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Laplace–Mellin transform of the non-holomorphic Eisenstein series

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1 Introduction and statement of the results

In this report, we give a Laplace-Mellin transform of the non-holomorphic Eisenstein series and the Fourier coefficients of the Eisenstein series respectively. The main theorem appeared in author's previous paper [5] (2007). We give a new proof by means of Mellin-Barnes type integral formulas in section 4.

Let $k \geqq 0$ be an even integer, Let i be the imaginary unit, s be a complex number whose real part σ and imaginary part t. As usual, H is the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ is defined by

$$E_k(z,s) = y^s \sum_{\{c,d\}} (cz+d)^{-k}|cz+d|^{-2s}. \tag{1}$$

Here z is a point of H, s is a complex variable and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$, a complete system of representation of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\right\} \backslash SL_2(\mathbb{Z})$. The right-hand side of (1) converges absolutely and locally uniformly on $\{(z,s) \mid z \in H,$ $\mathrm{Re}(s) > 1 - \frac{k}{2}\}$, and $E_k(z,s)$ has a meromorphic continuation to the whole s-plane.

Our main result is as follows:

Theorem 1 *([5] Theorem 1) (I) Assume that $k \geqq 4$. Then the Eisenstein series $E_k(z,s)$ is a C^∞-modular form of weight k, and of bounded growth for $2-k <$ $\mathrm{Re}(s) < -1$ except on the poles. Further, for $1-\frac{k}{2} < \mathrm{Re}(s) < -1$ and $n \in \mathbb{Z}_{>0}$,*

$$\int_0^1 \int_0^\infty E_k(z,s) \exp(-2\pi i n\bar{z}) y^{k-2} dy dx = 0. \tag{2}$$

(II) Let $E_k(z,s) = \sum_{n=-\infty}^{\infty} a(n;y,s)e^{2\pi i n x}$ be the Fourier expansion of the Eisenstein series. Then Fourier coefficients of the projection of $E_k(z,s)$ to the space of holomorphic cusp forms are zero, namely,

$$(2\pi n)^{k-1}\Gamma(k-1)^{-1} \int_0^\infty a(n;y,s)e^{-2\pi n y} y^{k-2} dy = 0 \tag{3}$$

for $1-k < \mathrm{Re}(s) < 0$ and $n \in \mathbb{Z}_{>0}$.

2 Projection to the space of cusp forms

The C^∞-automorphic forms of bounded growth are introduced by Sturm in the study of zeta-functions of Rankin type.
The function F is called a C^∞-modular form of weight k, if F satisfies the following conditions:

(A.1) F is a C^∞-function from H to $\mathbb{C}$,

(A.2) $F((az+b)(cz+d)^{-1}) = (cz+d)^k F(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

We denote by $\mathfrak{M}_k$ the set of all C^∞-modular forms of weight k. The function $F \in \mathfrak{M}_k$ is called of bounded growth if for every $\varepsilon > 0$

$$\int_0^1 \int_0^\infty |F(z)| y^{k-2} e^{-\varepsilon y} dy dx < \infty.$$

Let k be a positive even integer and S_k be the space of cusp forms of weight k on $SL_2(\mathbb{Z})$. For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the Petersson inner product as usual

$$(f,F) = \int_{SL_2(\mathbb{Z})\backslash H} f(z)\overline{F(z)} y^{k-2} dx dy.$$

In 1981, Sturm [7] constructed a certain kernel function by using Poincaré series, and showed the following theorem:

Theorem 2 *(Sturm 1981) Assume that $k > 2$. Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=-\infty}^{\infty} a(n,y) e^{2\pi i n x}$. Let*

$$c(n) = (2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n,y) e^{-2\pi n y} y^{k-2} dy.$$

Then $h(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \in S_k$ and

$$(g,F) = (g,h)$$

for all $g \in S_k$.

3 Fourier expansion of the Eisenstein series

Next, we recall the Fourier expansion and the growth condition of $E_k(z,s)$. Let $\mathrm{e}(u) := \exp(2\pi i u)$ for $u \in \mathbb{C}$. For $z \in H$ and $\mathrm{Re}(s) > 1 - \frac{k}{2}$, $E_k(z,s)$ has an expansion:

$$E_k(z,s) = y^s + a_0(s) y^{1-k-s} + \frac{y^s}{\zeta(k+2s)} \sum_{m \neq 0} \sigma_{1-k-2s}(m) a_m(y,s) \mathrm{e}(mx), \quad (4)$$

where

$$\begin{aligned} a_0(s) &= (-1)^{\frac{k}{2}} 2\pi \cdot 2^{1-k-2s} \frac{\zeta(k+2s-1)}{\zeta(k+2s)} \frac{\Gamma(k+2s-1)}{\Gamma(s)\Gamma(k+s)}, \\ \sigma_s(m) &= \sum_{d|m,\, d>0} d^s, \end{aligned}$$

and

$$a_m(y,s) = \int_{-\infty}^{\infty} \mathrm{e}(-mu)(u+iy)^{-k}|u+iy|^{-2s}du. \tag{5}$$

We call the first two terms of (4) are the constant term of $E(z,s)$. The integral (5) is entire function in s and of exponential decay in $y|m|$. This fact gives the meromorphical continuation of $E_k(z,s)$ to the whole s-plane, and shows that the constant terms represent the y-aspect of $E(z,s)$ when y tends to ∞. Namely, there exist positive constants A_1 and A_2 depending only on k and s such that

$$|E_k(z,s)| \leqq A_1 y^{\mathrm{Re}(s)} + A_2 y^{1-\mathrm{Re}(s)-k} \qquad (y \to \infty), \tag{6}$$

except on the poles. Further, the integral (5) is expressed in terms of special functions:

$$a_m(y,s) = \begin{cases} \dfrac{(-1)^{\frac{k}{2}}(2\pi)^{k+2s} m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi y m} \Psi(s, k+2s; 4\pi y m) & (m>0), \\ \dfrac{(-1)^{\frac{k}{2}}(2\pi)^{k+2s} |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi y|m|} \Psi(k+s, k+2s; 4\pi y|m|) & (m<0). \end{cases} \tag{7}$$

Here $\Psi(\alpha,\beta;z)$ is the confluent hypergeometric function defined for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\alpha) > 0$ by the following

$$\Psi(\alpha,\beta;z) := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zu} u^{\alpha-1}(1+u)^{\beta-\alpha-1}du,$$

(See for example [4] §7.2.) Then we have

Proposition 1 *Assume $E_k(z,s)$ is holomorphic at $s \in \mathbb{C}$. Then, there exist positive constants A_1 and A_2 depending only on k and s such that*

$$|E_k(x+iy,s)| \leqq \begin{cases} A_1(y^{-\mathrm{Re}(s)-k} + y^{\mathrm{Re}(s)}) & (\mathrm{Re}(s) > \frac{1-k}{2}) \\ A_2(y^{-1+\mathrm{Re}(s)} + y^{1-\mathrm{Re}(s)-k}) & (\mathrm{Re}(s) \leqq \frac{1-k}{2}) \end{cases}$$

for every $y > 0$.

Remark 1. The integral (5) plays a fundamental role in the study of automorphic forms. The initial work is due to Hecke [2]. His approach is explicated in

Schoenberg's book [6] (pp. 63-68). The representation (7) was originally investigated by Maass [3] (pp. 209-211). He used the Whittaker function to express the integral (7).

Remark 2. The estimation (6) is well-known, and Proposition 1 is the consequence of (6) and modularity of $y^{\frac{k}{2}}E_k(z,s)$.

4 Proof of Theorem 1

The orthogonality of $E_k(z,s)$ and cusp forms gives the equation (2) under the convergence of the integral. Here Proposition 1 is used, however, we omit the detail of the proof of Theorem 1 (I) (see [5]). In this section, we give a new proof of Theorem 1 (II) by using the **Mellin-Barnes integral**. The following proposition is crucial to evaluate the integral in (3). (See [1], Section 6.5.)

Proposition 2 *(Mellin-Barnes integral) $\Psi(\alpha,\beta;z)$ has an integral representation*

$$\Psi(\alpha,\beta;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+w)\Gamma(-w)\Gamma(1-\beta-w)}{\Gamma(\alpha)\Gamma(\alpha-\beta+1)} z^w dw,$$

where $|\arg(z)| \leqq \frac{3}{2}\pi$, $\alpha \notin \mathbb{Z}_{\leqq 0}$, $\alpha-\beta+1 \notin \mathbb{Z}_{\leqq 0}$ and the path of integration is indented so as to separate the poles of $\Gamma(\alpha+w)$ and $\Gamma(-w)\Gamma(1-\beta-w)$.

Proof of Theorem 1 (II) By Proposition 2, we have

$$\int_0^\infty \Psi(s,k+2s;4\pi ny)e^{-4\pi ny}y^{k+s-2}dy$$
$$= \frac{1}{2\pi i\Gamma(s)\Gamma(1-k-s)}\int_0^\infty\int_L \Gamma(s+w)\Gamma(-w)\Gamma(1-k-2s-w)$$
$$\times(4\pi n)^w e^{-4\pi ny}y^{w+k+s-2}dwdy$$

for $s \notin \mathbb{Z}$. Here the path of integration in w is denoted by L taken from $-i\infty$ to $i\infty$ so as to separate the poles of $\Gamma(s+w)$ and $\Gamma(-w)\Gamma(1-k-2s-w)$. Then the interchange of the order of integration (11) is justified by Fubini's theorem in the region $1-k<\mathrm{Re}(s)<0$.

We also employ Barnes' lemma:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(\alpha+w)\Gamma(\beta+w)\Gamma(\gamma-w)\Gamma(\delta-w)dw = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)},$$

where the path of integration is taken so as to separate the poles of $\Gamma(\alpha+w)\Gamma(\beta+w)$ and $\Gamma(\gamma-w)\Gamma(\delta-w)$. (See [8] Section 14.52.) By the above lemma, we have

$$\begin{aligned}
&\int_0^\infty \Psi(s,k+2s;4\pi ny)e^{-4\pi ny}y^{k+s-2}dy \\
&=\frac{(4\pi n)^{1-k-s}}{2\pi i}\Gamma(-s)\Gamma(-1+k+s)\cdot\Gamma(0)^{-1} \\
&=0,
\end{aligned}$$

for $1-k<\mathrm{Re}(s)<0$ and $s\notin\mathbb{Z}$.

If α is zero or a negative integer, $\Psi(\alpha,\beta;z)$ is a polynomial in z, which is called the (generalized) **Laguerre polynomial**. The Laguerre polynomials $L_n^a(x)$ for $n\in\mathbb{Z}_{\geqq 0}$ are defined by

$$L_n^a(x)=\sum_{m=0}^{n}\binom{n+a}{n-m}\frac{(-x)^m}{m!}=\frac{(-1)^n}{n!}\Psi(-n,a+1;x),$$

which are known as the orthogonal polynomials associated with the scalar product

$$(\varphi_1,\varphi_2):=\int_0^\infty \varphi_1(x)\varphi_2(x)e^{-x}x^a\,dx$$

for $a>-1$. Especially, for positive integer n,

$$(L_n^a,L_0^a)=\int_0^\infty L_n^a(x)e^{-x}x^a\,dx=0. \tag{8}$$

By (11), we have for $s\in\mathbb{Z}_{\leqq 0}$,

$$\begin{aligned}
&\int_0^\infty \Psi(s,k+2s;4\pi ny)e^{-4\pi ny}y^{k+s-2}dy \\
&=(-s)!(-1)^s\int_0^\infty L_{-s}^{k+2s-1}(4\pi ny)e^{-4\pi ny}y^{k+s-2}dy.
\end{aligned} \tag{9}$$

Applying the relation $L_n^{a-1}(x)=L_n^a(x)-L_{n-1}^a(x)$ for $(-s-1)$-times, we resolve (10) into (9) and complete the proof of Theorem 1 (II). □

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