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ON THE HERZ-TYPE SPACES WITH POWER WEIGHTS AND THE BOUNDEDNESS OF SOME SUBLINEAR OPERATORS

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1. Introduction

First, we state the notation which is used throughtout this paper. For a measurable set $E \subset \mathbb{R}^n$, we denote the Lebesgue measure of E by $|E|$ and the characteristic function of the set E by χ_E. Also, let for $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $P_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{P_k}$. And let for $k \in \mathbb{N}$, $\tilde{P}_k = P_k$, $\tilde{\chi}_k = \chi_{\tilde{P}_k}$ and $\tilde{P}_0 = B_0$, $\tilde{\chi}_0 = \chi_{\tilde{P}_0}$. Further, we denote the open ball in $\mathbb{R}^n$, having center 0 and radius $R > 0$, by $B(0, R)$.

Now, we define the homogeneous and non-homogeneous Herz spaces (see [LiY]).

Definition 1. *Let $\alpha \in \mathbb{R}$ and $0 < p \le \infty$.*

(*a*) *The homogeneous Herz space $\dot{K}^{\alpha}_{p,r}(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,*

$$\dot{K}^{\alpha}_{p,r}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha}_{p,r}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha r} \|f\chi_k\|^r_{L^p} \right)^{1/r} < \infty \right\};$$

$$\dot{K}^{\alpha}_{p,\infty}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha}_{p,\infty}} = \sup_{k\in\mathbb{Z}} 2^{k\alpha} \|f\chi_k\|_{L^p} < \infty \right\}.$$

(*b*) *The non-homogeneous Herz space $K^{\alpha}_{p,r}(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,*

$$K^{\alpha}_{p,r}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{K^{\alpha}_{p,r}} = \left(\sum_{k=0}^{\infty} 2^{k\alpha r} \|f\tilde{\chi}_k\|^r_{L^p} \right)^{1/r} < \infty \right\};$$

$$K^{\alpha}_{p,\infty}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{K^{\alpha}_{p,\infty}} = \sup_{k\ge 0} 2^{k\alpha} \|f\tilde{\chi}_k\|_{L^p} < \infty \right\}.$$

Here, throughout this talk, there are similar definitions and results for the non-homogeneous case as those for the homogeneous case. But, for simplicity, we only state the definitions and results for the homogeneous case.

Next, we recall the definition of the Hardy-Littlewood maximal operator M: that is, for any measurable function f on $\mathbb{R}^n$,

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^n$ containing x.

Moreover, we define the standard singular integral operator T.

Definition 2. We say that T is a standard singular integral operator, if there exists a function K which satisfies the following conditions:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) dy$$

exists almost everywhere, where $f \in L^2(\mathbb{R}^n)$;

$$|K(x)| \leq \frac{C_K}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \leq \frac{C_K}{|x|^{n+1}}, \quad x \neq 0;$$

$$\int_{\epsilon < |x| < N} K(x) dx = 0 \quad \text{for all} \quad 0 < \epsilon < N.$$

Then, the following strong-type estimates of the boundedness of the Hardy-Littlewood maximal operator M and a standard singular integral operator T on $L^p(\mathbb{R}^n)$ are well-known:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $1 < p \leq \infty$;

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $1 < p < \infty$.

Furthermore, let S be a sublinear operator satisfying for any integrable function f with a compact support,

$$(*) \qquad |Sf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \operatorname{supp} f,$$

where $c > 0$ is independent of f and x.

We remark that $(*)$ is satisfied by several operators in harmonic analysis, including the Hardy-Littlewood maximal operator M and a standard singular integral operator T.

Then, the following theorem was shown.

Theorem 3 ([LiY]). *Let $1 < p < \infty$, $0 < r \leq \infty$ and $-n/p < \alpha < n/p'$, where $1/p + 1/p' = 1$, and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then*

$$T : \dot{K}^{\alpha}_{p,r}(\mathbb{R}^n) \to \dot{K}^{\alpha}_{p,r}(\mathbb{R}^n).$$

Second, we define the weighted Herz spaces $\dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n)$ (see [K], [LuY] and [LYY]).

Now, for a nonnegative locally integrable function on $\mathbb{R}^n$, i.e. a weight (or a weight function), w, we write $w(E) = \int_E w(x) dx \quad (E \subset \mathbb{R}^n)$ and define

$$L^p(w)(\mathbb{R}^n) = \left\{ f : \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty \right\}.$$

Definition 4. *For $0 < \alpha < \infty$, $1 \le p < \infty$, $0 < r \le \infty$ and the weights w_1 and w_2,*

$$\dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(w_2)(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha}_{p,r}(w_1,w_2)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}^{\alpha}_{p,r}(w_1,w_2)} = \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha r/n} \|f\chi_k\|^r_{L^p(w_2)} \right\}^{1/r}.$$

In particular, when $w_1 = w_2 = w$, we put

$$\dot{K}^{\alpha}_{p,r}(w)(\mathbb{R}^n) = \dot{K}^{\alpha}_{p,r}(w, w)(\mathbb{R}^n).$$

Also, the following theorem was proved.

Theorem 5 ([LiY]). *Let $1 < p < \infty$, $0 < r < \infty$, $0 < \alpha < n/p'$, where $1/p + 1/p' = 1$, $w_1(x) = 1$, $w_2(x) = |x|^{-a}$ $(0 \le a < n)$, and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then*

$$T : \dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n) \to \dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n).$$

In this talk, we will introduce some weighted Herz-type space, $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which is a weighted Herz space $\dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n)$ with the critical index $\alpha = n/p'$, where $1/p + 1/p' = 1$, and show the boundedness of the sublinear operator T satisfying $(*)$ at the critical index $\alpha = n/p'$.

2. The boundedness on some weighted Herz-type spaces

First, we define the particular cases of the Herz spaces $\dot{K}^{\alpha}_{p,r}(\mathbb{R}^n)$ and the weighted Herz spaces $\dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n)$ (see [CL], [FW], [G], [GH], [LS$_1$], [LS$_2$] and [M]).

Definition 6. *For $1 \le p < \infty$*

$$\begin{aligned} \dot{A}^p(\mathbb{R}^n) &= \dot{K}^{n/p'}_{p,1}(\mathbb{R}^n) \\ &= \left\{ f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{A}^p} = \sum_{k=-\infty}^{\infty} 2^{kn/p'} \|f\chi_k\|_p < \infty \right\}, \end{aligned}$$

where $1/p + 1/p' = 1$.

Definition 7. *Let w_1 and w_2 be the weights. For $1 \le p < \infty$*

$$\begin{aligned} \dot{A}^p(w_1, w_2)(\mathbb{R}^n) &= \dot{K}^{n/p'}_{p,1}(w_1, w_2)(\mathbb{R}^n) \\ &= \left\{ f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{A}^p(w_1,w_2)} < \infty \right\}, \end{aligned}$$

where $1/p + 1/p' = 1$ and

$$\|f\|_{\dot{A}^p(w_1,w_2)} = \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1/p'} \|f\chi_k\|_{L^p(w_2)}.$$

In particular, when $w_1 = w_2 = w$, we put

$$\dot{A}^p(w)(\mathbb{R}^n) = \dot{A}^p(w, w)(\mathbb{R}^n).$$

Next, we define the central $(\alpha, p; w_1, w_2)$-block, and observe the block decomposition of $\dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n)$ (see [LS$_1$], [LS$_2$] and [LuY]).

Definition 8. *Let $0 < \alpha < \infty$ and $1 \le p < \infty$, and let w_1, w_2 be a weights. Then, we state that a measurable function $b(x)$ is a central $(\alpha, p; w_1, w_2)$-block, if the support of b is contained in a ball $B = B(0, R)$ $(R > 0)$, and so that*

$$\|b\|_{L^p(w_2)} \le [w_1(B)]^{-\alpha/n}.$$

Theorem 9. *Let $0 < \alpha < \infty$, $1 \le p < \infty$, and $0 < r < \infty$, and let $w_1 \in A_1$ and w_2 be a weight. Then, the following are equivalent:*

(i) $f \in \dot{K}^{\alpha}_{p,r}(w_1, w_2)(\mathbb{R}^n)$;

(ii) $f = \sum_{k=-\infty}^{\infty} \lambda_k b_k$ *where the b_k's are central $(\alpha, p; w_1, w_2)$-blocks and* $\sum_{k=-\infty}^{\infty} |\lambda_k|^r < \infty$.

Besides,

$$\|f\|_{\dot{K}^{\alpha}_{p,r}} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^r \right)^{1/r},$$

where the infimum is taken over all such decompositions.

Then, using the block decomposition of $\dot{A}^p(w)(\mathbb{R}^n)$, the boundedness of the sublinear operator satisfying $(*)$ on $\dot{A}^p(w)(\mathbb{R}^n)$ was shown.

Theorem 10 ([LS$_1$] and [LS$_2$]). *Let $1 < p < \infty$, $w(x) = |x|^{-a}$ $(0 < a < n)$, and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then*

$$T : \dot{K}^{n/p'}_{p,1}(w)(\mathbb{R}^n) \to \dot{K}^{n/p'}_{p,1}(w)(\mathbb{R}^n),$$

where $1/p + 1/p' = 1$, i.e.

$$T : \dot{A}^p(w)(\mathbb{R}^n) \to \dot{A}^p(w)(\mathbb{R}^n).$$

Now, we are in a position to show the result of our purpose, i.e. the boundedness of the sublinear operator satisfying $(*)$ on $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which extends the above results.

Theorem 11. *Let $1 < p < \infty$, $w_i(x) = |x|^{-a_i}$ such that $0 < a_i < n$ $(i = 1, 2)$, and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then*

$$T : \dot{K}^{n/p'}_{p,1}(w_1, w_2)(\mathbb{R}^n) \to \dot{K}^{n/p'}_{p,1}(w_1, w_2)(\mathbb{R}^n),$$

where $1/p + 1/p' = 1$, i.e.

$$T : \dot{A}^p(w_1, w_2)(\mathbb{R}^n) \to \dot{A}^p(w_1, w_2)(\mathbb{R}^n).$$

Proof. The proof of this theorem is similar to that of Theorem 2 of [LS$_2$].

By Theorem 9, it suffices to show that for any central $(n/p', p; w_1, w_2)$-block b,

$$\|Tb\|_{\dot{A}^p(w_1, w_2)} \le C,$$

where C is independent of b.

Now, let $B = B(0,R)$ be the supporting ball of b. Then, since we can choose a $j \in \mathbb{N}$ such that $2^{j-2} < R \le 2^{j-1}$. Therefore,

$$\begin{aligned}\|Tb\|_{\dot{A}^p(w_1,w_2)} &= \left(\sum_{k\le j} + \sum_{k>j}\right)[w_1(B_k)]^{1/p'}\|(Tb)\chi_k\|_{L^p(w_2)} \\ &= S_1 + S_2, \text{ say.}\end{aligned}$$

First, we estimate S_1. By the assumption, it follows that T maps $L^p(w_2)(\mathbb{R}^n)$ into $L^p(w_2)(\mathbb{R}^n)$ (see [SW]). Consequently,

$$\begin{aligned}\|(Tb)\chi_k\|_{L^p(w_2)} &\le C\left(\int_B |b(x)|^p w_2(x)dx\right)^{1/p} \\ &\le C[w_1(B_j)]^{1/p'}.\end{aligned}$$

Thus,

$$S_1 \le C\sum_{k\le j}\left[\frac{w_1(B_k)}{w_1(B_j)}\right]^{1/p'} \le C\sum_{k\le j} 2^{(k-j)(n-a_1)/p'} < \infty.$$

Next, in order to estimate S_2, note that if $x \in P_k$, $y \in B$ and $j < k$, then $|x-y| \sim |x|$. Hence, using the size condition of T, it follows that

$$\begin{aligned}\|(Tb)\chi_k\|^p_{L^p(w_2)} &\le C\int_{P_k}\left(\int_B \frac{|b(y)|}{|x-y|^n}dy\right)^p w_2(x)dx \\ &\le C\int_{P_k}\frac{1}{|x|^{np}}\left(\int_B |b(y)|^p dy\right)|B|^{p-1}w_2(x)dx \\ &\le C\int_{P_k}\frac{1}{|x|^{np}}\frac{1}{\operatorname{essinf}_{y\in B} w_2(y)}\left(\int_B |b(y)|^p w_2(y)dy\right)|B|^{p-1}w_2(x)dx.\end{aligned}$$

Since $w_2 \in A_1$,

$$\frac{w_2(B)}{|B|} \le C\operatorname{essinf}_{y\in B} w_2(y),$$

and therefore we have

$$\|(Tb)\chi_k\|_{L^p(w_2)} \le C[w_1(B)]^{-1/p'}[w_2(B)]^{-1/p}|B|\left(\int_{P_k}\frac{1}{|x|^{np}}w_2(x)dx\right)^{1/p}.$$

Thus, by the assumption,

$$\begin{aligned}S_2 &\le C\sum_{k>j}\left[\frac{w_1(B_k)}{w_1(B)}\right]^{1/p'}\left[\frac{w_2(B_k)}{w_2(B)}\right]^{1/p}|B|\,2^{-kn} \\ &\le C\sum_{k>j}2^{(k-j)(n-a_1)/p'}2^{(k-j)(n-a_2)/p}2^{(j-k)n} \\ &= C\sum_{k>j}2^{(j-k)(a_1/p'+a_2/p)} \\ &< \infty.\end{aligned}$$

□

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