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On arithmetic properties of q-series

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1 Introduction

In this paper we shall give a survey on recent results on arithmetic properties of q-series. These are values of solutions of q-difference equations, the analogues of differential equations, where the derivative is replaced by the q-difference operator Δ_q defined by

$$\Delta_q f(z) = \frac{f(qz) - f(z)}{qz - z} \quad (q \neq 1).$$

For example the differential equation $f'(z) = f(z)$ becomes

$$\Delta_q f(z) = f(z) \quad \text{or} \quad f(qz) = (1 + (q-1)z)f(z).$$

If $|q| < 1$, then the solution satisfying $f(0) = 1$ is

$$\prod_{j=0}^{\infty}(1 + q^j(q-1)z)^{-1}.$$

After replacing $(q-1)z$ by z we obtain the function

$$\exp_q(z) = \prod_{j=0}^{\infty}(1 + q^j z)^{-1} \quad (z \neq -q^{-j},\ j = 0, 1, \ldots).$$

A similar consideration of $\Delta_Q f(z) = f(z)$ with $|Q| > 1$ gives the function

$$E_Q(z) = \prod_{j=1}^{\infty}\left(1 + \frac{z}{Q^j}\right),$$

which is connected to $\exp_q(z)$ by the relation

$$E_{1/q}(z)\exp_q(qz) = 1.$$

In the following we shall assume that $0 < |q| < 1$ and denote $Q = q^{-1}$. As a q-analogue of the exponential function we choose the entire function

$$E_Q(z) = \prod_{j=1}^{\infty}\left(1+\frac{z}{Q^j}\right) = \sum_{n=0}^{\infty}\frac{z^n}{\prod_{k=1}^{n}(Q^k-1)}$$

satisfying $E_Q(Qz) = (1+z)E_Q(z)$. By using q we have

$$e_q(z) = \prod_{j=1}^{\infty}(1+q^j z) = \sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{q^k}{1-q^k}\right)z^n = E_Q(z).$$

This function is a special case $M(x) = qx$, $N(x) = 1-qx$ of more general series

$$f_q(z) = \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}R(q^k)\right)z^n, \quad R(x) = \frac{M(x)}{N(x)}, \tag{1}$$

where $M(x)$ and $N(x)$ are polynomials with $N(0) = 1$. This function satisfies a q-difference equation

$$\{N(J/q) - zM(J)\}f(z) = N(q^{-1}), \quad Jf(z) = f(qz),$$

and it has also many other interesting special cases, for example the q-hypergeometric series

$${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n(q)_n}q^{(s+1-r)\binom{n}{2}}z^n,$$

where $(a)_0 = 1$, $(a)_n = (1-a)\cdots(1-aq^{n-1})$, $n \geq 1$.

In this survey we are mainly interested in q-series obtained as values of different $f_q(z)$. Our aim is to concentrate mainly to recent results since there are already two excellent survey papers by Bundschuh [B1], [B2] on earlier results of this topic.

We shall denote by $\mathbb{Q}$ the field of rational numbers, by K an algebraic number field and by $\mathcal{O}_K$ the ring of integers of K. In particular, if K is an imaginary quadratic field it will be denoted by I.

2 Tschakaloff function

Historically the first q-series studied arithmetically are the values of Tschakaloff function

$$T_q(z) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} z^n,$$

which is closely related to Jacobi's theta series

$$\theta(q,z) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n.$$

Clearly $T_q(z)$ is the special case $M(x) = qx,\ N(x) \equiv 1$ of (1). Already in 1921 Tschakaloff [Ts] proved using Hermite's method the linear independence (over I) of the numbers

$$1,\ T_q(\alpha_j), \quad j = 1, \ldots, m,$$

if $Q \in \mathcal{O}_I$ (or more generally is "nearly an integer" meaning that in absolute values the denominator is small in comparision to the nominator) and $\alpha_j \in I^*$ satisfy

$$\frac{\alpha_i}{\alpha_j} \notin q^{\mathbb{Z}}, \quad \text{if} \quad i \neq j. \tag{α}$$

In 1949 Skolem [Sk] did the same, by using more arithmetic Hilbert's method, for the numbers

$$1,\ T_q^{(i)}(\alpha_j), \quad j = 1, \ldots, m;\ i = 0, 1, \ldots,$$

when I is replaced by $\mathbb{Q}$.

By a quantitative form of linear independence we mean a linear independence measure of linearly independent numbers $1, \beta_1, \ldots, \beta_m$, which is a lower bound of the form

$$|h_0 + h_1\beta_1 + \cdots + h_m\beta_m| > H^{-\mu}$$

for all $\underline{h} \in \mathbb{Z}^{m+1}$ with $H = \max|h_j|$ large enough, say $H \geq H_0$. By Dirichlet's box principle we know that $\mu \geq m$ for real numbers β_j. If we consider linear independence over K, then $\underline{h} \in K^{m+1}$ and H is replaced by the absolute height of $\underline{h}$ defined on p. 5.

A quantitative form of Tschakaloff's result was given in [BS] , and of Skolem's result in [Ka1] and [VW1], see also [VW2] for p-adic case and [KSV] for more general K. In these results the measures are rather sharp and μ is about twice the best possible value. We note that in all these results Q is assumed to be nearly an integer, but one naturally conjectures that this assumption should not be needed.

The transcendence of $T_q(\alpha)$ is an interesting open problem. In the special case $\alpha = q^k$, $k \in \mathbb{Z}$, the transcendence follows from the famous results of Nesterenko [Ne2] for all algebraic q, $0 < |q| < 1$, by using the connection of $T_q(z)$ and theta series, for further applications of [Ne2] see [DNNS]. For general α the only result in this direction is by Bézivin [Béz2], who proved that for a nearly integer $Q \in \mathbb{Q}^*$ both of the numbers $\alpha \neq 0$ and $T_q(\alpha)$ cannot belong to a quadratic number field, see also [Cho].

The first approach to the linear independence of $T_{q_i}(\alpha)$ with different integers q_i is given in [AV4]. This is essentially improved recently in [VZ3], where following result is proved by using a new scaling argument.

Theorem 1 ([VZ3]). *Assume that $Q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and let $t_1, \ldots, t_d$ be positive integers such that all the numbers $\sqrt{t_i/t_j}$ for $i \neq j$ are irrational. If $\beta_1, \ldots, \beta_d$ are nonzero rationals, then the numbers*

$$1, T_{q^{t_1}}(\beta_1), \ldots, T_{q^{t_d}}(\beta_d)$$

are linearly independent over $\mathbb{Q}$.

The case $d = 2$ is considered in [VZ2]. If some of the numbers $\sqrt{t_i/t_j}$, $i \neq j$, are rationals, then we need extra assumptions on multiplicative independence of β_i and q. For example the numbers $1, T_q(\beta), T_{q^2}(\beta)$ and $T_{q^4}(\beta)$ are linearly independent if $\beta \in \mathbb{Q}^*$ and q are multiplicatively independent.

These results imply also linear independence of the values of theta series.

Theorem 2 ([VZ3]). *Let $\beta \in \mathbb{Q}^*$ and q be multiplicatively independent, and let $t_1, \ldots, t_d$ be distinct. Then the numbers*

$$1, \theta(q^{t_1}, \beta), \ldots, \theta(q^{t_d}, \beta)$$

are linearly independent over $\mathbb{Q}$.

3 Results by using ideas of Siegel's method

Let us first note a useful connection between some functions $f_q(z)$ in (1) and the function

$$g(z) = g(z, \alpha) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{a(qz) \cdots a(q^n z)} \alpha^n, \tag{2}$$

where $a(z)$ is a polynomial satisfying $a(0) = 1$. This function $g(z)$ is defined for all z satisfying $a(q^k z) \neq 0$, $k = 1, 2, \ldots$, and it satisfies a Poincaré type functional equation

$$\alpha(qz)^s g(qz) = a(qz) f(z) - a(qz).$$

Clearly we have a connection

$$g(1, \alpha) = f(\alpha), \tag{3}$$

where $f(z) = f_q(z)$ with $M(x) = (qx)^s$ and $N(x) = a(qx)$, more precisely

$$f(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}}}{a(q) \cdots a(q^n)} z^n. \tag{4}$$

This suggests to study the solutions of a system of functional equations

$$\bar{y}(qz) = A(z)\bar{y}(z) + \bar{B}(z), \tag{5}$$

where the elements of the matrix $A(z)$ and the components of the vector $\bar{B}(z)$ are rational functions.

The arithmetic properties of E- and G-function solutions of the system of differential equations corresponding to (5) are successfully studied by Siegel's method (see e.g. [Sh]), and therefore it is natural to try to apply the ideas of this method also to the consideration of (5). The key result in the analytic part of this method is Shidlovskii's lemma, and in [AMV] an analogue of this result is given for the system (5) if $A(z)$ is nonsingular, and independently by Bertrand [Be], see also [AV1] for some special cases. To get the arithmetic part to work we use in [AMV] Padé-type approximations of the second kind constructed by Siegel's lemma together with Chudnovsky's [Ch] ideas for G- function considerations, and restrict our studies to the special case of (5), namely to

$$z^s\bar{y}(qz) = a(z)C\bar{y}(z) + \bar{b}(z), \tag{6}$$

where s is a positive integer, C is a nonsingular constant matrix with elements from K, $a(z)$ and the components $b_i(z)$ of $\bar{b}(z)$ belong to $K[z]$, $a(0) = 1$, $t = \deg a(z) \leq s$ and $\deg b_i(z) \leq s$.

To state the main result of [AMV] we introduce some notations. Let $d = [K : \mathbb{Q}]$, $d_w = [K_w : \mathbb{Q}_w]$, and define the (absolute) height of $\alpha \in K^*$ by

$$h(\alpha) = \prod_w \max\{1, |\alpha|_w^{d_w/d}\},$$

where the product is over all places of K and the valuations $|\cdot|_w$ are normalized in the usual way. Furthermore, for nonzero $\bar{l} = (l_1, \ldots, l_m) \in K^m$,

$$h(\bar{l}) = \prod_w \max\{1, |\bar{l}|_w^{d_w/d}\}, \quad |\bar{l}|_w = \max\{|l_i|_w\}.$$

We now fix a place v of K and assume that $q \in K^*$ satisfies $|q|_v < 1$. Then (6) has a unique analytic solution $\bar{f}(z)$ converging in some neighbourhood of the origin and by using (6) this can be continued to any $z \in K_v$ satisfying $a(zq^k) \neq 0$, $k = 0, 1, \ldots$. By defining

$$\lambda = \lambda(v, q) = \frac{d \log h(q)}{d_v \log |q|_v} \quad (\leq -1, \quad = -\lambda(v, Q))$$

we can state the following result for the components $f_i(z)$ of $\bar{f}(z)$.

Theorem 3 ([AMV]). *Assume that the functions $1, f_1(z), \ldots, f_m(z)$ are linearly independent over $K(z)$, and let $\alpha \in K^*$ satisfy $a(\alpha q^k) \neq 0$, $k = 0, 1, \ldots$. There exists an effective constant $\Lambda > 1$ such that the numbers $1, f_1(\alpha), \ldots, f_m(\alpha)$ $(\in K_v)$ are linearly independent over K, if $-\Lambda < \lambda \leq -1$. Moreover, for a given $\varepsilon > 0$, there exists a positive constant $H_0 = H_0((6), \varepsilon, \alpha)$ such that*

$$|l_0 + l_1 f_1(\alpha) + \cdots + l_m f_m(\alpha)|_v > H^{-\mu-\varepsilon}$$

for all $\bar{l} \in K^{m+1}$, $\bar{l} \neq \bar{0}$, where $H = \max\{h(\bar{l}), H_0\}$ and $\mu = \frac{d\Lambda}{d_v(\Lambda+\lambda)}$.

Remarks. 1) The constant H_0 above is not effective in general.

2) In many important cases $\lambda = -1$ (see [AMV]), and in this case

$$\mu \leq \frac{d}{d_v} \frac{8m}{8m-1}(8sm^2 + (s+4)m + \frac{s}{3} + 2) = \mathcal{O}(m^2).$$

3) Generally the condition $-\Lambda < \lambda \leq -1$ corresponds the condition "Q is nearly an integer" of the rational case. It restricts the possible values of q and an improvement removing this would be important.

The irrationality of the values of $g(z)$ satisfying a Poincaré type functional equation $z^s g(qz) = a(z)g(z) + b(z)$ with polynomials a and b was proved in [D1] and a quantitative measure with applications using (3) in [AKV1], [AKV2] and [AKV3]. The special case of (6) with diagonal matrix C was considered in [V1] and a measure $\mu = \mathcal{O}(m^3)$ was obtained, and this was improved in certain cases to $\mu = \mathcal{O}(m^2)$ in [VZ1].

We now give an application of (3) and Theorem 3, for the details see [AMV]. Let $\alpha_1, \ldots, \alpha_m \in K^*$ and define

$$f_{j\mu\nu}(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{a(qz)\cdots a(q^n z)} n^\nu (q^\mu \alpha_j)^n,$$

$j = 1, \ldots, m$; $\mu = 0, 1, \ldots, s-1$; $\nu = 0, 1, \ldots, l$. These functions satisfy a system of functional equations of type (6) and are together with 1 linearly independent over $K(z)$ if the condition (α) is satisfied and in the case $\deg a(z) = s$ also $\alpha_i q^n \neq a_s$ for all i and $n = s, s+1, \ldots$, where a_s is the leading coefficient of $a(z)$. Thus Theorem 3 can be applied and with (3) we get the following result on the values of

$$\phi_{\mu\nu}(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} n^\nu}{a(q)\cdots a(q^n)} (q^\mu z)^n,$$

$\mu = 0, 1, \ldots, s-1$; $\nu = 0, 1, \ldots$.

Theorem 4 ([AMV]). *Let $\alpha_1, \ldots, \alpha_m$ satisfy the above conditions and assume that $a(q^k) \neq 0$, $k = 1, 2, \ldots$. Then there exists a $\Lambda > 1$ such that the numbers $1, \phi_{\mu\nu}(\alpha_j)$ $(\in K_v)$, $j = 1, \ldots, m$; $\mu = 0, 1, \ldots, s-1$; $\nu = 0, 1, \ldots, l$, are linearly independent over K, if $-\Lambda < \lambda \leq -1$. Further, the measure of Theorem 3 holds true for these numbers with m replaced by $M = ms(l+1)$.*

In particular Theorem 4 applies to the Tschakaloff function $T_q(z)$, q-exponential function $e_q(z)$ and a q-analogue of the Bessel function by using the choices $s = 1$, $a(z) \equiv 1$; $s = 1$, $a(z) = 1 - z$; $s = 2$, $a(z) = (1-z)^2$, respectively. For the numbers 1, $e_q^{(\nu)}(\alpha_j)$, $j = 1, \ldots, m$; $\nu = 0, 1, \ldots, l$ it gives a measure $\mathcal{O}(M^2)$ with $M = m(l+1)$. In the case

$l = 0$ (no derivatives) this is very recently improved to $\mathcal{O}(m)$ in [M2] by using explicit Padé approximation construction and (3).

In the special case $t = \deg a(z) < s$, $a(z) = (1 - b_1 z) \cdots (1 - b_t z)$, $b_i \in K^*$, Stihl [St] proved Theorem 4 in the archimedean case without the derivatives, if $K = \mathbb{Q}$ or I, Katsurada [Ka2] did the same with derivatives, and the case of general K and v is studied in [SV]. All these works use explicit Padé approximations of the second kind and in the measure $\mu = \mathcal{O}(M)$ obtained there is better than $\mu = \mathcal{O}(M^2)$ in Theorem 4.

In the general case $t \leq s$, $a(z) \in K[z]$ Theorem 4 with $l = 0$ and without powers q^μ is proved in [V1] with $\mu = \mathcal{O}(m^3)$. The qualitative part without powers q^μ in the case $K = \mathbb{Q}$ or I follows already from the paper of Bézivin [Béz1], where a completely different method is used.

4 Results by Bézivin's method

In 1988 Bézivin [Béz1] studied the function

$$\phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=0}^{n} A(k)},$$

where $\{A(n)\}$ is a linear recurrence sequence of the form

$$A(n) = \lambda_1 \theta_1^n + \cdots + \lambda_h \theta_h^n, \quad n = 0, 1, \ldots,$$

$\theta_1, \ldots, \theta_h$ are nonzero algebraic integers and $\lambda_1, \ldots, \lambda_h$ are nonzero algebraic numbers. Assuming that $A(n) \in I^*$, $|\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h| \geq 1$, and $1 = \theta_h < |\theta_{h-1}|$ if $|\theta_h| = 1$, Bézivin proved the following result, where G denotes the multiplicative group generated by $\theta_1, \ldots, \theta_h$.

Theorem 5 ([Béz1]). *The numbers*

$$1, \phi^{(\nu)}(\alpha_j), \quad j = 1, \ldots, m; \ \nu = 0, 1, \ldots, l,$$

are linearly independent over I, if $\alpha_j \in I^*$ satisfy $\alpha_i/\alpha_j \notin G$ for $i \neq j$, and in the case $\theta_h = 1$ $\lambda_h \alpha_j^{-1} \notin G$.

On noting that

$$\sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}}}{a(q) \cdots a(q^n)} z^n = \sum_{n=0}^{\infty} \frac{z^n}{p(Q) \cdots p(Q^n)},$$

where $p(x) = x^s a(\frac{1}{x})$, we get linear independence of $1, \varphi_{0\nu}(\alpha_j)$ in Theorem 4, if Q is an integer in I such that $|Q| > 1$.

Let us now introduce a generalized linear independence. Assume that $\phi_1(z), \dots, \phi_m(z)$ are functions defined at $\alpha \in K$ with $\phi_j(\alpha) \in K_w$ for all $w \in M$, a finite set of places of K. We say that $\phi_1(\alpha), \dots, \phi_m(\alpha)$ are linearly independent over K with respect to M, if

$$a_1\phi_1(\alpha) + \cdots + a_m\phi_m(\alpha) = 0, \quad \bar{a} = (a_1, \dots, a_m) \in K^m,$$

in K_w for all $w \in M$ implies $\bar{a} = \bar{0}$. Clearly we have usual linear independence over K, if M is a set of one place.

Bézivin [Bézl] proved in fact linear independence of the numbers $1, \phi^{(\nu)}(\alpha_j)$ over K with respect to the set of all infinite places of K. In 2000 André [An] proved this kind of result for $T_q(\alpha_j)$ and $e_q(\alpha_j)$, when $M = \{w \mid |q|_w < 1\}$. The methods used in these proofs are based on rationality criterion of Borel and Dwork type, see [Am]. Recently in a joint work with Amou [AV3] we generalized Theorem 5 for algebraic θ_i (not necessarily integers), again by using Borel-Dwork rationality criterion. As a corollary we obtain the linear independence over K of the numbers in Theorem 4 with respect to $M = \{w \mid |q|_w < 1\}$. In particular, if Q is a Pisot number, then these numbers are linearly independent over K. We also have analogous results for more general series

$$F(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{\prod_{k=1}^{n} p(Q_1^k, \dots, Q_r^k)},$$

where $p \in K[x_1, \dots, x_r]$ and $Q_1, \dots, Q_r$ are multiplicatively independent elements of K^*.

To give a brief sketch of the main idea of these proofs we define for formal power series

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \in K[[z]]$$

the operation

$$g^*(z) = \sum_{n=0}^{\infty} B(n) c_n z^n, \quad B(n) = \prod_{k=0}^{n} A(k).$$

Then for any $\alpha \in K^*$

$$(z\, g(\alpha z))^* = z \sum_{i=1}^{h} \lambda_i \theta_i g^*(\alpha\, \theta_i z),$$

and

$$\phi^*(\alpha z) = \frac{1}{1 - \alpha z}.$$

Assuming now

$$a_0 + \sum_{j=1}^{m} a_j \phi(\alpha_j) = 0, \quad \bar{a} \in K^{m+1}, \ \bar{a} \neq \bar{0},$$

in K_w for all $w \in M = \left\{ w \mid \max_i \{ |\theta_i|_w > 1 \} \right\}$ we define

$$F(z) = a_0 + \sum_{j=1}^{m} a_j \, \phi(\alpha_j z), \quad G(z) = \frac{F(z)}{z-1}.$$

The Borel-Dwork rationality criterion can be applied to show that $G^*(z)$ is a rational function, say $G^*(z) = C(z)/D(z)$. By using the above properties of $*$-operation we now have

$$F^*(z) = z \sum_{i=1}^{h} \lambda_i \theta_i \frac{C(\theta_i z)}{D(\theta_i z)} - \frac{C(z)}{D(z)}.$$

Then a careful consideration of the poles gives a contradiction.

5 Other functions $f_q(z)$

The above results belong essentially to the case $M(x) = x^s$, $\deg N(x) \leq s$ of (1). Some other cases are considered for example in [AM], [BoZ], [CZ], [Chi], [M1] and [MV2]. Often these results say that at least a certain amount of the given numbers are linearly independent. One of the most general results of this type is a recent work of Bundschuh [B3], where he considers linear independence of certain values of the infinite product

$$F_0(z) = \prod_{j=0}^{\infty} M(zq^{mj})$$

and the series

$$F_h(z) = \sum_{j=0}^{\infty} q^{hmj} \prod_{i=0}^{j-1} M(zq^{mi}),$$

where $Q \in \mathcal{O}_I$, $|q| < 1$, $M \in I[x]$ with $M(0) = 1$ is of exact degree l, and $m, h \in \mathbb{N}$. Clearly this is essentially the case $N(x) \equiv 1$ of (1). Very recently the results of [B3] were slightly generalized and improved in [BV4], where we proved the following result.

Theorem 6 ([BV4]). *Assume that q is as above and G is an entire transcendental solution of*

$$G(Q^m z) = R_0(z)G(z) + R_1(z)$$

with $R_0, R_1 \in I[z]$ such that $G(0) = 1$ if $R_1(0) = 0$, and $R_0(0) \in q^{\mathbb{N}}$ if $R_1(0) \neq 0$. Let $\alpha \in I^*$ satisfy the condition $R_0(\alpha q^k) \neq 0$ for every $k \geq m$. Then, for every real number $\delta \in [0, 1]$ satisfying

$$(m+1)^2 > (1-\delta)2l(m+1) + \delta^2 l^2 m, \quad l = \deg R_0,$$

the dimension estimate

$$\dim_I\{I+\sum_{\mu=0}^{m-1} I\,G(\alpha q^{\mu})\} \geq \frac{(m+1)^2+l(m+2)-2(1-\delta)l(m+1)}{l(m+2+\delta^2 lm)}$$

holds.

The proof of Theorem 6 uses explicit approximation constructions and Nesterenko's [Ne1] dimension estimate. Note that the above $F_h(z)$ are special cases, where $R_0(z) = q^{-hm}M(q^m z)$ and $R_1(z) = 1-\delta_{h,0}$ ($\delta_{h,0}$ is the Kronecker symbol). Theorem 6 gives a nontrivial lower bound for the dimension if $m \geq 2l-2$. Asymptotically, for fixed l and $m \to \infty$, the best value m/l is obtained with $\delta = 0$.

In the case $l = 1$ [B3] gives nearly the best possible value m for the dimension. In this case the results of [AMV] can be used to prove that the dimension is maximal $m+1$, see [V2].

One further interesting class of functions under active studies recently are the functions connected to the special case $N(x) = M(qx)$ of $f_q(x)$ in (1). The q-logarithmic function

$$l_q(z) = \log_q(1-z) = \sum_{n=1}^{\infty} \frac{q^n z^n}{1-q^n} = z\sum_{n=1}^{\infty} \frac{q^n}{1-q^n z} \qquad (|z| < |Q|)$$

is the main example of such functions. Since

$$z\frac{e_q'(z)}{e_q(z)} = -l_q(-z),$$

the irrationality of $l_q(\alpha)$ with $Q \in \mathbb{Z}\setminus\{0,\pm1\}$ and $\alpha \in \mathbb{Q}^*$, $|\alpha| < |Q|$, follows already from [Béz1]. After that there are several quantitative refinements of this result, see e.g. [Bo1], [BV1] and [MV1]. The best known bound is given in [MVZ].

Theorem 7 ([MVZ]). *Let $Q \in \mathbb{Z}\setminus\{0,\pm1\}$ and $\alpha \in \mathbb{Q}^*$ be such that $|\alpha| < |Q|$. Then the irrationality exponent of $l_q(\alpha)$ satisfies*

$$\mu(l_q(\alpha)) \leq 3,7633\ldots.$$

As usual the irrationality exponent of a real irrational number γ is defined by

$$\begin{aligned}\mu(\gamma) = \inf\{c \in \mathbb{R} |\ &\text{the inequality } |\gamma - \frac{a}{b}| \leq b^{-c}\\ &\text{has only finitely many solutions } (a,b) \in \mathbb{Z}\times\mathbb{N}\}.\end{aligned}$$

Interesting special values of $l_q(z)$ are

$$l_q(1) = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n} = \zeta_q(1) \quad \text{and} \quad l_q(-1) = \sum_{n=1}^{\infty}\frac{(-q)^n}{1-q^n} = \log_q 2,$$

the q-analogs of the harmonic series $\zeta(1)$ and $\log 2$. In particular, these numbers and the q-analogs of $\zeta(2)$ and π, namely

$$l_q'(1) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \zeta_q(2) \quad \text{and} \quad \pi_q = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n-1}}{1-q^{2n-1}},$$

and some related numbers have been intensively studied recently, see [ATa], [Ass], [Bo1], [BV2], [BV3], [BZ1], [BZ2], [MVZ], [Ta], [Z1], [Z2], [Z3], [Z4]. Best estimates for irrationality exponents are

$$\mu(\log_q 2) \leq 2,9383\ldots, \quad \mu(\zeta_q(1)) \leq 2,4649\ldots,$$
$$\mu(\zeta_q(2)) \leq 4,0786\ldots, \quad \mu(\pi_q) \leq 6,5037\ldots$$

given in [MVZ], [Z3], [Z2] and [BZ2], respectively. Note that also π_q is closely connected to the special values of $l_q(z)$, namely the series in the definition of π_q equals $il_{q^2}(-1) - il_q(i)$. In the proofs of these results there are different ways to construct the needed approximation forms, but a common feature of all these proofs is a very careful and delicate arithmetic consideration of denominators by using the properties of cyclotomic polynomials. It is just at this point where one gets advantage in considering the points $\alpha = \pm 1$ instead of general α.

We note that even the trancendence of $\zeta_q(2)$ and π_q follows from Nesterenko's [Ne2] results for all algebraic q, $0 < |q| < 1$, but the transcendence of $\log_q 2$ and $\zeta_q(1)$ is still an open question. Furthermore, we refer to [KrRZ] and [P] for studies on q-analogs of more general ζ-values.

Concerning the question on linear independence of the values of $l_q(z)$ not much is known. The first result in this direction is given by Tachiya [Ta], who proved for $Q \in \mathbb{Z} \setminus \{0, \pm 1\}$ the linear independence over I of each of the following sets:

$$1,\ l_q(1),\ l_q(-1) \quad (\text{ or } 1, \zeta_q(1), \log_q 2);$$
$$1,\ l_q(1),\ l_{q^2}(1);$$
$$1,\ l_q(1),\ l_q'(1) \quad (\text{ or } 1, \zeta_q(1), \zeta_q(2)).$$

The quantitative refinement of this result is given in [BV2] and [Z4] with linear independence measure $\mu = 14,8369\ldots$, and for general K in [BV3].

One of the most interesting and important problems in this field is certainly the question on linear independence of

$$1,\ l_q^{(\nu)}(\alpha_j), \quad j = 1, \ldots, m;\ \nu = 0, 1, \ldots, l,$$

with $Q \in \mathbb{Z}\setminus\{0, \pm 1\}$ (or even more general Q) and $\alpha_j \in \mathbb{Q}^*$ satisfying (α) and $|\alpha_j| < Q$. Even some partial progress in this question would be of great interest.

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