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Combinatorial games
-A Research Project by High School Students Using Computer Algebra Systems II-

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1 Introduction and Combinatorial game

In this article we are going to present interesting combinatorial games that have been invented and studied by high school students. Combinatorial games are very good topic for high school students to study, and it is fairly easy for high school students to invent new combinatorial games, once they are introduced to some games. After students invent an interesting game, they can study the game by using computer algebra systems such as Mathematica. Mathematica has functions that are very useful to study combinatorial game theory. Therefore combinatorial game theory will give a very good chance for high

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school students to do creative things by discovering new facts, if they can make the best of computer algebra systems.

In Section 2,3 and 4 we are going to study games that have already been studied by many people. In Section 5, 6 and 7 we are going to study games that have been introduced by the authors of this article.

In the followings we use the word option to mean "choice of move". In a combinatorial game there are two players who take turns alternately. They continue playing until one of the players has no legal options available.

Traditionally the two players of a combinatorial game are called Left(or just L) and Right (R).

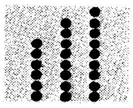
The left options and the right options of a position are always the same in some games, then we call such games impartial. In this article we study impartial games. For the details of the theory of Combinatorial games see [2] and [3].

2 The Traditional Nim

Definition 1

We are going to define a game called Nim. This game is played by the following rules. There are one or more piles, and the players alternate by taking all or some of the counters in a single heap. The player who takes the last counter or stack of counters is the winner.

In Graph 2.1 we have three piles with 6, 8, 9 counters.



Graph 2.1

The theory of Nim is equivalent to the theory of a chocolate problem in Section 3. Therefore we are going to study the theory of Nim in Section 3.

3 A Bitter Chocolate Problem 1

This is a bitter chocolate problem that is a very interesting variant of Nim, and it has been proposed in [4].

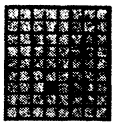
Definition 2

Given the below pieces of chocolate, where the light gray parts are sweet and the dark gray part is very bitter.Two players in turn break the chocolate (in a straight line along the grooves) and eats the piece he breaks off.The player to leave his opponent with the single bitter part is the winner.

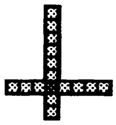
We are going to study this problem by using the chocolate in Graph 3.1 as an example.

Example 1

It is easy to see that the problem in Graph 3.1 is equivalent to the chocolate problem in Graph 3.2.



Graph 3.1



Graph 3.2

If you look at the chocolate in Graph 3.2, then you can easily see that the solution of this game is the same as that of the traditional Nim with piles of {6,4,2,3}.

Now we are going to study the theory of Nim using this chocolate problem as an example. In this game there are two kinds of positions. One kind is a P-position, a previous-player-winning position. The other is an N-position, a Next-player-winning position.

Let me explain about these positions. Our aim is to find all the P-positions.
By the definition of P-position and N-position it is clear that they have the following properties.

P - positions	N - Positions
Every **option** leads to an N - position	There is always at least one **option** leading to a **P -position**

Graph 3.3

This chocolate in Graph 3.1 and Graph 3.2 has 6 rows over the red part, 4 columns on the right side, 2 rows under the red parts and 3 columns on the left side of the red part. We partitioned these numbers 6,4,2,3 into powers of two. Therefore 6 = 2 + 4, 4 = 4, 2 = 2 and 3 = 1 + 2. Thus we can make Graph 3.4.

orientation	over	right	under	left
numbers	6	4	2	3
1	0	0	0	1
2	1	0	1	1
4	1	1	0	0

Graph 3.4

We are going to check the table for each power of 2. For $2^0 = 1$ we have only 1 occurrence. For $2^1 = 2$ we have 3 occurrences, and for $2^2 = 4$ we have 2 occurrences. C.Bouton [1] proved that we have a P-position when each power of 2 occurs evenly often. Therefore we have an N-position in Graph 3.4.

We have to remove one row over the red part if you want to move to a P-position. See Graph 3.5. Here each power of 2 occurs evenly often.

orientation	over	right	under	left
numbers	5	4	2	3
1	1	0	0	1
2	0	0	1	1
4	1	1	0	0

Graph 3.5

Another option is to remove 3 columns from the left side of the red part.

Now we are going to study the bitter chocolate problem in general. This problem is a typical example of traditional Nim. We can represent the game with 4-numbers (x_1, x_2, x_3, x_4). When it is your turn, you choose one of the 4 coordinates and subtract a number that is smaller than the coordinate. These 4 coordinates are independent, i.e., you can take a number from one coordinate without affecting other coordinates.

Theorem 3
In the game of above chocolate problem we have a P-position when each power of 2 occurs evenly often.

We omit the proof, because this is a well known fact about the game of Nim. This theorem was proved in [1] for the first time.

By Theorem 3 the P-positions of this chocolate problem can be obtained by mathematical theory, but in many combinatorial games it is difficult to get a strategy to win mathematically. Usually we can get the list of P-positions only by calculation of computer, and one the most important tool for that is the Grundy Number.

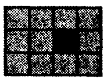
Here we are going to define Grundy Number using the Nim in this section as an example.
First we define a very important function Mex[ ].

Definition 4
The Mex of a set of nonnegative integers is the least nonnegative integer not in the set.

Example 2
$Mex[0,1,4,5,6] = 2$ *and* $Mex[1,4,5,6] = 0$.

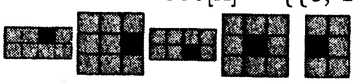
Definition 5
For any position $\mathbf{x}$ *we denote by* $Move[\mathbf{x}]$ *the set of all the positions that players can reach directly from the position* $\mathbf{x}$.

Example 3
Let $\mathbf{x} = \{1, 1, 1, 2\}$. *Then this position is the following chocolate. See Graph 3.6.*



Graph 3.6

From this position the player can reach {0, 1, 1, 2}, {1, 0, 1, 2}, {1, 1, 0, 2}, {1, 1, 1, 1}, {1, 1, 1, 0}. These positions are the following chocolates. See Graph 3.7.
Therefore $Move[\mathbf{x}] = \{\{0, 1, 1, 2\}, \{1, 0, 1, 2\}, \{1, 1, 0, 2\}, \{1, 1, 1, 1\}, \{1, 1, 1, 0\}\}$.



Graph 3.7

We are going to define the Grundy Number G(x) for any position $\mathbf{x}$.

Definition 6

Let P_0 be the set of positions from which the players can have no legal option.
For any position $\mathbf{x} \in P_0$ we define $G(\mathbf{x}) = 0$.
Let N_1 be the set of positions from which the players can choose a proper option that leads to P_0.
For any position $\mathbf{x} \in N_1$ we define $G(\mathbf{x}) = 1$.
For any position $\mathbf{x}$ we define $G(\mathbf{x})$ recursively.
$G(\mathbf{x}) = Mex[G[\mathbf{y}], \mathbf{y} \in Move[\mathbf{x}]]$.
For the details of the Grundy Number see [2].

By the theory of Grundy Number we know that $\mathbf{x}$ is a P-position if and only if $G(\mathbf{x}) = 0$. Therefore we can find P-positions by calculating Grundy Number $G(\mathbf{x})$
We are going to use Grundy number in Section 5.

4 A Strip with Coins.

This is a variant of Nim, and you can play the game with coins and a strip.

Definition 7

There are k coins and there are n squares in the strip, and the strips are numbered with 1,2,3,...,n.
We are going to move the coins by the following rules (1), (2) and (3).
(1)If there is a coin in the m th place and no coin in the (m+1)th place, then a player can move the coin to the (m+1)th place.
(2) If there are coins in the m th place and in the (m+1)th place and no coin in the (m+2)th place, then a player can move the coin in the m th place to the (m+2)th place.
(3) The game with k coins ends when these coins are one each on the right end of the strip.

Example 4

In the following example we let $n = 15$ and $k = 3$. See Graph 4.1.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Graph 4.1

By the rule (1) and (2) of Definition 7 players can move from the position of Graph 4.1 to the positions of Graph 4.2.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Graph 4.2

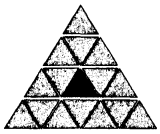
We are going to omit the theory of this game, since it has been studied by many people.

After the authors studied the theory of Section 1,2,3 and 4, they began to make new combinatorial games. In Sections 5,6 and 7 you are going to see the results of the research.

5 A Bitter Chocolate Problem 2

This variant of the chocolate problem has been introduced by the authors.
Since the game itself is very difficult to treat generally, we are going to study it by an example.

Example 5
Suppose that you have the following chocolate. See Graph 5.1. The light gray parts are sweet, but the dark gray part is very bitter. Two players in turn breaks the chocolate (in a straight line along the grooves) and eats the piece he breaks off. The player to leave his opponent with the single bitter part is the winner.



Graph 5.1

The problem in Graph 3.1 is different from the problem in Graph 5.1. In Graph 5.1 you can cut the chocolate in 6 ways, so it is appropriate to represent it with 6 numbers $(x_1, x_2, x_3, x_4, x_5, x_6)$. We represent the position in Graph 5.1 with {2,1,2,1,2,1} and the 6th position in Graph 5.2 with {0,1,0,1,1,1}.

Note that these 6 coordinates are not independent, i.e., in some cases you cannot take a number from one coordinate without affecting other coordinates.

In fact we have 6 inequalities between these 6 coordinates.

$x_1 \le x_2 + x_6, x_2 \le x_1 + x_3 + 1, x_3 \le x_2 + x_4, x_4 \le x_3 + x_5 + 1, x_5 \le x_4 + x_6, x_6 \le x_5 + x_1$

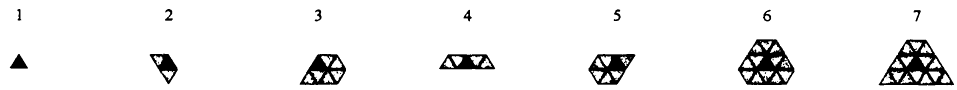
As far as we know there have been no research on the Nim conditioned by inequalities. Therefore we are studying a new kind of Nim.

Example 6
Here we are going to calculate all the P-positions of this game. Since there is no method to find all the P-position theoretically, we are going to find all of them by calculation of Mathematica. Because this game has a complicated structure, the Mathematica program for this game is a little bit complicated.

```
Clear[ss, al, allcases];
ss = 1; al =
Flatten[Table[{a,b,c,d,e,f},{a,0,ss + 2},{b,0,ss},{c,0,
ss + 2},{d,0,ss},{e,0,ss + 2},{f,0,ss}],5];
allcases =
Select[al,#[[1]]+#[[3]]+1>= #[[2]]&&#[[2]]+#[[4]]>=#[[
3]] && #[[3]]+#[[5]]+1>=#[[4]]&&#[[4]]+#[[6]]>=#[[
5]]&&#[[5]]+#[[1]]+1>=#[[6]]&&#[[6]]+#[[2]]>=#[[
1]]&];(*allcases are the set of all possible shapes of the
chocolate.  Note that the above inequality are the necessary
and sufficient conditions for {a,b,c,d,e,f} to be a possible
shape of the chocolate.*)
num=Length[allcases];
(*num is the number of all the cases*)
x1=allcases[[num]];
(*x1 is the case with which we start the fame*)
pos[x_List,y_List]:=
Block[{s,t,u,v},u=x;v=y;t=Apply[Plus,v];
s=Position[v,t][[1,1]];
u[[s]]=u[[s]]-t;{Min[u[[6]]+u[[2]],u[[1]]],
Min[u[[1]]+u[[3]]+1,u[[2]]],
Min[u[[2]]+u[[4]],u[[3]]],
Min[u[[3]]+u[[5]]+1,u[[4]]],
```

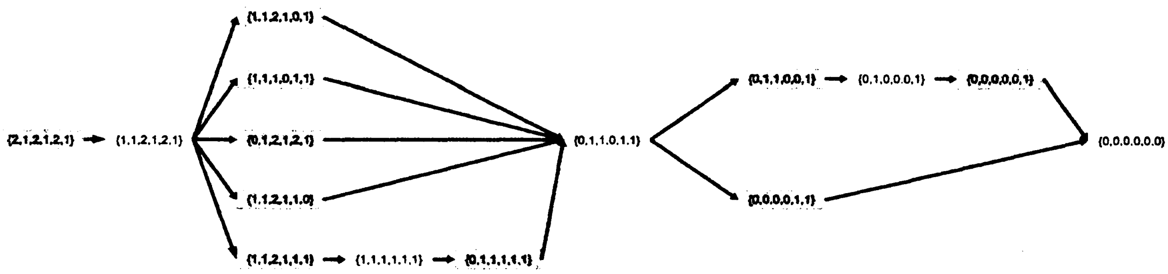
```
Min[u[[4]]+u[[6]],u[[5]]],
Min[u[[5]]+u[[1]]+1,u[[6]]]}];
(*pos[x,y]returns the case that you will get after removing
 y from x.
For example let x={1,2,1,2,1,2} and y={1,0,0,0,0,0},
then pos[x,y] denotes the case you get after removing one
from the 1st coordinate of x; that is {1,1,2,1,2,1}*)
Clear[move,z,p,t1,t2,t3,t4,t5,t6];
move[z_]:=Block[{p},p=z;
Union[Table[pos[p,{t1,0,0,0,0,0}],{t1,1,p[[1]]}],
Table[pos[p,{0,t2,0,0,0,0}],{t2,1,p[[2]]}],
Table[pos[p,{0,0,t3,0,0,0}],{t3,1,p[[3]]}],
Table[pos[p,{0,0,0,t4,0,0}],{t4,1,p[[4]]}],
Table[pos[p,{0,0,0,0,t5,0}],{t5,1,p[[5]]}],
Table[pos[p,{0,0,0,0,0,t6}],{t6,1,p[[6]]}]
]
]
(*move[x] returns all the cases you can get from x when
you remove a part of x*)
Mex[L_]:=Min[Complement[Range[0,Length[L]],L]];
Gr[z_]:=Gr[z]=Mex[Map[Gr,move[z]]];
```

By the Mathematica program in Example 6 we can find all the P-positions. (For this scale of problem you can find all the P-position only with pen and paper.) This Graph 5.2 contains all the P-positions. All the other positions are N-position. As you can see easily the initial position in Graph 5.1 is an N-position, because you cannot find the initial position in Graph 5.2. Therefore you are sure to win if you start the game as the first player.



Graph 5.2

By the P-positions in Graph 5.2 we can make Graph 5.3. In this graph N-positions are colored in dark gray and P-positions are colored in light gray. If you start with the original position in Graph 5.1, then by using Graph 5.3 you can win easily.



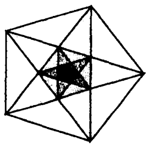
Graph 5.3

6 A Chocolate Problem 3

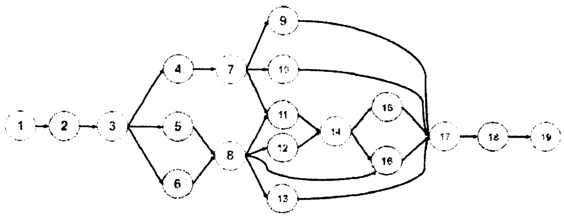
Definition 8

Suppose that we have the chocolate in Graph 6.1, where the dark gray part is very bitter. Two players in turn break the chocolate (in a straight line along the grooves) and eats the piece he breaks off.The player to leave his opponent with the single bitter part is the winner.

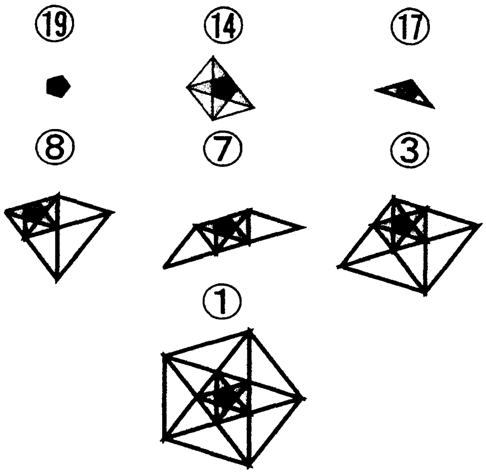
This variant of the Bitter Chocolate Problem is very different from the original one. This problem is interesting for people who use only their brains to solve, but this can be a challenging problem for those who are going to solve this with computers.
By using computer we can find all the P - positions and N-positions. For this scale of problem you can find all the P - position only with pen and paper.
We can make a chart of positions by which you can win the game. See Graph 6.2.
All positions are numbered. For P-positions see Graph 6.3, and for For N-positions see Graph 6.4.



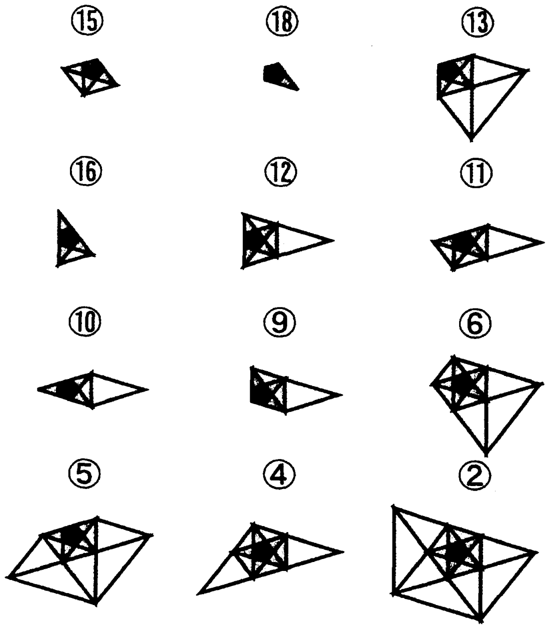
Graph 6.1



Graph 6.2



Graph 6.3



Graph 6.4

7 A Strip with Coins 2.

This is a new game you can play with coins and a strip, and it has been introduced by the authors.

Definition 9
We have k coins and a strip with n squares, and squares are numbered with 1,2,3,...,n. The shape of the strip is not a straight line here.
The strip consists of sub-strips {1,2,3,4,5} , {6,7,8,9,10}, {11,12,13,14,15},....
The rule of the game is very similar to that of the game in Section 4.
(1) The difference is that you cannot make a coin jump over another coin when they are at the corner.

We are going to study this game by Example 7.

Example 7
We let n = 15 and k = 3. See Graph 7.1.

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.1

Here we are going to study the rule (1) of Definition 9. If you have the position of Graph 7.2, then you can move to the position of Graph 7.3. You cannot move to the position of Graph 7.4.
Another option is move the coin in the 9th square to the 10th square.

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.2

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.3

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.4

If you have the position of Graph 7.5, then you can move to the position of Graph 7.6. You cannot move to the position of Graph 7.7.
Another option is move the coin in the 9th square to the 10th square.

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.5

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.6

1 2 3 4 5
6 7 8 9 10
11 12 13 14 15

Graph 7.7

Example 8
Here we are going to calculate all the P-positions of this game. Since the authors have not discovered a method to find all the P-position theoretically, we are going to find all of them by calculation of Mathematica. Mathematica has many functions that are very useful to calculate the Grundy Number, and it is fairly easy to find all the P-positions of this game.

```
Po[pos_] :=
  Union[ReplaceList[pos, {X___, 1, 0, Y___} -> {X, 0, 1, Y}],
   ReplaceList[pos, {X___, 1, 1, 0, Y___} -> {X, 0, 1, 1, Y}],
   ReplaceList[pos, {X___, 1, 2, 0, Y___} -> {X, 0, 2, 1, Y}]];
Mex[L_] := Min[Complement[Range[0, Length[L]], L]];
Gr[pos_] := Gr[pos] = Mex[Map[Gr, Po[pos]]];
Clear[data]; yy = {1,1,1,0,0,2,0,0,0,0,0,2,0,0,0,0,0};
 data[0] = {};
data[1] = Join[{yy}, Po[yy]]; n = 1; While[
 Length[data[n]] > Length[data[n - 1]],
 data[n + 1] = Union[data[n], Flatten[Map[Po, data[n]], 1]];
 n = n + 1]; gote = Select[data[n], Gr[#] == 0 &];
 gote2 = Reverse[gote];
ff[s_] := Which[s < 6, s, s < 12, s - 1, s > 11, s - 2]; gote3 =
 Map[Map[ff, Flatten[Position[#, 1]]] &, gote2];
```

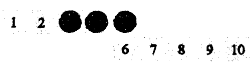
The following is the list of all the P-positions produced by Mathematica.
We have not discovered general formulas to produce this list of P-position, and hence calculation by computers is the only way to find P-positions.

{{1, 2, 3}, {1, 2, 5}, {1, 3, 6}, {1, 3, 8}, {1, 3, 10}, {1, 3, 12}, {1, 3, 14}, {1, 4, 5}, {1, 4, 7}, {1, 4, 9}, {1, 4, 11}, {1, 4, 13}, {1, 4, 15}, {1, 5, 6}, {1, 5, 8}, {1, 5, 10}, {1, 5, 12}, {1, 5, 14}, {1, 6, 9}, {1, 6, 11}, {1, 6, 13}, {1, 6, 15}, {1, 7, 8}, {1, 7, 10}, {1, 7, 12}, {1, 7, 14}, {1, 8, 11}, {1, 8, 13}, {1, 8, 15}, {1, 9, 10}, {1, 9, 12}, {1, 9, 14}, {1, 10, 11}, {1, 10, 13}, {1, 10, 15}, {1, 11, 14}, {1, 12, 13}, {1, 12, 15}, {1, 14, 15}, {2, 3, 4}, {2, 3, 7}, {2, 3, 9}, {2, 3, 11}, {2, 3, 13}, {2, 3, 15}, {2, 4, 6}, {2, 4, 8}, {2, 4, 10}, {2, 4, 12}, {2, 4, 14}, {2, 5, 7}, {2, 5, 9}, {2, 5, 11}, {2, 5, 13}, {2, 5, 15}, {2, 6, 8}, {2, 6, 10}, {2, 6, 12}, {2, 6, 14}, {2, 7, 9}, {2, 7, 11}, {2, 7, 13}, {2, 7, 15}, {2, 8, 10}, {2, 8, 12}, {2, 8, 14}, {2, 9, 11}, {2, 9, 13}, {2, 9, 15}, {2, 10, 12}, {2, 10, 14}, {2, 11, 13}, {2, 11, 15}, {2, 12, 14}, {2, 13, 15}, {3, 4, 5}, {3, 5, 6}, {3, 5, 8}, {3, 5, 10}, {3, 5, 12}, {3, 5, 14}, {3, 6, 9}, {3, 6, 11}, {3, 6, 13}, {3, 6, 15}, {3, 7, 8}, {3, 7, 10}, {3, 7, 12}, {3, 7, 14}, {3, 8, 11}, {3, 8, 13}, {3, 8, 15}, {3, 9, 10}, {3, 9, 12}, {3, 9, 14}, {3, 10, 11}, {3, 10, 13}, {3, 10, 15}, {3, 11, 14}}

Since the game of Definition 9is not simple enough, we are going to study a simpler version.

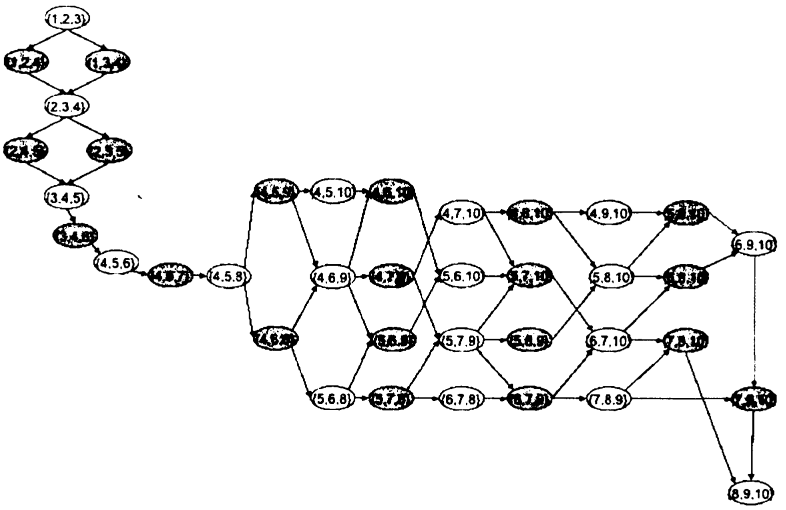
Definition 10

We use the strip and coins in Graph 7.8. The rule is the same as that of Definition 9.



Graph 7.8

By the same method we used Example 8 we can find all the P-positions and the N-positions, and from them we can make a chart in Graph 7.9. By using this chart you can win the game.
In this chart P-positions are colored in light gray and N-positions are colored in dark gray.



Graph 7.9

If you have read this article, you can appreciate the rich possibilities of combinatorial games in education and research. With a proper use of computer algebra system, even high school students can do many creative research.

References

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