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An equivalence problem of homogeneous sub-Riemannian structures

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1 Introduction

A sub-Riemannian manifold $(M, D, g)$ is a differential manifold $M$ equipped with a subbundle $D$ of the tangent bundle $TM$ of $M$ and a Riemannian metric $g$ on $D$. In particular, it is called a sub-Riemannian contact manifold if $D$ is a contact structure, i.e., a subbundle of codimension 1 and non-degenerate.

An infinitesimal automorphism of a sub-Riemannian manifold $(M, D, g)$ is a local vector field $X$ on $M$ such that $L_X D \subset D$ and $L_X g = 0$. Denote by $\mathcal{L}$ the sheaf of the germs of infinitesimal automorphisms of $(M, D, g)$ and by $\mathcal{L}_a$ the stalk of $\mathcal{L}$ at $a \in M$. We say that $\mathcal{L}$ is transitive, or $(M, D, g)$ is homogeneous if the evaluation map $\mathcal{L}_a \ni [X]_a \mapsto X_a \in T_a M$ is surjective for all $a \in M$.

In this paper we study the structure of the Lie algebra $\mathcal{L}_a$ for a point $a$ of a homogeneous sub-Riemannian contact manifold $(M, D, g)$ from the viewpoint of nilpotent geometry. We show that the formal algebra $L$ of $\mathcal{L}_a$ (and therefore $\mathcal{L}_a$) is of finite dimension less than or equal to $(n + 1)^2$ if $\dim M = 2n + 1$. We then completely determine the structures of the Lie algebras $L$ which attain the maximal dimension, which then leads to the determination of the Lie algebras $\mathcal{L}_a$ which attain the maximal dimension. We also describe the standard concrete subriemannian manifolds on which these Lie algebra sheaves are realized.
2 Sub-Riemannian contact transitive filtered Lie algebras

Let \((M, D, g)\) be a homogeneous sub-Riemannian contact manifold of dimension \((2n+1)\) and \(\mathcal{L}\) the sheaf of germs of infinitesimal automorphisms of \((M, D, g)\) as defined in Introduction. First of all let us introduce the contact filtration \(\{\mathcal{L}_a^p\}_{p \in \mathbb{Z}}\) of \(\mathcal{L}_a\) defined inductively as follows:

(i) \(\mathcal{L}_a^p = \mathcal{L}_a \ (p \leq -2)\)

(ii) \(\mathcal{L}_a^{-1} = \{[X]_a \in \mathcal{L}_a; X_a \in D_a\}\)

(iii) \(\mathcal{L}_a^0 = \{[X]_a \in \mathcal{L}_a; X_a = 0\}\)

(iv) \(\mathcal{L}_a^{p+1} = \{\xi \in \mathcal{L}_a^p; \ [[\xi, \eta]] \in \mathcal{L}_a^{p+q+1} \text{ for all } \eta \in \mathcal{L}_a^q, q < 0\} \ (p \geq 0)\).

Then it is easy to see that

\[ [\mathcal{L}_a^p, \mathcal{L}_a^q] \subset \mathcal{L}_a^{p+q} \quad \text{for all } p, q \in \mathbb{Z}, \]

and that

\[ \dim \frac{\mathcal{L}_a^p}{\mathcal{L}_a^{p+1}} < \infty. \]

Passing to the projective limit by setting

\[ L = \lim_{\longrightarrow k} \mathcal{L}_a/\mathcal{L}_a^k, \]

we obtain a Lie algebra \(L\), which also carries a filtration \(\{L^p\}_{p \in \mathbb{Z}}\) given by

\[ L^p = \lim_{\longrightarrow k} \mathcal{L}_a^p/\mathcal{L}_a^k. \]

Then we see that \((L, \{L^p\})\) is a transitive filtered Lie algebra of depth 2 in the sense of Morimoto[6]: A transitive filtered Lie algebra (TFLA) of depth \(\mu\), with \(\mu\) being a positive integer, is a Lie algebra \(L\) endowed with a filtration \(\{L^p\}_{p \in \mathbb{Z}}\) of subspaces of \(L\) satisfying the following conditions:

(F1) \(L = L^{-\mu}\),

(F2) \(L^p \supset L^{p+1}\),
(F3) $[L^p, L^q] \subset L^{p+q}$,

(F4) $\cap_{p \in \mathbb{Z}} L^p = 0$,

(F5) $\dim L^p / L^{p+1} < \infty$,

(F6) $L^{p+1} = \{ X \in L^p ; [X, L^a] \subset L^{p+a+1} \text{ for all } a < 0 \}$, for any $p \geq 0$.

The TFLA $(L, \{L^p\})$ thus obtained is called the formal algebra of $\mathcal{L}$ at $a$.

Let $\mathfrak{l} = \bigoplus \mathfrak{l}_p$ be the graded Lie algebra associated to the TFLA $(L, \{L^p\})$ defined by

$$\mathfrak{l}_p = L^p / L^{p+1}.$$  

Then it is easy to see that $\mathfrak{l} = \bigoplus \mathfrak{l}_p$ satisfies the following properties:

(i) $\mathfrak{l}_- = \bigoplus_{p<0} \mathfrak{l}_p$ is isomorphic to the $(2n+1)$-dimensional Heisenberg Lie algebra $c_-(n) = c_{-2}(n) \oplus c_{-1}(n)$, where $c_{-2}(n) = \mathbb{R}$, $c_{-1}(n) = \mathbb{R}^{2n}$, and the bracket operation is given by $[e_i, e_j] = \delta_{n+j-i} f$ for $i < j$ and trivial for the other pairs with respect to the standard bases $\{f\}$ and $\{e_1, e_2, \ldots, e_{2n}\}$ of $c_{-2}(n)$ and $c_{-1}(n)$ respectively.

(ii) $\bigoplus \mathfrak{l}_p$ is transitive, that is, the condition that $p \geq 0$, $x \in \mathfrak{l}_p \ [x, \mathfrak{l}_-] = 0$ implies $x = 0$.

(iii) There exists a positive definite inner product $g : \mathfrak{l}_- \times \mathfrak{l}_- \to \mathbb{R}$ such that

$$g([A, x], y) + g(x, [A, y]) = 0 \text{ for all } A \in \mathfrak{l}_0 \text{ and } x, y \in \mathfrak{l}_-.$$   

A graded Lie algebra $\bigoplus \mathfrak{l}_p$ satisfying the above conditions will be called a sub-Riemannian contact transitive graded Lie algebra (TGLA) and a filtered Lie algebra $(L, \{L^p\})$ whose associated graded Lie algebra is a sub-Riemannian contact TGLA will be called a sub-Riemannian contact transitive filtered Lie algebra (TFLA).
3 Sub-Riemannian contact graded Lie algebras

We call a pair \((I_-, g)\) a sub-Riemannian Heisenberg Lie algebra if \(I_- = I_-2 \oplus I_-1\) is a graded Lie algebra isomorphic to the Heisenberg Lie algebra \(c_- (n)\) and \(g\) is an inner product on \(I_-1\). Such pairs are classified as follows: For an \(n\)-tuple of positive numbers \(\lambda = (\lambda_1, \ldots, \lambda_n)\) such that \(\lambda_1 \geq \cdots \geq \lambda_n\) and \(\lambda_1 \cdots \lambda_n = 1\), we define an inner product \(g_\lambda\) on \(c_- (n)\) by

\[
\begin{align*}
g_\lambda(e_i, e_j) &= 0 \quad (i \neq j), \\
g_\lambda(e_k, e_k) &= 1, \\
g_\lambda(e_{n+k}, e_{n+k}) &= \lambda_k \quad (1 \leq k \leq n),
\end{align*}
\]

where \(\{e_1, \ldots, e_{2n}\}\) is the basis of \(c_- (n)\). From the normal form of a skew symmetric matrix under the orthogonal group, we see:

**Proposition 1** For an sub-Riemannian Heisenberg Lie algebra \((I_-, g)\), there is a unique \(\lambda = (\lambda_1, \ldots, \lambda_n)\) such that \((I_-, g)\) is isomorphic to \((c_- (n), g_\lambda)\).

Next we define \(c_0 (n, g_\lambda)\) to be the Lie algebra consisting of all \(\alpha \in \text{Hom}(I_-, I_-)\) such that

\[
\begin{align*}
(i) \quad & \alpha(I_p) \subset I_p, \quad p < 0 \\
(ii) \quad & \alpha([x,y]) = [\alpha(x), y] + [x,\alpha(y)], \quad x,y \in I_- \\
(iii) \quad & g(\alpha(x), y) + g(x, \alpha(y)) = 0, \quad x,y \in I_-1.
\end{align*}
\]

From (i) and (ii) the matrix representation of \(X \in c_0 (n, g_\lambda)\) with respect to the basis \(\{f, e_1, \ldots, e_{2n}\}\) has the following form.

\[
X = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} + c \begin{pmatrix} 2 & 0 \\ 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \end{pmatrix},
\]

where

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \text{sp}(n, \mathbb{R}),
\]
that is, $A_{22} = -{}^tA_{11}$, $A_{12}$ and $A_{21}$ are symmetric matrices of degree $n$. Then by (iii) we have

$^t\tilde{A}K + K\tilde{A} = 0$,

where

$\tilde{A} = A + cI_{2n}, \quad K = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda_1 & \\ & & & & \ddots \\ & & & & \lambda_n \end{pmatrix}$.

It follows from this that the trace of $\tilde{A}$ vanishes, but $A \in \text{sp}(n, \mathbb{R})$ is also traceless, therefore we see that the constant $c = 0$. Using these facts, we have the following proposition.

**Proposition 2** If $l = \bigoplus_p \mathfrak{l}_p$ is a subriemannian contact TGLA, then $\mathfrak{l}_p = 0$ for $p \geq 1$, and therefore $l$ is finite dimensional.

The dimension of $c_0(n, g_\lambda)$ will be maximal, when all the eigenvalues coincide, i.e., $\lambda = (1, \ldots, 1)$. Then $X \in c_0(n, g_\lambda)$ can be expressed as:

$$X = \begin{pmatrix} 0 & 0 \\ A_{11} & A_{12} \\ 0 & -A_{12} & A_{11} \end{pmatrix},$$

where $A_{11}$ is skew symmetric and $A_{12}$ is symmetric. It then turns out that $c_0(n, g_{(1,\ldots,1)})$ is isomorphic to $\mathfrak{u}(n)$, the Lie algebra of unitary group. Thus we have shown:

**Proposition 3** If a sub-Rriemannian contact TGLA $l$ has the maximal dimension $(n+1)^2$, it is isomorphic to the TGLA $\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1} \oplus \mathfrak{k}_0$, where $\mathfrak{k}_{-2} = \mathbb{R}$, $\mathfrak{k}_{-1} = \mathbb{C}^n \cong \mathbb{R}^{2n}$, $\mathfrak{k}_0 = \mathfrak{u}(n)$, and the bracket operation is given by

(i) $[\cdot, \cdot] : \mathfrak{k}_{-2} \times \mathfrak{k}_0 \to 0$
\[
\begin{align*}
\text{(ii)} \hspace{0.5cm} [\cdot, \cdot] : \mathfrak{k}_0 \times \mathfrak{k}_{-1} &\rightarrow \mathfrak{k}_{-1}; \quad [A, x] := Ax \quad (A \in \mathfrak{k}_0, x \in \mathfrak{k}_{-1}) \\
\text{(iii)} \hspace{0.5cm} [\cdot, \cdot] : \mathfrak{k}_0 \times \mathfrak{k}_0 &\rightarrow \mathfrak{k}_0; \quad [X, Y] := XY - YX \quad (X, Y \in \mathfrak{k}_0) \\
\text{(iv)} \hspace{0.5cm} [\cdot, \cdot] : \mathfrak{k}_{-1} \times \mathfrak{k}_{-1} &\rightarrow \mathfrak{k}_{-2}; \quad [Z, W] := \text{Im} h(Z, W)
\end{align*}
\]

where \( h(, , ) \) is the canonical Hermitian product on \( \mathbb{C}^n \).

### 4 Cohomology group \( H(\mathfrak{k}_- , \mathfrak{k}) \)

In order to determine the TFLA's whose associated graded Lie algebras are isomorphic to \( \mathfrak{k} \), we need to study the cohomology group \( H(\mathfrak{k}_- , \mathfrak{k}) \). Let us now recall the definition of the cohomology group \( H(\mathfrak{g}_-, \mathfrak{g}) \) for a transitive graded Lie algebra \( \mathfrak{g} \). We set \( \mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p \), which is a nilpotent subalgebra of \( \mathfrak{g} \), and consider the cohomology group associated with the adjoint representation of \( \mathfrak{g}_- \) on \( \mathfrak{g} \), namely the cohomology group \( H(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus H^p(\mathfrak{g}_-, \mathfrak{g}) \) of the cochain complex \( (\text{Hom}(\wedge^p \mathfrak{g}_-, \mathfrak{g}), \partial) \), where the coboundary operator \( \partial : \text{Hom}(\wedge^p \mathfrak{g}_-, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^{p+1} \mathfrak{g}_-, \mathfrak{g}) \) is defined by

\[
(\partial \omega)(X_1, X_2, \ldots, X_{p+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} [X_i, \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})] + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})
\]

for \( \omega \in \text{Hom}(\wedge^p \mathfrak{g}_-, \mathfrak{g}), X_1, X_2, \ldots, X_{p+1} \in \mathfrak{g}_- \). Since both \( \mathfrak{g}_- \) and \( \mathfrak{g} \) are graded, we can define a bigradation \( \bigoplus H^p_r(\mathfrak{g}_-, \mathfrak{g}) \) of \( H(\mathfrak{g}_-, \mathfrak{g}) \) as follows: Denote by \( \text{Hom}(\wedge^p \mathfrak{g}_-, \mathfrak{g})_r \) the set of all homogeneous \( p \)-cochains \( \omega \) of degree \( r \) (i.e., \( \omega(\mathfrak{g}_{a_1} \wedge \cdots \wedge \mathfrak{g}_{a_p}) \subset \mathfrak{g}_{a_1+\cdots+a_p+r} \) for any \( a_1, \ldots, a_p \leq 0 \)), and set

\[
\text{Hom}(\wedge \mathfrak{g}_-, \mathfrak{g})_r = \bigoplus_p \text{Hom}(\wedge^p \mathfrak{g}_-, \mathfrak{g})_r.
\]

Note that \( \partial \) preserves the degree. Hence \( \text{Hom}(\wedge \mathfrak{g}_-, \mathfrak{g})_r \) is a subcomplex and the direct sum decomposition

\[
\text{Hom}(\wedge \mathfrak{g}_-, \mathfrak{g}) = \bigoplus_r \text{Hom}(\wedge \mathfrak{g}_-, \mathfrak{g})_r.
\]
yields that of the cohomology group:

\[ H(\mathfrak{g}, \mathfrak{g}) = \bigoplus H_r(\mathfrak{g}, \mathfrak{g}) = \bigoplus H_r^p(\mathfrak{g}, \mathfrak{g}). \]

On the other hand we note that \( \mathfrak{g}_0 \) naturally acts on \( \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g}) \), and we denote its representation by \( \rho \), which is given by: for \( X_1, \ldots, X_p \in \mathfrak{g} \),

\[(\rho(A)\alpha)(X_1, \ldots, X_p) = [A, \alpha(X_1, \ldots, X_p)] - \sum_{i=1}^{p} \alpha(X_1, \ldots, [A, X_i], \ldots, X_p).\]

Then we have

\[ \partial \rho(A) = \rho(A)\partial \quad \text{for any } A \in \mathfrak{g}_0. \]

Therefore it induces the representation \( \bar{\rho} \) of \( \mathfrak{g}_0 \) on \( H_r^p(\mathfrak{g}, \mathfrak{g}) \). Now we define the set of all \( \mathfrak{g}_0 \)-invariant elements by

\[ IH_r^p(\mathfrak{g}, \mathfrak{g}) = \{ \alpha \in H_r^p(\mathfrak{g}, \mathfrak{g}); \bar{\rho}(A)\alpha = 0 \quad \text{for all } A \in \mathfrak{g}_0 \}. \]

Then we have the following proposition for the subriemannian contact TGLA \( \xi \) of dimension \((n+1)^2\):

**Proposition 4**

(i) \( IH_1^2(\xi, \xi) = 0 \).

(ii) \( IH_2^2(\xi, \xi) \) is 1-dimensional and generated by the equivalence class \([\omega]\) of a cocycle \( \omega \in \text{Hom}(\wedge^2 \xi_{-1}, \xi_0) \) given by:

\[
\begin{cases}
\omega(e_i \wedge e_j) = \omega(e_{n+i} \wedge e_{n+j}) = -E_{ij} + E_{ji} \\
\omega(e_i \wedge e_{n+j}) = \sqrt{-1}(E_{ij} + E_{ji} + 2\delta_{ij}I_n),
\end{cases}
\]

where \( \{e_1, e_2, \ldots, e_{2n}\} \) is the standard basis of \( \xi_{-1} \) and \( E_{ij} \) denotes the \( (i, j) \) matrix unit in \( \text{gl}(n, \mathbb{C}) \). Moreover, \( \omega \) itself is \( \xi_0 \)-invariant, that is, \( \rho(A)\omega = 0 \) for \( A \in \xi_0 \), where \( \rho \) is the representation of \( \xi_0 \) on \( \text{Hom}(\xi, \xi) \).

(iii) \( H_r^2(\xi, \xi) = 0 \) for \( r \geq 3 \).

The proof of the proposition is based on the decomposition of the complex

\[ \text{Hom}(\xi, \xi)_r \longrightarrow \text{Hom}(\wedge^2 \xi, \xi)_r \longrightarrow \text{Hom}(\wedge^3 \xi, k)_r \]
into

\[
\begin{align*}
\text{Hom}(\mathfrak{t}_{-2}, \mathfrak{t}_{r-2}) & \rightarrow \text{Hom}(\mathfrak{t}_{-2} \otimes \mathfrak{t}_{-1}, \mathfrak{t}_{r-3}) \rightarrow \text{Hom}(\mathfrak{t}_{-2} \otimes \wedge^{2} \mathfrak{t}_{-1}, \mathfrak{t}_{r-4}) \\
\downarrow & \downarrow \\
\text{Hom}(\mathfrak{t}_{-1}, \mathfrak{t}_{r-1}) & \rightarrow \text{Hom}(\wedge^{2} \mathfrak{t}_{-1}, \mathfrak{t}_{r-2}) \rightarrow \text{Hom}(\wedge^{3} \mathfrak{t}_{-1}, \mathfrak{t}_{r-3})
\end{align*}
\]

and uses the knowledge on irreducible \(u(n)\)-modules informed from Y. Agaoka. A detailed proof of the proposition will be published elsewhere.

## 5 Maximal sub-Riemannian contact transitive filtered Lie algebras

### 5.1 Main theorem

We define, for each \(\epsilon \in \mathbb{R}\), a TFLA \(K_\epsilon\) as follows: Let the underlying vector space of \(K_\epsilon\) to be the graded vector space \(\mathfrak{k} = \mathfrak{t}_{-2} \oplus \mathfrak{t}_{-1} \oplus \mathfrak{t}_{0}\), and define the filtration \(\{K_\epsilon^p\}_{p \in \mathbb{Z}}\) of \(K_\epsilon\) by \(K_\epsilon^p = \bigoplus_{i \geq p} \mathfrak{t}_i\), and the bracket operation \([,]_\epsilon : K_\epsilon \times K_\epsilon \rightarrow K_\epsilon\) by

\[
[x, y]_\epsilon = [x, y]_\mathfrak{k} + \epsilon \omega(x, y)
\]

for \(x, y \in K_\epsilon\),

where \([x, y]_\mathfrak{k}\) denotes the bracket of the graded Lie algebra \(\mathfrak{k}\) and \(\omega\) is the cocycle in \(\text{Hom}(\wedge^{2} \mathfrak{t}_{-1}, \mathfrak{t}_{0})\) given in Proposition 4 (ii) (regarded as an element of \(\text{Hom}(\wedge^{2} \mathfrak{t}, \mathfrak{k})\) in an obvious manner). Now our main theorem may be stated as follows:

**Theorem 1** If \(K\) is a TFLA and if there is an isomorphism \(\phi : grK \rightarrow \mathfrak{k}\) of graded Lie algebras, then there exists a unique real number \(\epsilon\) and an isomorphism \(\Phi : K \rightarrow K_\epsilon\) of filtered Lie algebras such that the associated map \(gr\Phi\) equals to \(\phi\).

By using proposition 4 it is shown that the theorem holds. A detailed proof of the theorem is given in [3].
5.2 Realizations

Let us see how the filtered Lie algebras $K_\varepsilon$ are realized on sub-Riemannian manifolds.

If $\varepsilon = 0$, then the filtered Lie algebra $K_\varepsilon$ is isomorphic to $\mathfrak{t}_{-2} \oplus \mathfrak{t}_{-1} \oplus \mathfrak{t}_{0}$. It is realized as the Lie algebra of the infinitesimal automorphisms of the space $(\mathbb{R}^{2n+1}, D, g)$, where $D$ is the contact structure on $\mathbb{R}^{2n+1}(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ defined by

$$dz - \frac{1}{2} \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j) = 0,$$

and the metric $g$ on $D$ is given by

$$g = (dx_1|_D)^2 + \cdots + (dx_n|_D)^2 + (dy_1|_D)^2 + \cdots + (dy_n|_D)^2.$$

If $\varepsilon$ is positive, then the filtered Lie algebra $K_\varepsilon$ is isomorphic to $(\mathfrak{u}(n+1), \{F^p\}_{p \in \mathbb{Z}})$, where $\{F^p\}_{p \in \mathbb{Z}}$ is a filtration of $\mathfrak{u}(n+1)$ given by:

\[
F^p = \left\{ \begin{pmatrix} \lambda i & \xi \\ -i \bar{\xi} & A \end{pmatrix} \middle| \lambda \in \mathbb{R}, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, A \in \mathfrak{u}(n) \right\} \quad (p \leq -2),
\]

\[
F^{-1} = \left\{ \begin{pmatrix} 0 & \xi \\ -i \bar{\xi} & A \end{pmatrix} \middle| \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, A \in \mathfrak{u}(n) \right\},
\]

\[
F^0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{u}(n) \right\}, \quad F^q = 0 \quad (q \geq 1).
\]
It is realized as the Lie algebra of the infinitesimal automorphisms of the sphere $(S^{2n+1}, D, g|_D)$, where $S^{2n+1}$ is the set of all $(x_1, y_1, \ldots, x_{n+1}, y_{n+1}) \in \mathbb{R}^{2n+2}$ such that

$$(x_1)^2 + (y_1)^2 + \cdots + (x_{n+1})^2 + (y_{n+1})^2 = 1,$$

and $D$ is defined by

$$\sum_{i}^{n+1} x_i dy_i - y_i dx_i |_{S^{2n+1}} = 0$$

and

$$g = (dx_1)^2 + (dy_1)^2 + \cdots + (dx_{n+1})^2 + (dy_{n+1})^2.$$

If $\varepsilon$ is negative, then the filtered Lie algebra $K_{\varepsilon}$ is isomorphic to $(u(n, 1), \{F^p\}_{p \in \mathbb{Z}})$, where $\{F^p\}_{p \in \mathbb{Z}}$ is a filtration of $u(n, 1)$ given by:

$$F^p = \left\{ \begin{pmatrix} \lambda i & \xi \\ \xi^t & A \end{pmatrix} \middle| \lambda \in \mathbb{R}, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, A \in u(n) \right\} \quad (p \leq -2),$$

$$F^{-1} = \left\{ \begin{pmatrix} 0 & \xi \\ i\xi^t & A \end{pmatrix} \middle| \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, A \in u(n) \right\},$$

$$F^0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \middle| A \in u(n) \right\}, \quad F^q = 0 \quad (q \geq 1).$$

It is realized as the Lie algebra of infinitesimal automorphisms of the hypersurface $(\Sigma^{2n+1}, D, g|_D)$, where $\Sigma^{2n+1}$ is the set of all $(x_1, y_1, \ldots, x_{n+1}, y_{n+1}) \in \mathbb{R}^{2n+2}$ such that

$$(x_1)^2 + (y_1)^2 + \cdots - (x_{n+1})^2 - (y_{n+1})^2 = -1.$$
and $D$ is defined by

$$\sum_{j=1}^{n}(y_{j}dx_{j} - x_{j}dy_{j}) - (y_{n+1}dx_{n+1} - x_{n+1}dy_{n+1}) = 0,$$

and

$$g = (dx_{1})^{2} + (dy_{1})^{2} + \cdots + (dx_{n})^{2} + (dy_{n})^{2} - (dx_{n+1})^{2} - (dy_{n+1})^{2}$$

is a pseudo-Riemannian metric on $\mathbb{R}^{2n+2}(x_{1}, y_{1}, \ldots, x_{n+1}, y_{n+1})$, whose restriction $g|_{D}$ on $D$ is a positive definite inner product.

Summarizing the above discussion, we have, in particular:

**Theorem 2** If $K$ is a maximal sub-Riemannian contact TFLA, then $K$ is isomorphic to $K_{\epsilon}$ for $\epsilon = -1, 0$ or 1.

It should be noted that there exists a Cartan connection associated with a sub-Riemannian structure (satisfying certain regularity conditions)[8]. By using this Cartan connection we can prove that $L_{a}^{p} = 0$ if $p$ is large enough, which implies that $L_{a}$ is in fact isomorphic to $L$. Thus the results above for $L$ hold also for $L_{a}$, and we have:

**Theorem 3** Let $(M, D, g)$ be a homogeneous sub-Riemannian contact manifold of dimension $2n + 1$, and let $L_{a}$ be the stalk at $a \in M$ of the sheaf $L$ the of infinitesimal automorphisms of $(M, D, g)$. If $L_{a}$ attains the maximal dimension $(n + 1)^{2}$, then $L_{a}$ is isomorphic to $K_{\epsilon}$ for $\epsilon = -1, 0$ or 1.

### 6 A remark on transitive filtered Lie algebras

In [6] Morimoto studied transitive filtered Lie algebras (TFLA’s) of depth $\mu \geq 1$ and established the fundamental structure theorems which describe how a TFLA is built on its associated transitive graded Lie algebra (TGLA).

In this paper we have followed his method to study the structure of sub-Riemannian contact TFLA’s. While applying it to our concrete problems we have obtained some improvement of his general theorems. In particular, we can extend Theorem 4.3 ([6], p.69) as follows:
Theorem 4 Let $L_i$ $(i = 1, 2)$ be complete TFLA's, and let $k$ be an integer $\geq 0$ such that

$$H^1_r(gr_-L, grL) = IH^2_r(gr_-L, grL) = 0 \text{ for } i = 1, 2, \ r \geq k + 1.$$ 

Then $L_1$ and $L_2$ are isomorphic if and only if $\text{Trun}_kL_1$ and $\text{Trun}_kL_2$ are isomorphic.

Here we follow the notation of [6]. In particular, we refer to it for the definition of a truncated transitive filtered Lie algebra $\text{Trun}_kL$ of order $k$ ([6], p.57). As defined in section 4, $IH^2_r(gr_-L, grL)$ denotes the space of $gr_0L$-invariant elements in $H^2_r(gr_-L, grL)$

Our theorem asserts that the condition $H^2_r(gr_-L, grL) = 0$ in the original theorem can be replaced by the weaker condition $IH^2_r(gr_-L, grL) = 0$. Roughly speaking, given a TGLA $\mathfrak{g}$, we can take the smaller space $IH^2_r(\mathfrak{g}_-, \mathfrak{g})$ instead of $H^2_r(\mathfrak{g}_-, \mathfrak{g})$ as a parameter space of the moduli of the TFLA's whose associated TGLA's are equal to $\mathfrak{g}$.

The proof of the theorem is similar to that of the original one if we properly interpret that the formula (2.21), ii) ([6], p.67) actually leads to our condition $IH^2_r(gr_-L, grL) = 0$.

The improvement observed here seems useful also in other applications of the theorem. As a corollary of the theorem above, we have also:

Corollary 1 If $L$ is a TFLA satisfying $H^1_r(gr_-L, grL) = IH^2_r(gr_-L, grL) = 0$ for $r \geq 1$, then $L$ is graded, that is, $L$ can be embedded into the completion of the graded Lie algebra $grL$. 
References


