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An equivalence problem of homogeneous sub-Riemannian structures

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1 Introduction

A sub-Riemannian manifold (M, D, g) is a differential manifold M equipped with a subbundle D of the tangent bundle TM of M and a Riemannian metric g on D. In particular, it is called a sub-Riemannian contact manifold if D is a contact structure, i.e., a subbundle of codimension 1 and non-degenerate.

An infinitesimal automorphism of a sub-Riemannian manifold (M, D, g) is a local vector field X on M such that $L_X D \subset D$ and $L_X g = 0$. Denote by $\mathcal{L}$ the sheaf of the germs of infinitesimal automorphisms of (M, D, g) and by $\mathcal{L}_a$ the stalk of $\mathcal{L}$ at $a \in M$. We say that $\mathcal{L}$ is transitive, or (M, D, g) is homogeneous if the evaluation map $\mathcal{L}_a \ni [X]_a \mapsto X_a \in T_a M$ is surjective for all $a \in M$.

In this paper we study the structure of the Lie algebra $\mathcal{L}_a$ for a point a of a homogeneous sub-Riemannian contact manifold (M, D, g) from the viewpoint of nilpotent geometry. We show that the formal algebra L of $\mathcal{L}_a$ (and therefore $\mathcal{L}_a$) is of finite dimension less than or equal to $(n+1)^2$ if $\dim M = 2n+1$. We then completely determine the structures of the Lie algebras L which attain the maximal dimension, which then leads to the determination of the Lie algebras $\mathcal{L}_a$ which attain the maximal dimension. We also describe the standard concrete subriemannian manifolds on which these Lie algebra sheaves are realized.

2 Sub-Riemannian contact transitive filtered Lie algebras

Let (M, D, g) be a homogeneous sub-Riemannian contact manifold of dimension $(2n+1)$ and $\mathcal{L}$ the sheaf of germs of infinitesimal automorphisms of (M, D, g) as defined in Introduction. First of all let us introduce the contact filtration $\{\mathcal{L}_a^p\}_{p\in\mathbf{Z}}$ of $\mathcal{L}_a$ defined inductively as follows:

(i) $\mathcal{L}_a^p = \mathcal{L}_a \ (p \leq -2)$

(ii) $\mathcal{L}_a^{-1} = \{[X]_a \in \mathcal{L}_a; X_a \in D_a\}$

(iii) $\mathcal{L}_a^0 = \{[X]_a \in \mathcal{L}_a; X_a = 0\}$

(iv) $\mathcal{L}_a^{p+1} = \{\xi \in \mathcal{L}_a^p; [\xi, \eta] \in \mathcal{L}_a^{p+q+1} \text{ for all } \eta \in \mathcal{L}_a^q, q < 0\} \ (p \geq 0)$.

Then it is easy to see that

$$[\mathcal{L}_a^p, \mathcal{L}_a^q] \subset \mathcal{L}_a^{p+q} \quad \text{for all } p, q \in \mathbf{Z},$$

and that

$$\dim \mathcal{L}_a^p/\mathcal{L}_a^{p+1} < \infty.$$

Passing to the projective limit by setting

$$L = \varprojlim_k \mathcal{L}_a/\mathcal{L}_a^k,$$

we obtain a Lie algebra L, which also carries a filtration $\{L^p\}_{p\in\mathbf{Z}}$ given by

$$L^p = \varprojlim_k \mathcal{L}_a^p/\mathcal{L}_a^k.$$

Then we see that $(L, \{L^p\})$ is a transitive filtered Lie algebra of depth 2 in the sense of Morimoto[6]: A transitive filtered Lie algebra (TFLA) of depth μ, with μ being a positive integer, is a Lie algebra L endowed with a filtration $\{L^p\}_{p\in\mathbf{Z}}$ of subspaces of L satisfying the following conditions:

(F1) $L = L^{-\mu}$,

(F2) $L^p \supset L^{p+1}$,

(F3) $[L^p, L^q] \subset L^{p+q}$,

(F4) $\bigcap_{p\in\mathbf{Z}} L^p = 0$,

(F5) $\dim L^p/L^{p+1} < \infty$,

(F6) $L^{p+1} = \{X \in L^p; [X, L^a] \subset L^{p+a+1} \text{ for all } a < 0\}$, for any $p \geq 0$.

The TFLA $(L, \{L^p\})$ thus obtained is called the formal algebra of $\mathcal{L}$ at a.

Let $\mathfrak{l} = \bigoplus \mathfrak{l}_p$ be the graded Lie algebra associated to the TFLA $(L, \{L^p\})$ defined by

$$\mathfrak{l}_p = L^p/L^{p+1}.$$

Then it is easy to see that $\mathfrak{l} = \bigoplus \mathfrak{l}_p$ satisfies the following properties:

(i) $\mathfrak{l}_- = \bigoplus_{p<0} \mathfrak{l}_p$ is isomorphic to the (2n+1)-dimensional Heisenberg Lie algebra $\mathfrak{c}_-(n) = \mathfrak{c}_{-2}(n) \oplus \mathfrak{c}_{-1}(n)$, where $\mathfrak{c}_{-2}(n) = \mathbf{R}, \mathfrak{c}_{-1}(n) = \mathbf{R}^{2n}$, and the bracket operation is given by $[e_i, e_j] = \delta_{n,j-i} f$ for $i < j$ and trivial for the other pairs with respect to the standard bases $\{f\}$ and $\{e_1, e_2, \ldots, e_{2n}\}$ of $\mathfrak{c}_{-2}(n)$ and $\mathfrak{c}_{-1}(n)$ respectively.

(ii) $\bigoplus \mathfrak{l}_p$ is transitive, that is, the condition that $p \geq 0$, $x \in \mathfrak{l}_p$ $[x, \mathfrak{l}_-] = 0$ implies $x = 0$.

(iii) There exists a positive definite inner product $g : \mathfrak{l}_{-1} \times \mathfrak{l}_{-1} \to \mathbf{R}$ such that

$$g([A, x], y) + g(x, [A, y]) = 0 \quad \text{for all } A \in \mathfrak{l}_0 \text{ and } x, y \in \mathfrak{l}_{-1}.$$

A graded Lie algebra $\bigoplus \mathfrak{l}_p$ satisfying the above conditions will be called a *sub-Riemannian contact* transitive graded Lie algebra (TGLA) and a filtered Lie algebra $(L, \{L^p\})$ whose associated graded Lie algebra is a sub-Riemannian contact TGLA will be called a *sub-Riemannian contact* transitive filtered Lie algebra (TFLA).

3 Sub-Riemannian contact graded Lie algebras

We call a pair $(\mathfrak{l}_-, g)$ *a sub-Riemannian Heisenberg Lie algebra* if $\mathfrak{l}_- = \mathfrak{l}_{-2} \oplus \mathfrak{l}_{-1}$ is a graded Lie algebra isomorphic to the Heisenberg Lie algebra $\mathfrak{c}_-(n)$ and g is an inner product on $\mathfrak{l}_{-1}$. Such pairs are classified as follows: For an n-tuple of positive numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1 \cdots \lambda_n = 1$, we define an inner product g_λ on $\mathfrak{c}_{-1}(n)$ by

$$g_\lambda(e_i, e_j) = 0 \ (i \neq j), \ g_\lambda(e_k, e_k) = 1 \ , \ g_\lambda(e_{n+k}, e_{n+k}) = \lambda_k \ (1 \leq k \leq n),$$

where $\{e_1, \ldots, e_{2n}\}$ is the basis of $\mathfrak{c}_{-1}(n)$. From the normal form of a skew symmetric matrix under the orthogonal group, we see:

Proposition 1 *For an sub-Riemannian Heisenberg Lie algebra $(\mathfrak{l}_-, g)$, there is a unique $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $(\mathfrak{l}_-, g)$ is isomorphic to $(\mathfrak{c}_-(n), g_\lambda)$.*

Next we define $\mathfrak{c}_0(n, g_\lambda)$ to be the Lie algebra consisting of all $\alpha \in \mathrm{Hom}(\mathfrak{l}_-, \mathfrak{l}_-)$ such that

$$\begin{cases} \text{(i)} \ \alpha(\mathfrak{l}_p) \subset \mathfrak{l}_p, \ p < 0 \\ \text{(ii)} \ \alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)], \ x, y \in \mathfrak{l}_- \\ \text{(iii)} \ g(\alpha(x), y) + g(x, \alpha(y)) = 0, \ x, y \in \mathfrak{l}_{-1}. \end{cases}$$

From (i) and (ii) the matrix representation of $X \in \mathfrak{c}_0(n, g_\lambda)$ with respect to the basis $\{f, e_1, \ldots, e_{2n}\}$ has the following form.

$$X = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array} \right) + c \left(\begin{array}{c|ccc} 2 & & 0 & \\ \hline & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{array} \right),$$

where

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \in sp(n, \mathbf{R}),$$

that is, $A_{22} = -{}^tA_{11}$, A_{12} and A_{21} are symmetric matrices of degree n. Then by (iii) we have

$$ {}^t\tilde{A}K + K\tilde{A} = 0, $$

where

$$ \tilde{A} = A + cI_{2n}, \quad K = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ \hline & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{array}\right). $$

It follows from this that the trace of $\tilde{A}$ vanishes, but $A \in sp(n, \mathbf{R})$ is also traceless, therefore we see that the constant $c = 0$. Using these facts, we have the following proposition.

Proposition 2 *If* $\mathfrak{l} = \bigoplus_p \mathfrak{l}_p$ *is a subriemannian contact TGLA, then* $\mathfrak{l}_p = 0$ *for* $p \geq 1$, *and therefore* $\mathfrak{l}$ *is finite dimensional.*

The dimension of $\mathfrak{c}_0(n, g_\lambda)$ will be maximal, when all the eigenvalues coincide, i.e., $\lambda = (1, \ldots, 1)$. Then $X \in \mathfrak{c}_0(n, g_\lambda)$ can be expressed as:

$$ X = \left(\begin{array}{c|cc} 0 & & 0 \\ \hline & A_{11} & A_{12} \\ 0 & & \\ & -A_{12} & A_{11} \end{array}\right), $$

where A_{11} is skew symmetric and A_{12} is symmetric. It then turns out that $\mathfrak{c}_0(n, g_{(1,\ldots,1)})$ is isomorphic to $\mathfrak{u}(n)$, the Lie algebra of unitary group. Thus we have shown:

Proposition 3 *If a sub-Rriemannian contact TGLA* $\mathfrak{l}$ *has the maximal dimension* $(n+1)^2$, *it is isomorphic to the TGLA* $\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1} \oplus \mathfrak{k}_0$, *where* $\mathfrak{k}_{-2} = \mathbf{R}$, $\mathfrak{k}_{-1} = \mathbf{C}^n \cong \mathbf{R}^{2n}$, $\mathfrak{k}_0 = \mathfrak{u}(n)$, *and the bracket operation is given by*

(i) $[,] : \mathfrak{k}_{-2} \times \mathfrak{k}_0 \longrightarrow 0$

(ii) $[,]:\mathfrak{k}_0\times\mathfrak{k}_{-1}\to\mathfrak{k}_{-1}$; $[A,x]:=Ax\ (A\in\mathfrak{k}_0, x\in\mathfrak{k}_{-1})$

(iii) $[,]:\mathfrak{k}_0\times\mathfrak{k}_0\to\mathfrak{k}_0$; $[X,Y]:=XY-YX\quad(X,Y\in\mathfrak{k}_0)$

(iv) $[,]:\mathfrak{k}_{-1}\times\mathfrak{k}_{-1}\to\mathfrak{k}_{-2}$; $[Z,W]:=\mathrm{Im}h(Z,W)$, *where* $h(,)$ *is the canonical Hermitian product on* $\mathbf{C}^n$.

4 Cohomology group $H(\mathfrak{k}_-,\mathfrak{k})$

In order to determine the TFLA's whose associated graded Lie algebras are isomorphic to $\mathfrak{k}$, we need to study the cohomology group $H(\mathfrak{k}_-,\mathfrak{k})$. Let us now recall the definition of the cohomology group $H(\mathfrak{g}_-,\mathfrak{g})$ for a transitive graded Lie algebra $\mathfrak{g}$. We set $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$, which is a nilpotent subalgebra of $\mathfrak{g}$, and consider the cohomology group associated with the adjoint representation of $\mathfrak{g}_-$ on $\mathfrak{g}$, namely the cohomology group $H(\mathfrak{g}_-,\mathfrak{g})=\bigoplus H^p(\mathfrak{g}_-,\mathfrak{g})$ of the cochain complex $(\mathrm{Hom}(\wedge^p\mathfrak{g}_-,\mathfrak{g}),\partial)$, where the coboundary operator $\partial:\mathrm{Hom}(\wedge^p\mathfrak{g}_-,\mathfrak{g})\to\mathrm{Hom}(\wedge^{p+1}\mathfrak{g}_-,\mathfrak{g})$ is defined by

$$\begin{aligned}&(\partial\omega)(X_1,X_2,\dots,X_{p+1})\\ =\ &\sum_{i=1}^{n+1}(-1)^{i-1}[X_i,\omega(X_1,\dots,\hat{X}_i,\dots,X_{p+1})]\\ +&\sum_{1\le i<j\le p+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1})\end{aligned}$$

for $\omega\in\mathrm{Hom}(\wedge^p\mathfrak{g}_-,\mathfrak{g}), X_1,X_2,\dots,X_{p+1}\in\mathfrak{g}_-$. Since both $\mathfrak{g}_-$ and $\mathfrak{g}$ are graded, we can define a bigradation $\oplus H^p_r(\mathfrak{g}_-,\mathfrak{g})$ of $H(\mathfrak{g}_-,\mathfrak{g})$ as follows: Denote by $\mathrm{Hom}(\wedge^p\mathfrak{g}_-,\mathfrak{g})_r$ the set of all homogeneous p-cochains ω of degree r (i.e., $\omega(\mathfrak{g}_{a_1}\wedge\cdots\wedge\mathfrak{g}_{a_p})\subset\mathfrak{g}_{a_1+\cdots+a_p+r}$ for any $a_1,\dots,a_p\le 0$), and set

$$\mathrm{Hom}(\wedge\mathfrak{g}_-,\mathfrak{g})_r=\bigoplus_p\mathrm{Hom}(\wedge^p\mathfrak{g}_-,\mathfrak{g})_r.$$

Note that ∂ preserves the degree. Hence $\mathrm{Hom}(\wedge\mathfrak{g}_-,\mathfrak{g})_r$ is a subcomplex and the direct sum decomposition

$$\mathrm{Hom}(\wedge\mathfrak{g}_-,\mathfrak{g})=\bigoplus_r\mathrm{Hom}(\wedge\mathfrak{g}_-,\mathfrak{g})_r$$

yields that of the cohomology group:

$$H(\mathfrak{g}_-,\mathfrak{g}) = \bigoplus H_r(\mathfrak{g}_-,\mathfrak{g}) = \bigoplus H_r^p(\mathfrak{g}_-,\mathfrak{g}).$$

On the other hand we note that $\mathfrak{g}_0$ naturally acts on $\mathrm{Hom}(\wedge^p\mathfrak{g}_-,\mathfrak{g})_r$, and we denote its representation by ρ, which is given by: for $X_1,\dots,X_p \in \mathfrak{g}_-$,

$$(\rho(A)\alpha)(X_1,\dots,X_p) = [A,\alpha(X_1,\dots,X_p)] - \sum_{i=1}^{p}\alpha(X_1,\dots,[A,X_i],\dots,X_p).$$

Then we have

$$\partial\rho(A) = \rho(A)\partial \quad \text{for any } A \in \mathfrak{g}_0.$$

Therefore it induces the representation $\bar\rho$ of $\mathfrak{g}_0$ on $H_r^p(\mathfrak{g}_-,\mathfrak{g})$. Now we define the set of all $\mathfrak{g}_0$-invariant elements by

$$IH_r^p(\mathfrak{g}_-,\mathfrak{g}) = \{\alpha \in H_r^p(\mathfrak{g}_-,\mathfrak{g}); \bar\rho(A)\alpha = 0 \text{ for all } A \in \mathfrak{l}_0\}.$$

Then we have the following proposition for the subriemannian contact TGLA $\mathfrak{k}$ of dimension $(n+1)^2$:

Proposition 4 (i) $IH_1^2(\mathfrak{k}_-,\mathfrak{k}) = 0$.

(ii) $IH_2^2(\mathfrak{k}_-,\mathfrak{k})$ *is 1-dimensional and generated by the equivalence class* $[\omega]$ *of a cocycle* $\omega \in \mathrm{Hom}(\wedge^2\mathfrak{k}_{-1},\mathfrak{k}_0)$ *given by:*

$$\begin{cases} \omega(e_i \wedge e_j) = \omega(e_{n+i} \wedge e_{n+j}) = -E_{ij} + E_{ji} \\ \omega(e_i \wedge e_{n+j}) = \sqrt{-1}(E_{ij} + E_{ji} + 2\delta_{ij} I_n), \end{cases}$$

where $\{e_1, e_2, \dots, e_{2n}\}$ *is the standard basis of* $\mathfrak{k}_{-1}$ *and* E_{ij} *denotes the* (i,j) *matrix unit in* $gl(n,\mathbf{C})$. *Moreover,* ω *itself is* $\mathfrak{k}_0$*-invariant, that is,* $\rho(A)\omega = 0$ *for* $A \in \mathfrak{k}_0$, *where* ρ *is the representation of* $\mathfrak{k}_0$ *on* $\mathrm{Hom}(\mathfrak{k}_-,\mathfrak{k})$.

(iii) $H_r^2(\mathfrak{k}_-,\mathfrak{k}) = 0$ *for* $r \geq 3$.

The proof of the proposition is based on the decomposition of the complex

$$\mathrm{Hom}(\mathfrak{k}_-,\mathfrak{k})_r \longrightarrow \mathrm{Hom}(\wedge^2\mathfrak{k}_-,\mathfrak{k})_r \longrightarrow \mathrm{Hom}(\wedge^3\mathfrak{k}_-,k)_r$$

into

$$\begin{array}{ccccc} \mathrm{Hom}(\mathfrak{k}_{-2}, \mathfrak{k}_{r-2}) & \longrightarrow & \mathrm{Hom}(\mathfrak{k}_{-2} \otimes \mathfrak{k}_{-1}, \mathfrak{k}_{r-3}) & \longrightarrow & \mathrm{Hom}(\mathfrak{k}_{-2} \otimes \wedge^2 \mathfrak{k}_{-1}, \mathfrak{k}_{r-4}) \\ & \searrow & & \searrow & \\ \mathrm{Hom}(\mathfrak{k}_{-1}, \mathfrak{k}_{r-1}) & \longrightarrow & \mathrm{Hom}(\wedge^2 \mathfrak{k}_{-1}, \mathfrak{k}_{r-2}) & \longrightarrow & \mathrm{Hom}(\wedge^3 \mathfrak{k}_{-1}, \mathfrak{k}_{r-3}) \end{array}$$

and uses the knowledge on irreducible $\mathfrak{u}(n)$-modules informed from Y. Agaoka. A detailed proof of the proposition will be published elsewhere.

5 Maximal sub-Riemannian contact transitive filtered Lie algebras

5.1 Main theorem

We define, for each $\varepsilon \in \mathbf{R}$, a TFLA K_ε as follows: Let the underlying vector space of K_ε to be the graded vector space $\mathfrak{k} = \mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1} \oplus \mathfrak{k}_0$, and define the filtration $\{K_\varepsilon^p\}_{p \in \mathbf{Z}}$ of K_ε by $K_\varepsilon^p = \bigoplus_{i \geq p} \mathfrak{k}_i$, and the bracket operation $[,]_\varepsilon : K_\varepsilon \times K_\varepsilon \to K_\varepsilon$ by

$$[x, y]_\varepsilon = [x, y]_{\mathfrak{k}} + \varepsilon\omega(x, y) \quad \text{for } x, y \in K_\varepsilon,$$

where $[x, y]_{\mathfrak{k}}$ denotes the bracket of the graded Lie algebra $\mathfrak{k}$ and ω is the cocycle in $\mathrm{Hom}(\wedge^2 \mathfrak{k}_{-1}, \mathfrak{k}_0)$ given in Proposition 4 (ii) (regarded as an element of $\mathrm{Hom}(\wedge^2 \mathfrak{k}, \mathfrak{k})$ in an obvious manner). Now our main theorem may be stated as follows:

Theorem 1 *If K is a TFLA and if there is an isomorphism $\phi : grK \to \mathfrak{k}$ of graded Lie algebras, then there exists a unique real number ε and an isomorphism $\Phi : K \to K_\varepsilon$ of filtered Lie algebras such that the associated map $gr\Phi$ equals to ϕ.*

By using proposition 4 it is shown that the theorem holds. A detailed proof of the theorem is given in [3].

5.2 Realizations

Let us see how the filtered Lie algebras K_ε are realized on sub-Riemannian manifolds.

If $\varepsilon = 0$, then the filtered Lie algebra K_ε is isomorphic to $\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1} \oplus \mathfrak{k}_0$. It is realized as the Lie algebra of the infinitesimal automorphisms of the space $(\mathbf{R}^{2n+1}, D, g)$, where D is the contact structure on $\mathbf{R}^{2n+1}(x_1, \dots, x_n, y_1, \dots, y_n, z)$ defined by

$$dz - \frac{1}{2}\sum_{j=1}^{n+1}(x_j dy_j - y_j dx_j) = 0,$$

and the metric g on D is given by

$$g = (dx_1|_D)^2 + \cdots + (dx_n|_D)^2 + (dy_1|_D)^2 + \cdots + (dy_n|_D)^2.$$

If ε is positive, then the filtered Lie algebra K_ε is isomorphic to $(\mathfrak{u}(n+1), \{F^p\}_{p\in\mathbf{Z}})$, where $\{F^p\}_{p\in\mathbf{Z}}$ is a filtration of $\mathfrak{u}(n+1)$ given by:

$$F^p = \left\{ \left(\begin{array}{c|c} \lambda i & \xi \\ \hline -{}^t\bar{\xi} & A \end{array} \right) \middle| \lambda \in \mathbf{R}, \xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n, A \in \mathfrak{u}(n) \right\} \quad (p \leq -2),$$

$$F^{-1} = \left\{ \left(\begin{array}{c|c} 0 & \xi \\ \hline -{}^t\bar{\xi} & A \end{array} \right) \middle| \xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n, A \in \mathfrak{u}(n) \right\},$$

$$F^0 = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array} \right) \middle| A \in \mathfrak{u}(n) \right\}, \quad F^q = 0 \quad (q \geq 1).$$

It is realized as the Lie algebra of the infinitesimal automorphisms of the sphere $(S^{2n+1}, D, g|_D)$, where S^{2n+1} is the set of all $(x_1, y_1, \ldots, x_{n+1}, y_{n+1}) \in \mathbf{R}^{2n+2}$ such that

$$(x_1)^2 + (y_1)^2 + \cdots + (x_{n+1})^2 + (y_{n+1})^2 = 1,$$

and D is defined by

$$\sum_{i}^{n+1} x_i dy_i - y_i dx_i|_{S^{2n+1}} = 0$$

and

$$g = (dx_1)^2 + (dy_1)^2 + \cdots + (dx_{n+1})^2 + (dy_{n+1})^2.$$

If ε is negative, then the filtered Lie algebra K_ε is isomorphic to $(\mathfrak{u}(n,1), \{F^p\}_{p\in\mathbf{Z}})$, where $\{F^p\}_{p\in\mathbf{Z}}$ is a filtration of $\mathfrak{u}(n,1)$ given by:

$$F^p = \left\{ \left(\begin{array}{c|c} \lambda i & \xi \\ \hline {}^t\bar{\xi} & A \end{array} \right) \middle| \lambda \in \mathbf{R}, \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{C}^n, A \in \mathfrak{u}(n) \right\} \quad (p \leq -2),$$

$$F^{-1} = \left\{ \left(\begin{array}{c|c} 0 & \xi \\ \hline {}^t\bar{\xi} & A \end{array} \right) \middle| \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{C}^n, A \in \mathfrak{u}(n) \right\},$$

$$F^0 = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array} \right) \middle| A \in \mathfrak{u}(n) \right\}, \quad F^q = 0 \quad (q \geq 1).$$

It is realized as the Lie algebra of infinitesimal automorphisms of the hypersurface $(\Sigma^{2n+1}, D, g|_D)$, where Σ^{2n+1} is the set of all $(x_1, y_1, \ldots, x_{n+1}, y_{n+1}) \in \mathbf{R}^{2n+2}$ such that

$$(x_1)^2 + (y_1)^2 + \cdots - (x_{n+1})^2 - (y_{n+1})^2 = -1$$

and D is defined by

$$\sum_{j=1}^{n}(y_j dx_j - x_j dy_j) - (y_{n+1}dx_{n+1} - x_{n+1}dy_{n+1}) = 0,$$

and

$$g = (dx_1)^2 + (dy_1)^2 + \cdots + (dx_n)^2 + (dy_n)^2 - (dx_{n+1})^2 - (dy_{n+1})^2$$

is a pseudo-Riemannian metric on $\mathbf{R}^{2n+2}(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$, whose restriction $g|_D$ on D is a positive definite inner product.

Summarizing the above discussion, we have, in particular:

Theorem 2 *If K is a maximal sub-Riemannian contact TFLA, then K is isomorphic to K_ε for $\varepsilon = -1, 0$ or 1.*

It should be noted that there exists a Cartan connection associated with a sub-Riemannian structure (satisfying certain regularity conditions)[8]. By using this Cartan connection we can prove that $\mathcal{L}_a^p = 0$ if p is large enough, which implies that $\mathcal{L}_a$ is in fact isomorphic to L. Thus the results above for L hold also for $\mathcal{L}_a$, and we have:

Theorem 3 *Let (M, D, g) be a homogeneous sub-Riemannian contact manifold of dimension $2n+1$, and let $\mathcal{L}_a$ be the stalk at $a \in M$ of the sheaf $\mathcal{L}$ the of infinitesimal automorphisms of (M, D, g). If $\mathcal{L}_a$ attains the maximal dimension $(n+1)^2$, then $\mathcal{L}_a$ is isomorphic to K_ε for $\varepsilon = -1, 0$ or 1.*

6 A remark on transitive filtered Lie algebras

In [6] Morimoto studied transitive filtered Lie algebras (TFLA's) of depth $\mu \geq 1$ and established the fundamental structure theorems which describe how a TFLA is built on its associated transitive graded Lie algebra (TGLA).

In this paper we have followed his method to study the structure of sub-Riemannian contact TFLA's. While applying it to our concrete problems we have obtained some improvement of his general theorems. In particular, we can extend Theorem 4.3 ([6], p.69) as follows:

Theorem 4 *Let L_i $(i = 1, 2)$ be complete TFLA's, and let k be an integer ≥ 0 such that*

$$H^1_r(gr_-L, grL) = IH^2_r(gr_-L, grL) = 0 \quad \textit{for } i = 1,\ 2,\ r \geq k+1.$$

Then L_1 and L_2 are isomorphic if and only if $Trun_k L_1$ and $Trun_k L_2$ are isomorphic.

Here we follow the notation of [6]. In particular, we refer to it for the definition of a truncated transitive filtered Lie algebra $\mathrm{Trun}_k L$ of order k ([6], p.57). As defined in section 4, $IH^2_r(gr_-L, grL)$ denotes the space of gr_0L-invariant elements in $H^2_r(gr_-L, grL)$

Our theorem asserts that the condition $H^2_r(gr_-L, grL) = 0$ in the original theorem can be replaced by the weaker condition $IH^2_r(gr_-L, grL) = 0$. Roughly speaking, given a TGLA $\mathfrak{g}$, we can take the smaller space $IH^2_r(\mathfrak{g}_-, \mathfrak{g})$ instead of $H^2_r(\mathfrak{g}_-, \mathfrak{g})$ as a parameter space of the moduli of the TFLA's whose associated TGLA's are equal to $\mathfrak{g}$.

The proof of the theorem is similar to that of the original one if we properly interpret that the formula $(2.21)_k$, ii) ([6], p.67) actually leads to our condition $IH^2_r(gr_-L, grL) = 0$.

The improvement observed here seems useful also in other applications of the theorem. As a corollary of the theorem above, we have also:

Corollary 1 *If L is a TFLA satisfying $H^1_r(gr_-L, grL) = IH^2_r(gr_-L, grL) = 0$ for $r \geq 1$, then L is graded, that is, L can be embedded into the completion of the graded Lie algebra grL.*

References

[1] É. Cartan, Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, Ann. École Norm. Sup. 27 (1910), pp. 109–192.

[2] E. Falbel and C. Gorodski, On contact sub-Riemannian symmetric spaces, Ann.Sc. Ec. Norm. Sup. 28 (4) (1995), pp. 571–589.

[3] Y.Kitagawa, The infinitesimal automorphisms of a subriemannan contact manifolds, Annual reports of Graduate School of Humanities and Sciences, vol.20 (2005), pp.147-163.

[4] R. Montgomery, Abnormal Minimizers, SIAM J. Control and Optimization, vol. 32, no. 6 (1994), pp. 1605–1620.

[5] R. Montgomery, A Survey of Singular Curves in Sub-Riemannian Geometry, Journal of Dynamical and Control Systems, vol.1 (1995), pp. 49–90.

[6] R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications, American Mathematical Society (Mathematical Surveys and Monographs Volume 91), 2000.

[7] T. Morimoto, Transitive Lie algebras admitting differential systems, Hokkaido Math. J. vol.17 (1988), pp. 45–81.

[8] T. Morimoto, Lie algebras, geometric structures and differential equations on filtered manifolds, Advance Studies in Pure Mathematics 37 (2002), pp. 205–252.

[9] T. Morimoto, Cartan connection associated with a subriemannan structure, Differential Geometry and its applications 26 (2008)pp.75–78

[10] R. Stricharz, Sub-Riemannian Geometry, J. Diff. Geom. 24 (1986), pp. 221–263.

[11] R. Stricharz, Corrections to Sub-Riemannian Geometry, J. Diff. Geom. 30 (1989), pp. 595–596.

[12] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Lectures in Mathmatics, Department of Mathmatics, Kyoto Univ., 9, 1975.

[13] S. Tanno, The automorphism groups of almost contact Riemannian manifolds. Tohoku Math. J. 21 (1969), pp. 21–38.