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Nonlinear Operators, Nonlinear Projections and Geometry of Banach Spaces

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Abstract
Our purpose in this article is to discuss nonlinear operators in Banach spaces which are related to the resolvents of m-accretive operators and maximal monotone operators. We first deal with nonexpansive mappings which are deduced from the resolvents of m-accretive operators in Banach spaces. Fixed point theorems for nonexpansive mappings are well-known. Next, we define nonlinear operators and nonlinear projections which are deduced from the the resolvents of maximal monotone operators in Banach spaces. These operators in Banach spaces are very new. We discuss fixed point theorems for such nonlinear operators in Banach spaces. Further, using nonlinear projections, we obtain some results which are related to conditional expectations in the probability theory. Finally, we deal with duality theorems for nonlinear operators in Banach spaces.

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1 Introduction

Let E be a Banach space and let E^* be the dual space of E. Then, the duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $A \subset E \times E$ be a multi-valued operator with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$. Then, $A \subset E \times E$ is called accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. An accretive operator A is m-accretive if and only if $R(I + rA) = E$ for all $r > 0$. If $A \subset E \times E$ is m-accretive, then for each $r > 0$ and $x \in E$, we can define the resolvent $J_r \colon R(I + rA) \to D(A)$ by $J_r x = \{z \in E : x \in z + rAz\}$. A multi-valued operator $A \colon E \to E^*$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then, A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$; see [47]. If $A \subset E \times E^*$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the following resolvents:

$$J_\lambda x = \{z \in E : 0 \in J(z - x) + \lambda A(z)\}$$

and

$$Q_\lambda x = \{z \in E : Jx \in Jz + \lambda A(z)\}.$$

Further, if $B \subset E^* \times E$ be a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the resolvent

$$R_\lambda x = \{z \in E : x \in z + \lambda BJ(z)\}.$$

These four resolvents are important and have interesting properties.

Our purpose in this article is to discuss nonlinear operators in Banach spaces which are related to the resolvents of m-accretive operators and maximal monotone operators. In Section 3, we first consider nonlinear operators which are directly deduced from the resolvents of m-accretive operators and maximal monotone operators in Banach spaces. Further, from these operators, we define four nonlinear projections (retractions) and then obtain some results for the nonlinear projections in Banach spaces. In particular, we obtain results which are related to conditional expectations in the probability theory. In Section 4, from the nonlinear operators defined in Section 3, we define more general nonlinear operators in Banach spaces. One of them is a nonexpansive mapping. The other nonlinear operators are new. In this section, we obtain fixed point theorems which are different from the fixed point theorems for nonexpansive mappings. Further, we deal with duality theorems for nonlinear operators in Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. We know the following result: Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty closed convex subset of E and let P_C be the metric projection of E onto C. Let $x_0 \in C$ and $x_1 \in E$. Then, $x_0 = P_C(x_1)$ if and only if

$$\langle x_0 - y, J(x_1 - x_0) \rangle \geq 0$$

for all $y \in C$, where J is the duality mapping of E.

A Banach space E is said to satisfy Opial's condition [33] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n \to \infty} \|x_n - y\| < \liminf_{n \to \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let C be a closed convex subset of E. A mapping $T \colon C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be sunny if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a retraction if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{co}D$.

Let E be a Banach space and let $A \subset E \times E$ be a multi-valued operator. Then, $A \subset E \times E$ is called accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. An accretive operator A is m-accretive if and only if $R(I + rA) = E$ for all $r > 0$. If $A \subset E \times E$ is m-accretive, then for each $r > 0$ and $x \in E$, we can define $J_r \colon R(I + rA) \to D(A)$ by $J_r x = \{z \in E : x \in z + rAz\}$. We call such $J_r = (I + rA)^{-1}$ the accretive resolvent of A for $r > 0$.

A multi-valued operator $A \colon E \to E^*$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; see, for instance, [47].

Theorem 2.1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A \colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Theorem 2.2. *Let E be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Then, for $\lambda > 0$ and $x \in E$, consider

$$J_\lambda x = \{z \in E : 0 \in J(z - x) + \lambda A(z)\}$$

and

$$Q_\lambda x = \{z \in E : Jx \in Jz + \lambda A(z)\}.$$

We denote J_λ and Q_λ by $J_\lambda = (I + \lambda J^{-1}A)^{-1}$ and $Q_\lambda = (J + \lambda A)^{-1}J$, respectively. We call such J_λ and Q_λ the metric resolvent and the relative resolvent of A for $\lambda > 0$, respectively. We also consider another resolvent of a maximal monotone operator. Let $B \subset E^* \times E$ be a maximal monotone operator. Then, for $\lambda > 0$ and $x \in E$, consider

$$R_\lambda x = \{z \in E : x \in z + \lambda BJ(z)\}.$$

We denote R_λ by $R_\lambda = (I + \lambda BJ)^{-1}$. We call such R_λ the generalized resolvent of B for $\lambda > 0$.

3 Nonlinear Operators and Nonlinear Projections

In this section, we first define nonlinear operators which are deduced from m-accretive operators and maximal monotone operators in a Banach space. Let E be a reflexive, strictly convex and smooth Banach space. The function ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [19]. If $A \subset E \times E$ is m-accretive, then for each $\lambda > 0$ and $x \in E$, we can define the accretive resolvent $J_\lambda\colon E \to D(A)$ by $J_\lambda x = \{z \in E : x \in z + \lambda Az\}$. If $J_\lambda = (I + \lambda A)^{-1}$ is the accretive resolvent, then we can show that

$$0 \le \langle x - J_\lambda x - (y - J_\lambda y), J(J_\lambda x - J_\lambda y)\rangle$$

for all $x, y \in E$. Let C be a subset of E. Then, a nonlinear operator $T : C \to C$ is called firmly nonexpansive if

$$0 \le \langle x - Tx - (y - Ty), J(Tx - Ty)\rangle$$

for all $x, y \in C$. If $A \subset E \times E^*$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we define the metric resolvent $J_\lambda x = \{z \in E : 0 \in J(z - x) + \lambda A(z)\}$. If $J_\lambda = (I + \lambda J^{-1}A)^{-1}$ is the metric resolvent, then we have

$$0 \le \langle J_\lambda x - J_\lambda y, J(x - J_\lambda x) - J(y - J_\lambda y)\rangle$$

for all $x, y \in E$; see, for instance, [2]. In general, a nonlinear operator $T : C \to C$ is called firmly metric if

$$0 \le \langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle$$

for all $x, y \in C$. If $A \subset E \times E^*$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the relative resolvent $Q_\lambda x = \{z \in E : Jx \in Jz + \lambda A(z)\}$. If $Q_\lambda = (J + \lambda A)^{-1}J$ is the relative resolvent, then we have

$$0 \le \langle J_\lambda x - J_\lambda y, Jx - JJ_\lambda x - (Jy - JJ_\lambda y)\rangle$$

for all $x, y \in E$. In general, a nonlinear operator $T : C \to C$ is firmly relative nonexpansive if

$$0 \le \langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle$$

for all $x, y \in C$. We can define another nonlinear operator. If $B \subset E^* \times E$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the generalized resolvent $R_\lambda x = \{z \in E : x \in z + \lambda BJ(z)\}$. If $R_\lambda = (I + \lambda BJ)^{-1}$ is the generalized resolvent, then we know that

$$0 \leq \langle x - J_\lambda x - (y - J_\lambda y), JJ_\lambda x - JJ_\lambda y\rangle$$

for all $x, y \in E$. In general, a nonlinear operator $T : C \to C$ is firmly generalized nonexpansive if

$$0 \leq \langle x - Tx - (y - Ty), JTx - JTy\rangle$$

for all $x, y \in C$.

Next, we define four projections in a Banach space. Let E be a reflexive, smooth and strictly convex Banach space. We know that $T : C \to C$ is firmly nonexpansive if

$$0 \leq \langle x - Tx - (y - Ty),\ J(Tx - Ty)\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have that

$$0 \leq \langle x - Tx,\ J(Tx - y)\rangle$$

for all $x \in C$ and $y \in F(T)$. If P is a retraction of E onto C, then P is called sunny nonexpansive if

$$0 \leq \langle x - Px,\ J(Px - y)\rangle$$

for all $x \in E$ and $y \in C$. We know that $T : C \to C$ is a firmly metric operator if

$$0 \leq \langle Tx - Ty,\ J(x - Tx) - J(y - Ty)\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have that

$$0 \leq \langle Tx - y,\ J(x - Tx)\rangle$$

for all $x \in C$ and $y \in F(T)$. A retraction P of E onto C is called metric if

$$0 \leq \langle Px - y,\ J(x - Px)\rangle$$

for all $x \in E$ and $y \in C$. If $T : C \to C$ is firmly relative nonexpansive, then we have

$$0 \leq \langle Tx - Ty,\ Jx - JTx - (Jy - JTy)\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have that

$$0 \leq \langle Tx - y,\ Jx - JTx\rangle$$

for all $x \in C$ and $y \in F(T)$. A retraction Π_C of E onto C is called generalized if

$$0 \leq \langle \Pi_C x - y,\ Jx - J\Pi_C x\rangle$$

for all $x \in E$ and $y \in C$. If $T : C \to C$ is firmly generalized nonexpansive, we have

$$0 \leq \langle x - Tx - (y - Ty),\ JTx - JTy\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have

$$0 \le \langle x - Tx,\ JTx - Jy \rangle$$

for all $x \in C$ and $y \in F(T)$. A retraction R of E onto C is called sunny generalized nonexpansive if

$$0 \le \langle x - Rx,\ JRx - Jy \rangle$$

for all $x \in E$ and $y \in C$.

Kohsaka and Takahashi [23] proved the following two theorems.

Theorem 3.1 (Kohsaka and Takahashi [23]). *Let E be a smooth, strictly convex and reflexive Banach space and let C^* be a nonempty closed convex subset of E^*. Suppose that Π_{C_*} is the generalized projection of E^* onto C_*. Then, R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Theorem 3.2 (Kohsaka and Takahashi [23]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent*

(1) D is a sunny generalized nonexpansive retract of E;
(2) D is a generalized nonexpansive retract of E;
(3) JD is closed and convex.

In this case, D is closed.

Motivated by these theorems, we define the following nonlinear operator: Let E be a reflexive, strictly convex and smooth Banach space and let J be the normalized duality mapping from E onto E^*. Suppose that Y^* is a closed linear subspace of the dual space E^* of E. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is defined as follows:

$$E_{Y^*} = J^{-1}\Pi_{Y^*}J,$$

where Π_{Y^*} is the generalized projection from E^* onto Y^*.

Let E be a normed linear space and let $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James, denoted by $x \perp y$, if

$$\|x\| \le \|x + \lambda y\|$$

for all $\lambda \in \mathbf{R}$. We know that for $x, y \in E$, $x \perp y$ if and only if there exists $f \in J(x)$ with $\langle y, f \rangle = 0$. In general, $x \perp y$ does not imply $y \perp x$. An operator T of E into itself is called left-orthogonal (resp. right-orthogonal) if for each $x \in E$, $Tx \perp (x - Tx)$ (resp. $(x - Tx) \perp Tx$).

The following theorems are in Honda and Takahashi [10].

Theorem 3.3 (Honda and Takahashi [10]). *Let E be a normed linear space and let T be an operator of E into itself such that*

$$T(Tx + \beta(x - Tx)) = Tx$$

for any $x \in E$ and $\beta \in \mathbf{R}$. Then, the following conditions are equivalent:

(1) $\|Tx\| \le \|x\|$ for all $x \in E$;
(2) T is left-orthogonal.

Theorem 3.4 (Honda and Takahashi [10]). *Let E be a reflexive, strictly convex and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^*. Then, E_{Y^*} with respect to Y^* is left-orthogonal, i.e., for any $x \in E$,*

$$E_{Y^*}x \perp (x - E_{Y^*}x).$$

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

The following theorems are also in Honda and Takahashi [10].

Theorem 3.5 (Honda and Takahashi [10]). *Let E be a reflexive, strictly convex and smooth Banach space and let I be the identity operator of E into itself. Suppose that Y^* is a closed linear subspace of the dual space E^* and E_{Y^*} is the generalized conditional expectation with respect to Y^*. Then, the mapping $I - E_{Y^*}$ is the metric projection of E onto $Y^*_\perp$.*

Further, suppose that Y is a closed linear subspace of E and P_Y is the metric projection of E onto Y. Then, $I - P_Y$ is the generalized conditional expectation $E_{Y^\perp}$, i.e., $I - P_Y = E_{Y^\perp}$.

Let E be a normed linear space and let Y_1 and $Y_2 \subset E$ be closed linear subspaces. If $Y_1 \cap Y_2 = \{0\}$ and for any $x \in E$ there exists a unique pair $y_1 \in Y_1, y_2 \in Y_2$ such that

$$x = y_1 + y_2,$$

and any element of Y_1 is BJ-orthogonal to any element of Y_2, i.e., $y_1 \perp y_2$ for any $y_1 \in Y_1, y_2 \in Y_2$, then we represent the space E as

$$E = Y_1 \oplus Y_2 \text{ and } Y_1 \perp Y_2.$$

The kernel of an operator $T : E \to E$ is denoted by $\ker(T)$, i.e.,

$$\ker(T) = \{x \in E : Tx = 0\}.$$

Theorem 3.6 (Honda and Takahashi [10]). *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E such that for any $y_1, y_2 \in J^{-1}Y^*$, $y_1 + y_2 \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a closed linear subspace of E and the generalized conditional expectation E_{Y^*} is a norm one linear projection from E to $J^{-1}Y^*$. Further, the following hold:*

(1) $E = J^{-1}Y^* \oplus \ker(E_{Y^*})$ and $J^{-1}Y^* \perp \ker(E_{Y^*})$;
(2) $I - E_{Y^*}$ is the metric projection of E onto $\ker(E_{Y^*})$.

Using Theorem 3.6, Honda and Takahashi [11] obtained the following two theorems.

Theorem 3.7 (Honda and Takahashi [11]). *Let E be a strictly convex, reflexive and smooth Banach space and let $P : E \to E$ be a norm one projection with $Y = \{Px : x \in E\}$. Then, JY is a closed linear subspace of E^* and P is the generalized conditional expectation E_{JY} with respect to JY, i.e., $P = J^{-1}\Pi_{JY}J$.*

Theorem 3.8 (Honda and Takahashi [11]). *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of E^* Suppose that P is a projection of E onto $J^{-1}Y^*$ such that $\|Px - m\| \leq \|x - m\|$ for all $x \in E$ and $m \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a closed linear subspace of E and P is the generalized conditional expectation E_{Y^*}. Further, P is a norm one linear projection.*

4 Four Nonlinear Operators in Banach Spaces

Let E be a reflexive, smooth and strictly convex Banach space. Let C be a closed convex subset of E and let T be a mapping of C into itself. Then, since

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, we know that for any $x, y \in C$,

$$\begin{aligned}
0 \leq \langle x - Tx - (y - Ty),\ J(Tx - Ty)\rangle& \\
&\iff \|Tx - Ty\|^2 \leq \langle x - y,\ J(Tx - Ty)\rangle \\
&\iff 2\|Tx - Ty\|^2 \leq 2\langle x - y,\ J(Tx - Ty)\rangle \\
&\iff 2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 - \phi(x - y, Tx - Ty).
\end{aligned}$$

So, from a firmly nonexpansive mapping T of C into itself, we can define a nonexpansive mapping. That is, $T : C \to C$ is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. An operator $T : C \to C$ is firmly metric if

$$0 \leq \langle Tx - Ty,\ J(x - Tx) - J(y - Ty)\rangle$$

for all $x, y \in C$. Since

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$$

for $x, y, z \in E$, we have that for any $x, y \in C$,

$$\begin{aligned}
0 \leq \langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle& \\
&\iff 0 \leq 2\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \\
&\iff 2\langle x - Tx - (y - Ty),\ J(x - Tx) - J(y - Ty)\rangle \\
&\qquad \leq 2\langle x - y,\ J(x - Tx) - J(y - Ty)\rangle \\
&\iff \phi(x - y - Tx, Sy) + \phi(y - Ty, x - Tx) \\
&\qquad \leq \phi(x, y - Ty) + \phi(y, x - Tx) - \phi(x, x - Tx) - \phi(y, y - Ty) \\
&\implies \phi(x - Tx, y - Ty) + \phi(y - Ty, x - Tx) \leq \phi(x, y - Ty) + \phi(y, x - Tx).
\end{aligned}$$

So, from a firmly metric operator, we can define a metric operator. That is, $T : C \to C$ is called a metric operator if

$$\begin{aligned}
\phi(x - Tx, y - Ty)+&\phi(y - Ty, x - Tx) \\
&\leq \phi(x, y - Ty) + \phi(y, x - Ty)
\end{aligned}$$

for all $x, y \in C$. In the case that H is a Hilbert space and C is a closed convex subset of H, $T : C \to C$ is firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty,\ x - y\rangle$$

for all $x, y \in C$. Further, $T : C \to C$ is a metric operator if for any $x, y \in C$,

$$2\|x - Tx - (y - Ty)\|^2 \leq \|x - (y - Ty)\|^2 + \|y - (x - Ty)\|^2.$$

This inequality is equivalent to

$$2\langle x - y, Tx - Ty\rangle + 2\langle Tx,\ Ty\rangle \geq \|Tx - Ty\|^2.$$

An operator $T : C \to C$ is firmly relatively nonexpansive if

$$0 \leq \langle Tx - Ty,\ Jx - JTx - (Jy - JTy)\rangle$$

for all $x, y \in C$. Then, we know that for any $x, y \in C$,

$$\begin{aligned} 0 \leq \langle Tx - Ty,\ & Jx - JTx - (Jy - JTy)\rangle \\ & \iff \langle Tx - Ty,\ JTx - JTy\rangle \leq \langle Tx - Ty,\ Jx - Jy\rangle \\ & \iff \phi(Tx, Ty) + \phi(Ty, Tx) \\ & \qquad \leq \phi(Tx, y) + \phi(Ty, x) - \phi(Tx, x) - \phi(Ty, y). \end{aligned}$$

So, from a firmly relatively nonexpansive operator, we can define a nonspreading operator. That is, $T : C \to C$ is a nonspreading operator if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. In the case that H is a Hilbert space and C is a closed convex subset of H, an operator $T : C \to C$ is firmly nonexpansive if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 - \|Tx - x\|^2 - \|Ty - y\|^2$$

for all $x, y \in C$. Further, an operator $T : C \to H$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. This inequality is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. An operator $T : C \to C$ is firmly generalized nonexpansive if

$$0 \leq \langle x - Tx - (y - Ty),\ JTx - JTy\rangle$$

for all $x, y \in C$. Then, we know that for any $x, y \in C$,

$$\begin{aligned} 0 \leq \langle x - Tx - (y - Ty),\ & JTx - JTy\rangle \\ & \iff \langle Tx - Ty,\ JTx - JTy\rangle \leq \langle x - y,\ JTx - JTy\rangle \\ & \iff \phi(Tx, Ty) + \phi(Ty, Tx) \\ & \qquad \leq \phi(x, Ty) + \phi(y, Tx) - \phi(x, Tx) - \phi(y, Ty). \end{aligned}$$

So, from a firmly generalized nonexpansive operator, we can define a generalized nonexpansive type operator. That is, $T : C \to C$ is a generalized nonexpansive type operator if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$.

The following is Kohsaka and Takahashi's fixed point theorem [25].

Theorem 4.1 (Kohsaka and Takahashi [25]). *Let E be a smooth, strictly convex, and reflexive Banach space and let C is a closed convex subset of E. Suppose that T: $C \to C$ is nonspreading, i.e., for all $x, y \in C$,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x).$$

Then the following are equivalent:

(1) There exists $x \in C$ such that $\{T^n x\}$ is bounded;
(2) $F(T)$ is nonempty.

In the case that E is a Hilbert space, we have the following theorem.

Theorem 4.2 (Kohsaka and Takahashi [25]). *Let H be a Hilbert space and let C be a closed convex subset of H. Suppose that T: $C \to C$ is nonspreading, i.e., for all $x, y \in C$,*

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2.$$

Then the following are equivalent:

(1) There exists $x \in C$ such that $\{T^n x\}$ is bounded;
(2) $F(T)$ is nonempty.

5 Four Nonlinear Operators with Fixed Points

Let E be a reflexive, smooth and strictly convex Banach space and let C be a closed convex subset of E. Let $T : C \to C$ be nonexpansive, i.e.,

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. If $F(T) \ne \emptyset$, then

$$\|Tx - y\| \le \|x - y\|$$

for all $x \in C$ and $y \in F(T)$. Such T is called quasi-nonexpansive. Let $T : C \to C$ be a metric operator, i.e.,

$$\phi(x - Tx, y - Ty) + \phi(y - Ty, x - Tx) \le \phi(x, y - Ty) + \phi(y, x - Tx)$$

for all $x, y \in C$. If $F(T) \ne \emptyset$, then

$$2\|x - Tx\|^2 \le \|x\|^2 + \|y - (x - Tx)\|^2$$

for all $x \in C$ and $y \in F(T)$. Such T is called a quasi-metric operator. Let $T : C \to C$ be nonspreading, i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then

$$\phi(Tx, y) + \phi(y, Tx) \leq \phi(Tx, y) + \phi(y, x)$$

for all $x, y \in C$. This implies that

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$. Such T is called a quasi relatively nonexpansive operator. Let $T : C \to C$ be a generalized nonexpansive type operator, i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then

$$\phi(Tx, y) + \phi(y, Tx) \leq \phi(x, y) + \phi(y, Tx)$$

for all $x \in C$ and $y \in F(T)$. This implies that

$$\phi(Tx, y) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$. Such T is called a generalized nonexpansive operator. Let E be a Banach space and let C be a closed convex subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T if there exists $\{x_n\}$ such that $x_n \rightharpoonup p$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $F(T)$ the set of fixed points of T and by $\hat{F}(T)$ the set of asymptotic fixed points of T. Matsushita and Takahashi [28] also gave the following definition: An operator $T : C \to C$ is relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$.

The following theorems are in Kohsaka and Takahashi [25].

Theorem 5.1 (Kohsaka and Takahashi [25]). *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Suppose that T: $C \to C$ is nonspreading, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. Then, $\hat{F}(T) = F(T)$.

Theorem 5.2 (Kohsaka and Takahashi [25]). *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Suppose $T : C \to C$ is nonspreading, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$ and $F(T)$ is nonempty. Then, $T : C \to C$ is relatively nonexpansive.

Finally, we deal with the duality theorems for nonlinear operators in a Banach space. Let E be a smooth, strictly convex, and reflexive Banach space and let T be a mapping of E into itself. Define $T^* : E^* \to E^*$ as follows:

$$T^* x^* = JTJ^{-1} x^*,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the duality mapping of T. Let E be a smooth Banach space and let C be a closed convex subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called a generalized asymptotic fixed point of T if there exists $\{x_n\} \subset C$ such that $Jx_n \rightharpoonup Jp$, $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. We denote by $\check{F}(T)$ the set of generalized asymptotic fixed points of T.

Theorem 5.3 (Honda, Ibaraki and Takahashi [9]). *Let E be a smooth, strictly convex, and reflexive Banach space and let T be a mapping of E into itself. Then the following hold:*

(i) $JF(T) = F(T^*)$;
(ii) $J\hat{F}(T) = \check{F}(T^*)$;
(iii) $J\check{F}(T) = \hat{F}(T^*)$.

Theorem 5.4 (Honda, Ibaraki and Takahashi [9]). *Let E be a smooth, strictly convex, and reflexive Banach space and let T be a relatively nonexpansive mapping of E into itself. Let T^* be the duality mapping of T. Then T^* is generalized nonexpansive and $\check{F}(T^*) = F(T^*)$.*

Theorem 5.5 (Honda, Ibaraki and Takahashi [9]). *Let E be a smooth, strictly convex, and reflexive Banach space and let T be a generalized nonexpansive mapping of E into itself such that*

$$\check{F}(T) = F(T^*)$$

is nonempty. Let T^* be the duality mapping of T. Then T^* is relatively nonexpansive and

$$\hat{F}(T) = F(T^*)$$

is nonempty.

Using ideas of such duality theorems, we can prove the following theorem.

Theorem 5.6 (Dhompongsa, Fupinwong and Takahashi). *Let E be a smooth, strictly convex, and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Suppose that T: $C \to C$ is a generalized nonexpansive type operator, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$. Then the following are equivalent:

(1) There exists $x \in C$ such that $\{T^n x\}$ is bounded;
(2) $F(T)$ is nonempty.

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