Laplace–Mellin transform of the non-holomorphic Eisenstein series

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1 Introduction and statement of the results


Let \( k \geqq 0 \) be an even integer, \( i \) be the imaginary unit, \( s \) be a complex number whose real part \( \sigma \) and imaginary part \( t \). As usual, \( H \) is the upper half plane. The non-holomorphic Eisenstein series for \( SL_2(Z) \) is defined by

\[
E_k(z,s) = y^s \sum_{\{c,d\}}(cz+d)^{-k}|cz+d|^{-2s}.
\]

Here \( z \) is a point of \( H \), \( s \) is a complex variable and the summation is taken over \( \left( \begin{smallmatrix} c \end{smallmatrix} \begin{smallmatrix} d \end{smallmatrix} \right) \), a complete system of representation of \( \{ \left( \begin{smallmatrix} 0 \end{smallmatrix} \begin{smallmatrix} * \end{smallmatrix} \end{smallmatrix} \right) \in SL_2(Z) \} \setminus SL_2(Z) \). The right-hand side of (1) converges absolutely and locally uniformly on \( \{(z,s)|z\in H, \quad \text{Re}(s)>1-\frac{k}{2}\} \), and \( E_k(z,s) \) has a meromorphic continuation to the whole \( s \)-plane.

Our main result is as follows:

**Theorem 1** ([5] Theorem 1) (I) Assume that \( k \geqq 4 \). Then the Eisenstein series \( E_k(z,s) \) is a \( C^\infty \)-modular form of weight \( k \), and of bounded growth for \( 2-k<\text{Re}(s)<-1 \) except on the poles. Further, for \( 1-\frac{k}{2}<\text{Re}(s)<-1 \) and \( n \in \mathbb{Z}_{>0} \),

\[
\int_0^1 \int_0^\infty E_k(z,s) \exp(-2\pi in\overline{z})y^{k-2}dydx = 0.
\]

(II) Let \( E_k(z,s) = \sum_{n=\infty} a(n;y,s)e^{2\pi inx} \) be the Fourier expansion of the Eisenstein series. Then Fourier coefficients of the projection of \( E_k(z,s) \) to the space of holomorphic cusp forms are zero, namely,

\[
(2\pi n)^{k-1}\Gamma(k-1)^{-1}\int_0^\infty a(n;y,s)e^{-2\pi ny}y^{k-2}dy = 0
\]

for \( 1-k<\text{Re}(s)<0 \) and \( n \in \mathbb{Z}_{>0} \).
2 Projection to the space of cusp forms

The $C^\infty$-automorphic forms of bounded growth are introduced by Sturm in the study of zeta-functions of Rankin type.

The function $F$ is called a $C^\infty$-modular form of weight $k$, if $F$ satisfies the following conditions:

(A.1) $F$ is a $C^\infty$-function from $H$ to $\mathbb{C}$,

(A.2) $F((az+b)(cz+d)^{-1})=(cz+d)^{k}F(z)$ for all $(a b c d) \in SL_2(\mathbb{Z})$.

We denote by $\mathfrak{M}_k$ the set of all $C^\infty$-modular forms of weight $k$. The function $F \in \mathfrak{M}_k$ is called of bounded growth if for every $\varepsilon > 0$

$$\int_0^1 \int_0^{\infty} |F(z)|y^{k-2}e^{-\varepsilon y}dydx < \infty.$$

Let $k$ be a positive even integer and $S_k$ be the space of cusp forms of weight $k$ on $SL_2(\mathbb{Z})$. For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the Petersson inner product as usual

$$(f,F) = \int_{SL_2(\mathbb{Z})\backslash H} f(z)\overline{F(z)}y^{k-2}dxdy.$$

In 1981, Sturm [7] constructed a certain kernel function by using Poincaré series, and showed the following theorem:

**Theorem 2 (Sturm 1981)** Assume that $k > 2$. Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=-\infty}^{\infty} a(n,y)e^{2\pi inz}$. Let

$$c(n) = (2\pi n)^{k-1}\Gamma(k-1)^{-1}\int_0^{\infty} a(n,y)e^{-2\pi ny}y^{k-2}dy.$$

Then $h(z) = \sum_{n=1}^{\infty} c(n)e^{2\piinz} \in S_k$ and

$$(g,F) = (g,h)$$

for all $g \in S_k$.

3 Fourier expansion of the Eisenstein series

Next, we recall the Fourier expansion and the growth condition of $E_k(z,s)$. Let $e(u) := \exp(2\pi iu)$ for $u \in \mathbb{C}$. For $z \in H$ and $\text{Re}(s) > 1 - \frac{k}{2}$, $E_k(z,s)$ has an expansion:

$$E_k(z,s) = y^s + a_0(s)y^{1-k-s} + \frac{y^s}{\zeta(k+2s)}\sum_{m \neq 0} \sigma_1-k-2s(m)a_m(y,s)e(mx),$$

(4)
where
\[ a_0(s) = (-1)^{\frac{s}{2}} 2\pi \cdot 2^{1-k-2s} \zeta(k+2s-1) \frac{\Gamma(k+2s-1)}{\zeta(k+2s) \Gamma(s) \Gamma(k+s)}, \]
\[ \sigma_s(m) = \sum_{d|m, d>0} d^s, \]
and
\[ a_m(y, s) = \int_{-\infty}^{\infty} e(-mu)(u+iy)^{-k}|u+iy|^{-2s} du. \]  

We call the first two terms of (4) are the constant term of \( E(z,s) \). The integral (5) is entire function in \( s \) and of exponential decay in \( y|m| \). This fact gives the meromorphical continuation of \( E_k(z,s) \) to the whole \( s \)-plane, and shows that the constant terms represent the \( y \)-aspect of \( E(z,s) \) when \( y \) tends to \( \infty \). Namely, there exist positive constants \( A_1 \) and \( A_2 \) depending only on \( k \) and \( s \) such that
\[ |E_k(z,s)| \leq A_1 y^\text{Re}(s) + A_2 y^{1-\text{Re}(s)-k} \quad (y \to \infty), \]
except on the poles. Further, the integral (5) is expressed in terms of special functions:
\[ a_m(y, s) = \begin{cases} 
\frac{(-1)^{\frac{s}{2}} (2\pi)^k 2s m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi y m} \Psi(s, k+2s; 4\pi y m) & (m > 0), \\
\frac{(-1)^{\frac{s}{2}} (2\pi)^k 2s |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi y |m|} \Psi(k+s, k+2s; 4\pi y |m|) & (m < 0).
\end{cases} \]

Here \( \Psi(\alpha, \beta; z) \) is the confluent hypergeometric function defined for \( \text{Re}(z) > 0 \) and \( \text{Re}(\alpha) > 0 \) by the following
\[ \Psi(\alpha, \beta; z) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} du, \]
(See for example [4] §7.2.) Then we have

**Proposition 1** Assume \( E_k(z,s) \) is holomorphic at \( s \in \mathbb{C} \). Then, there exist positive constants \( A_1 \) and \( A_2 \) depending only on \( k \) and \( s \) such that
\[ |E_k(x+iy,s)| \leq \begin{cases} 
A_1 (y^{-\text{Re}(s)-k} + y^{\text{Re}(s)}) & (\text{Re}(s) > \frac{1-k}{2}), \\
A_2 (y^{-1+\text{Re}(s)} + y^{1-\text{Re}(s)-k}) & (\text{Re}(s) \leq \frac{1-k}{2}).
\end{cases} \]
for every \( y > 0 \).

**Remark 1.** The integral (5) plays a fundamental role in the study of automorphic forms. The initial work is due to Hecke [2]. His approach is explicated in
Schoenberg's book [6] (pp. 63-68). The representation (7) was originally investigated by Maass [3] (pp. 209-211). He used the Whittaker function to express the integral (7).

**Remark 2.** The estimation (6) is well-known, and Proposition 1 is the consequence of (6) and modularity of $y^{\frac{k}{2}} E_k(z, s)$.

## 4 Proof of Theorem 1

The orthogonality of $E_k(z, s)$ and cusp forms gives the equation (2) under the convergence of the integral. Here Proposition 1 is used, however, we omit the detail of the proof of Theorem 1 (I) (see [5]). In this section, we give a new proof of Theorem 1 (II) by using the **Mellin-Barnes integral.** The following proposition is crucial to evaluate the integral in (3). (See [1], Section 6.5.)

**Proposition 2 (Mellin-Barnes integral)** \( \Psi(\alpha, \beta; z) \) has an integral representation

\[
\Psi(\alpha, \beta; z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+w)\Gamma(-w)\Gamma(1-\beta-w)}{\Gamma(\alpha)\Gamma(\alpha-\beta+1)} z^w dw
\]

where \( |\arg(z)| \leq \frac{3}{2}\pi, \alpha \notin \mathbb{Z}_{\leq 0}, \alpha - \beta + 1 \notin \mathbb{Z}_{\leq 0} \) and the path of integration is indented so as to separate the poles of \( \Gamma(\alpha+w) \) and \( \Gamma(-w)\Gamma(1-\beta-w) \).

**Proof of Theorem 1 (II)** By Proposition 2, we have

\[
\int_{0}^{\infty} \Psi(s, k+2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy = \frac{1}{2\pi i \Gamma(s) \Gamma(1-k-s)} \int_{0}^{\infty} \Gamma(s+w)\Gamma(-w)\Gamma(1-k-2s-w) \times (4\pi n)^w e^{-4\pi ny} y^{w+k+s-2} dwdy
\]

for \( s \notin \mathbb{Z} \). Here the path of integration in \( w \) is denoted by \( L \) taken from \(-i\infty\) to \( i\infty \) so as to separate the poles of \( \Gamma(s+w) \) and \( \Gamma(-w)\Gamma(1-k-2s-w) \). Then the interchange of the order of integration (11) is justified by Fubini's theorem in the region \( 1-k < \text{Re}(s) < 0 \).

We also employ Barnes' lemma:

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+w)\Gamma(\beta+w)\Gamma(\gamma-w)\Gamma(\delta-w) dw}{\Gamma(\alpha+w)\Gamma(\beta+w)\Gamma(\gamma-w)\Gamma(\delta-w) dw} = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}
\]
where the path of integration is taken so as to separate the poles of \( \Gamma(\alpha + w)\Gamma(\beta + w) \) and \( \Gamma(\gamma - w)\Gamma(\delta - w) \). (See [8] Section 14.52.) By the above lemma, we have

\[
\int_{0}^{\infty} \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy = \frac{(4\pi n)^{1-k-s}}{2\pi i} \Gamma(-s) \Gamma(-1 + k + s) \cdot \Gamma(0)^{-1} = 0,
\]

for \( 1 - k < \text{Re}(s) < 0 \) and \( s \notin \mathbb{Z} \).

If \( \alpha \) is zero or a negative integer, \( \Psi(\alpha, \beta; z) \) is a polynomial in \( z \), which is called the (generalized) Laguerre polynomial. The Laguerre polynomials \( L_n^a(x) \) for \( n \in \mathbb{Z}_{\geq 0} \) are defined by

\[
L_n^a(x) = \sum_{m=0}^{n} \binom{n+a}{n-m} \frac{(-x)^m}{m!} = \frac{(-1)^n}{n!} \Psi(-n, a+1; x),
\]

which are known as the orthogonal polynomials associated with the scalar product

\[
(\varphi_1, \varphi_2) := \int_{0}^{\infty} \varphi_1(x) \varphi_2(x) e^{-x} x^a dx
\]

for \( a > -1 \). Especially, for positive integer \( n \),

\[
(L_n^a, L_0^a) = \int_{0}^{\infty} L_n^a(x) e^{-x} x^a dx = 0. \quad (8)
\]

By (11), we have for \( s \in \mathbb{Z}_{\leq 0} \),

\[
\int_{0}^{\infty} \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy = (-s)! (-1)^s \int_{0}^{\infty} L_{k+s-1}^2(4\pi ny) e^{-4\pi ny} y^{k+s-2} dy. \quad (9)
\]

Applying the relation \( L_n^{a-1}(x) = L_n^a(x) - L_{n-1}^a(x) \) for \( (-s-1) \)-times, we resolve (10) into (9) and complete the proof of Theorem 1 (II). \( \square \)

References


