Cauchy characteristic system $\mathrm{Ch}\,(D)$ of a differential system (R, D) is defined by

$$\mathrm{Ch}\,(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \quad (\mathrm{mod}\ \ \omega_1, \ldots, \omega_s) \quad \text{for } i = 1, \ldots, s\},$$

where $D = \{\, \omega_1 = \cdots = \omega_s = 0 \,\}$ is defined locally by defining 1-forms $\{\omega_1, \ldots, \omega_s\}$.

(2) Case $m \geq 2$. Since $F = \mathrm{Ker}\,\pi_*$ is a covariant system of $(J(M,n), C)$, we define $J^2(M,n) \subset J(J(M,n),n)$ by

$$J^2(M,n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J(M,n), C), \text{ transversal to } F\},$$

C^2 is defined as the restriction to $J^2(M,n)$ of the canonical system on $J(J(M,n),n)$.

Now the higher order (geometric) jet spaces $(J^{k+1}(M,n), C^{k+1})$ for $k \geq 2$ are defined (simultaneously for all m) by induction on k. Namely, for $k \geq 2$, we define $J^{k+1}(M,n) \subset J(J^k(M,n),n)$ and C^{k+1} inductively as follows:

$$J^{k+1}(M,n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J^k(M,n), C^k), \text{ transversal to } \mathrm{Ker}\ (\pi^k_{k-1})_* \},$$

where $\pi^k_{k-1} : J^k(M,n) \to J^{k-1}(M,n)$ is the projection. Here we have

$$\mathrm{Ker}\ (\pi^k_{k-1})_* = \mathrm{Ch}\,(\partial C^k),$$

and C^{k+1} is defined as the restriction to $J^{k+1}(M,n)$ of the canonical system on $J(J^k(M,n),n)$.

Here we observe that, if we drop the transversality condition in our definition of $J^k(M,n)$ and collect all n-dimensional integral elements, we may have some singularities in $J^k(M,n)$ in general. Namely, a set of all n-dimensional integral elements of $(J^k(M,n), C^k)$ may be a variety.

Remark 2.1. In this paper, the notation $J^2(M,n)$ is used for the geometric 2-jet spaces, not for the ordinary 2-jet spaces $J^2(\mathbb{R}^n, \mathbb{R}^m)$. But $J^2(M,n)$ is locally isomorphic to $J^2(\mathbb{R}^n, \mathbb{R}^m)$, that is, local isomorphisms act on $J^2(M,n)$ transitively. Therefore the results are independent of the difference.

3. MAIN THEOREM

Let m, n be positive integers and M a manifold of dimension $m+n$. We denote $J^k(M,n)$ the k-jet space over M with n-independent variables and by C^k the canonical system on it (see §2).

We define, for $x \in J^k(M,n)$, the set Σ_x of n-dimensional integral elements of C^k through x;

$$\Sigma_x = \{\text{n-dim. integral elements of } (J^k(M,n), C^k)\},$$

and the subset

$$\Sigma(J^k(M,n)) := \bigcup_{x \in J^k(M,n)} \Sigma_x$$

of the Grassmannian $J(C^k, n) = \mathrm{Gr}(C^k, n)$ of n-dimensional linear subspaces of the distribution C^k;

$$J(C^k, n) = \bigcup_{x \in J^k} C_x, \qquad C_x = \mathrm{Gr}(C^k(x), n).$$

Here the integral elements of a differential system on a manifold are generally defined as follows;

Let (R, D) be a differential system, i.e. R is a manifold and D is a subbundle of TR. We take a system of local defining 1-forms $\{\varpi_1, \cdots, \varpi_s\}$ of D. An n-dimensional integral element of D at $x \in R$ is an n-dimensional subspace v of T_xR such that

$$\varpi_i|_v = d\varpi_i|_v = 0 \qquad (i = 1, \cdots, s).$$

That is, n-dimensional integral elements are candidates for the tangent spaces at x of n-dimensional integral manifolds of D.

By definition,

$$J^{k+1}(M,n) \subset \Sigma(J^k(M,n)) \subset J(C^k, n).$$

The set $\Sigma(J^k(M,n))$ of integral elements is the candidate for the extension of the notion "Monster Goursat manifolds" introduced in [MZ] to the case of several independent variables. However the subset $\Sigma(J^k(M,n))$ of $J(C^k, n) = \mathrm{Gr}(C^k, n)$ may not be a submanifold of $J(C^k, n)$. This situation is quite different from the case of 1 independent variable. One of main results of [S1] and [S2] is to check when the set $\Sigma(J^k(M,n))$ of integral elements of C^k becomes a submanifold of $J(C^k, n)$ or not in the case $n \geq 2$. If $\Sigma(J^k(M,n))$ is a submanifold of $J(C^k, n)$, then we define the canonical differential system D on $\Sigma(J^k(M,n))$. In this case, we regard $\Sigma(J^k(M,n))$ endowed with the canonical differential system as an extension of procedure to construct "Monster Goursat manifolds" or the procedure of "prolongation" of the jet space.

Theorem 3.1 ([S2]) *The set $\Sigma(J^k(M^{m+n}, n))$ of integral elements of the canonical system C^k on the jet space $J^k(M^{m+n}, n)$ over the $m{+}n$-dimensional manifold M with n-independent variables is a submanifold of the Grassmannian $J(C^k, n) = Gr(C^k, n)$ if and only if $(k, n, m) = (2, 2, 1), (k, 1, m), (1, n, 1)$.*

4. Review of Tanaka theory

Next we will consider the local equivalence problem of $(\Sigma(J^2(M^{1+2}, 2)), D)$, where D is a canonical system on $\Sigma(J^2(M^{1+2}, 2))$ (see §5). To this purpose, we first recall Tanaka theory of weakly regular differential systems in this section(see [T],[Y1]).

4.1. **Weak derived system.** Let D be a differential system on a manifold R. We denote by $\mathcal{D}$ the sheaf of sections to D. Then we define k-th weak higher derived system $\partial^{(k)}\mathcal{D}$ by ;

$$\partial^{(1)}\mathcal{D} = \partial\mathcal{D} \ , \ \ \partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}]$$

where $\mathcal{D} = \Gamma(D)$. A differential system D is called *weakly regular*, if $\partial^{(i)}\mathcal{D}$ is a sheaf of sections for a subbundle $\partial^{(i)}D$, for any i. If D is not weakly regular around $x \in R$, then x is called singular point in the sense of Tanaka theory.

We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)}D$ $(k \geq 2)$, for a weakly regular differential system D. Then we have;

$(S1)$ There exists a positive integer μ such that
$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$, for any negative integers p, q,
i.e. $[X, Y] \in \mathcal{D}^{p+q}$, $X \in \mathcal{D}^p, Y \in \mathcal{D}^q$.

4.2. **Symbol algebra.** Let (R, D) be a weakly regular differential system such that

$$T(R) = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} = D.$$

For all $x \in R$, we put $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$, and put

$$\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Then dim $\mathfrak{m}(x)$=dim R. For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, we take extensions $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ of representatives for X, Y ($\tilde{X}_x, \tilde{Y}_x$ give X, Y in $\mathfrak{g}_p(x), \mathfrak{g}_q(x)$) respectively. Then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$ and $[\tilde{X}, \tilde{Y}]_x$ does not depend on the choice of extensions up to D_x^{p+q+1} because of the equation

$$[f\tilde{X}, g\tilde{Y}] = fg[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X} \quad (f, g \in C^\infty(R)).$$

Therefore we define $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{p+q}(x)$, which makes $\mathfrak{m}(x)$ a graded Lie algebra. We call $(\mathfrak{m}(x), [\ ])$ the symbol algebra of (R, D) at x.

Note that the Symbol Algebra $(\mathfrak{m}(x), [\ ])$ satisfies the generating conditions

$$[\mathfrak{g}^p, \mathfrak{g}^{-1}] = \mathfrak{g}^{p-1} \qquad (p < 0).$$

Later, T. Morimoto introduced the notion of a filtered manifold as generalization of the weakly regular differential system in [Mori].

We define a filtered manifold (R, F) by a pair of a manifold R and a tangential filtration F. Here, a tangential filtration F on R is a sequence $\{F^p\}_{p<0}$ of subbundles of the tangent bundle TR such that the following conditions are satisfied;

(i) $TR = F^k = \cdots = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^0 = 0$

(ii) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q} \qquad \forall p, q < 0$

where $\mathcal{F}^p = \Gamma(F^p)$ is the set of sections of F^p.

Let (R, F) be a filtered manifold, for $x \in R$, we put

$$\mathfrak{f}^p(x) := F^p(x)/F^{p+1}(x)$$

and

$$\mathfrak{f}(x) := \bigoplus_{p<0} \mathfrak{f}_p(x).$$

For $X \in \mathfrak{f}_p(x)$, $Y \in \mathfrak{f}_q(x)$, Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined by ;

Let $\tilde{X} \in \mathcal{F}^p$, $\tilde{Y} \in \mathcal{F}^q$ be extensions ($\tilde{X}_x = X$, $\tilde{Y}_x = Y$), then $[\tilde{X}, \tilde{Y}] \in \mathcal{F}^{p+q}$ $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{f}_{p+q}(x)$ does not depend on the extensions.

Then we call $(\mathfrak{f}(x), [\ ])$ the (associated nilpotent) graded Lie algebra of (R, F) at $x \in R$.

In general $(\mathfrak{f}(x), [\ ])$ does not satisfy the generating conditions.

Remark 4.1. Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on a manifold R. We denote by $D^\perp$ the annihilator subbundle of D in T^*R, namely,

$$\begin{aligned} D^\perp(x) &= \{\omega \in T_x^*R \mid \omega(X) = 0 \text{ for any } X \in D(x)\} \\ &= < \varpi_1, \cdots, \varpi_s > . \end{aligned}$$

Then the annihilator $(\partial D)^\perp$ of the first derived system of D is given by

$$(\partial D)^\perp = \{\varpi \in D^\perp \mid d\varpi \equiv 0 \ (\mathrm{mod}\ D^\perp)\}.$$

Moreover the annihilator $(\partial^{(k+1)} D)^\perp$ of the $(k+1)$-th weak derived system of D is given by

$$\begin{aligned} (\partial^{(k+1)} D)^\perp = \{\varpi \in (\partial^{(k)} D)^\perp \mid d\varpi \equiv 0 \ (\mathrm{mod}\ &(\partial^{(k)} D)^\perp, \\ &(\partial^{(p)} D)^\perp \wedge (\partial^{(q)} D)^\perp,\ 2 \le p, q \le k-1)\}. \end{aligned}$$

5. Equivalence problem of $(\Sigma(J^2(M^{1+2}, 2)), D)$

Since $\Sigma(J^2)$ is a manifold from theorem 3.1, we can define the canonical system D on $\Sigma(J^2)$ as follows;

For any $u \in \Sigma(J^2)$ with $p(u) = x \in J^2$, we put

$$D(u) = p_*^{-1}(u) \subset T_u(\Sigma(J^2)) \xrightarrow{p_*} T_x(J^2)$$

where $p : \Sigma(J^2) \to J^2(M^{1+2}, 2)$ is the projection.

In this section, we will consider the equivalence problem of $(\Sigma(J^2), D)$. Namely we will give the orbit decomposition under the action of the $\mathrm{Aut}(\Sigma(J^2), D)$, where

$$\mathrm{Aut}(\Sigma(J^2), D) = \{\varphi : \Sigma(J^2) \to \Sigma(J^2) \mid \varphi\text{: local diffeomorpfhism such that } \varphi_*(D) = D\}.$$

First, the set $\Sigma(J^2)$ has a geometric decomposition;

$$\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \text{ (disjoint union)},$$

where $\Sigma_i = \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = i\}$ $(i = 0, 1, 2)$, and the fiber means that of $T(J^2) \supset C^2 \to T(J^1)$.

5.1. **Classification of Σ_2.**

Proposition 5.1. *The differential system D on $\Sigma(J^2) = \Sigma(J^2(M^{1+2}, 2))$ is regular, but is not weakly regular. Precisely we obtain that*

$$D \subset \partial D \subset \partial^2 D \subset \partial^3 D = T\Sigma(J^2).$$

Moreover $\partial^2 D = \partial^{(2)} D$ and

$$\begin{cases} \partial^{(3)} D & = T\Sigma(J^2) \quad \text{on } \Sigma_0 \cup \Sigma_1 \\ \partial^{(3)} D & = \partial^{(2)} D \quad \text{on } \Sigma_2 \end{cases}$$

From this proposition, $(\Sigma(J^2), D)$ is locally weak regular around $w \in \Sigma_1$. So we can define symbol algebra at $w \in \Sigma_1$ and the following holds;

Proposition 5.2. *For $w \in \Sigma_1$, the symbol algebra $\mathfrak{m}(w)$ is isomorphic to $\mathfrak{m}$, $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ and $[,]$ is given by;*

$$X_y = [X_a, X_x] = [X_B, X_t] \ , \ X_r = [X_c, X_x] \ , \ X_s = [X_e, X_x] = -[X_a, X_t]$$
$$X_p = [X_r, X_x] \ , \ X_q = [X_s, X_x] = -[X_y, X_t]$$
$$X_z = [X_p, X_x] \ , \qquad \text{the other is trivial,}$$

where $\{X_z, X_p, X_q, X_y, X_r, X_s, X_x, X_t, X_a, X_B, X_c, X_e\}$ are basis, and

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle\{X_x, X_t, X_a, X_B, X_c, X_e\}\rangle \\ \mathfrak{g}_{-2} &= \langle\{X_y, X_r, X_s\}\rangle \\ \mathfrak{g}_{-3} &= \langle\{X_p, X_q\}\rangle \\ \mathfrak{g}_{-4} &= \langle\{X_z\}\rangle \end{aligned}$$

Especially, for $w \in \Sigma_1$, the symbol algebra $(\mathfrak{m}(w), [,])$ is not isomorphic to the jet type symbol algebra $\mathfrak{m}_{jet}$.

Here, $\mathfrak{m}_{jet}$ is given as follows; for any $x \in J^3$,
$\mathfrak{g}_{-1}(x) := C^3 = \langle X_{111}, X_{112}, X_{122}, X_{222}, X_{x_1}, X_{x_2}\rangle$, $\mathfrak{g}_{-2}(x) := \langle X_{11}, X_{12}, X_{22}\rangle$,
$\mathfrak{g}_{-3}(x) := \langle X_1, X_2\rangle$, $\mathfrak{g}_{-4}(x) := \langle X_y\rangle$,

$$\mathfrak{m}_{jet}(x) = \bigoplus_{p=-1}^{-4} \mathfrak{g}_p(x) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}.$$

The bracket relations are;
$[X_{jkl}, X_{x_i}] = \delta_{il} X_{jk}$, $[X_{jk}, X_{x_i}] = \delta_{ik} X_j$, $[X_j, X_{x_i}] = \delta_{ij} X_y$, the other relations are given by 0.

Theorem 5.3. (normal form)
For any $w \in \Sigma_1$, the differential system $(\Sigma(J^2(M^{1+2}, 2)), D)$ around w is locally isomorphic to the germ at the origin of $(\mathbb{R}^{12}, \hat{D})$ given as follows;

We define the differential system $\hat{D}$ on $\mathbb{R}^{12}$ with coordinate $(x, y, z, p, q, r, s, t, a, B, c, e)$ by

$$\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$$

where

$$\left\{\begin{array}{ll} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{array}\right.$$

5.2. **Classification of Σ_2.** Finally, we will classify points in Σ_2. From the proposition 5.1, $w \in \Sigma_2$, we can not define the symbol algebra at w. But $\partial^{(1)}D$ and $\partial^{(2)}D$ are subbundle, so we can define graded Lie algebra at w as follows;

For $w \in \Sigma_2$, we put $\mathfrak{g}_{-1}(w) := D^{-1}(w) = D(w), \mathfrak{g}_{-2}(w) := D^{-2}(w)/D^{-1}(w), \mathfrak{g}_{-3}(w) := D^{-3}(w)/D^{-2}(w), \mathfrak{g}_{-4}(w) := T_w(\Sigma(J^2))/D^{-3}(w)$.

$$\mathfrak{m}(w) = \mathfrak{g}_{-1}(w) \oplus \mathfrak{g}_{-2}(w) \oplus \mathfrak{g}_{-3}(w) \oplus \mathfrak{g}_{-4}(w).$$

We define Lie bracket by the same way of the usual symbol algebra except for $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$. For $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$, we define $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}] = 0$.

Note that this graded Lie algebra does not satisfy the generating condition $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}] = \mathfrak{g}_{-4}$.

Remark 5.4. Note that the above graded Lie algebra at $w \in \Sigma_2$ is an example of the associated Lie algebra of filtered manifold $(\Sigma(J^2), F)$ by setting ;

$$F^{-4}(w) = T_w(J^2)\ ,\ F^{-3}(w) = \partial^{(2)}D(w)\ ,\ F^{-2}(w) = \partial^{(1)}D(w)\ ,\ F^{-1}(w) = D(w).$$

Proposition 5.5. *For $w \in \Sigma_2$, graded Lie algebra $\mathfrak{m}(w)$ is isomorphic to $\mathfrak{m}(E, F)$, $\mathfrak{m}(E, F) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ and $[,]$ is given by;*

$$[X_B, X_r] = X_y - FX_x\ ,\ [X_D, X_s] = X_y\ ,\ [X_B, X_s] = X_x\ ,\ [X_D, X_r] = EX_x$$
$$[X_E, X_r] = X_t\ ,\ [X_F, X_s] = X_t$$
$$[X_r, X_x] = X_p\ ,\ [X_s, X_y] = X_p + FX_q\ ,\ [X_s, X_x] = X_q\ ,\ [X_r, X_y] = EX_q$$

the other is trivial,

where $\{X_z, X_p, X_q, X_x, X_y, X_t, X_r, X_s, X_B, X_D, X_E, X_F\}$ are basis which satisfy

$$\begin{array}{rcl} \mathfrak{g}_{-1} &=& \langle\{X_r, X_s, X_B, X_D, X_E, X_F\}\rangle \\ \mathfrak{g}_{-2} &=& \langle\{X_x, X_y, X_t\}\rangle \\ \mathfrak{g}_{-3} &=& \langle\{X_p, X_q\}\rangle \\ \mathfrak{g}_{-4} &=& \langle\{X_z\}\rangle, \end{array}$$

and $E, F \in \mathbb{R}$ are parameters.

For the graded Lie algebra $\mathfrak{m}(E, F)$, the followings are intrinsic;

$$\begin{array}{rcl} \mathfrak{g}^V_{-1} &=& \{X \in \mathfrak{g}_{-1} \mid ad(X)|_{\mathfrak{g}_{-2}} = 0\} = < X_B, X_D, X_E, X_F > \\ \mathfrak{g}^V_{-2} &=& \{X \in \mathfrak{g}_{-2} \mid ad(X)|_{\mathfrak{g}_{-1}} = 0\} = < X_t > \\ \tilde{\mathfrak{g}}_{-1} &=& \{X \in \mathfrak{g}_{-1} \mid \mathrm{Im}\ ad(X)|_{\mathfrak{g}_{-1}} \in \mathfrak{g}^V_{-2}\} = < X_E, X_F >, \end{array}$$

i.e. the above subalgebras are preserved by Lie algebra isomorphisms induced by isomorphisms of differential systems.

Lemma 5.6. *For the graded Lie algebra* $\mathfrak{m}(E,F)$, *let* $Ch(\mathfrak{m}(E,F))$ *be a set of the characteristic directions, that is,*

$$Ch(\mathfrak{m}(E,F)) = \{V \subset \mathfrak{g}_{-1} : 1\text{-}dimensional\ subspace \mid X \in V, X \neq 0, \mathrm{rank}\, ad(X)|_{\mathfrak{g}_{-2}} = 1\}.$$

Then

$$\#Ch(\mathfrak{m}(E,F)) = \begin{cases} 2 & (F^2+4E>0) \\ 1 & (F^2+4E=0) \\ 0 & (F^2+4E<0). \end{cases}$$

Remark 5.7. For $w \in \Sigma_2$, w is said to be *hyperbolic, elliptic* or *parabolic* according to whether F^2+4E is positive, negative or zero, respectively.

From above lemma, Σ_2 has at least 3 components. We put

$$\begin{aligned} \Sigma_h &= \{w \in \Sigma_2 \mid w \textit{ is a hyperbolic point}\} \\ \Sigma_e &= \{w \in \Sigma_2 \mid w \textit{ is a elliptic point}\} \\ \Sigma_p &= \{w \in \Sigma_2 \mid w \textit{ is a parabolic point}\}. \end{aligned}$$

Then this classification is sufficient by the following theorem.

Theorem 5.8. *There exists a decomposition of* Σ_2

$$\Sigma_2 = \Sigma_h \cup \Sigma_e \cup \Sigma_p$$

into disjoint three subsets such that, if $w \in \Sigma_h$, *then* $(\Sigma(J^2(M^{1+2},2),D)$ *around* w *is locally isomorphic to the germ of* $(\mathbb{R}^{12},\bar{D})$ *at* $(0,\cdots,0,1,0)$, *if* $w \in \Sigma_e$, *then,* $(0,\cdots,0,-1,0)$, *and if* $w \in \Sigma_p$, *then,* $(0,\cdots,0,0,0)$. *Here* $(\mathbb{R}^{12},\bar{D})$ *is given as follows;*

We define the differential system $\bar{D}$ *on* $\mathbb{R}^{12}$ *with coordinate* $(x,y,z,p,q,r,s,t,B,D,E,F)$ *by*

$$\bar{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_t = 0\}$$

where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE-BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

We summarize

Corollary 5.9.

$$\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup (\Sigma_h \cup \Sigma_e \cup \Sigma_p)$$

where

$$\begin{aligned}
\Sigma_0 &= \{w \in \Sigma(J^2) \mid \dim(w \cap fiber) = 0\} = J^3 \\
\Sigma_1 &= \{w \in \Sigma(J^2) \mid \dim(w \cap fiber) = 1\} \\
\Sigma_2 &= \{w \in \Sigma(J^2) \mid \dim(w \cap fiber) = 2\} \\
\Sigma_2 &= \Sigma_h \cup \Sigma_p \cup \Sigma_e \\
\Sigma_h &= \Sigma_2 \cap \{w : \textit{hyperbolic point}\} \\
\Sigma_e &= \Sigma_2 \cap \{w : \textit{elliptic point}\} \\
\Sigma_p &= \Sigma_2 \cap \{w : \textit{parabolic point}\}
\end{aligned}$$

Σ_0 is an open set in $\Sigma(J^2)$. Σ_1 is an codimemsion 1 submanifold in $\Sigma(J^2)$ Σ_2 is an codimemsion 2 submanifold in $\Sigma(J^2)$ and P^2-bundle over J^2. Σ_h, Σ_e are also codimemsion 2 submanifolds in $\Sigma(J^2)$. Σ_p is an codimemsion 3 submanifold in $\Sigma(J^2)$.

Moreover, the each component have the following normal forms;
(0) Σ_0 has jet type normal form.

(1) $w \in \Sigma_1$ is locally isomorphic to a germ at the origin in $(\mathbb{R}^{12}, \hat{D})$ where $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, a, B, c, e)$ is coordinate and $\hat{D}$ is expressed by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$, where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

(2)
$w \in \Sigma_h$ is locally isomorphic to a germ at $(0, \cdots, 0, 1, 0)$ in $(\mathbb{R}^{12}, \bar{D})$.
$w \in \Sigma_e$ is locally isomorphic to a germ at $(0, \cdots, 0, -1, 0)$ in $(\mathbb{R}^{12}, \bar{D})$,
$w \in \Sigma_p$ is locally isomorphic to a germ at $(0, \cdots, 0, 0, 0)$ in $(\mathbb{R}^{12}, \bar{D})$.
where $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, B, D, E, F)$ is coordinate and $\bar{D}$ is expressed by $\bar{D} = \{\varpi^0 = \varpi^1 = \varpi^2 = \varpi_x = \varpi_y = \varpi_t = 0\}$ where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

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