ON THE HERZ-TYPE SPACES WITH POWER WEIGHTS
AND THE BOUNDEDNESS OF SOME SUBLINEAR OPERATORS
(The geometrical structure of Banach spaces and Function spaces and its applications)

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ON THE HERZ-TYPE SPACES WITH POWER WEIGHTS AND THE BOUNDEDNESS OF SOME SUBLINEAR OPERATORS

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1. INTRODUCTION

First, we state the notation which is used throughout this paper. For a measurable set $E \subset \mathbb{R}^n$, we denote the Lebesgue measure of $E$ by $|E|$ and the characteristic function of the set $E$ by $\chi_E$. Also, let for $k \in \mathbb{Z}$, $B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \}$, $P_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{P_k}$. And let for $k \in \mathbb{N}$, $\tilde{P}_k = P_k$, $\tilde{\chi}_k = \chi_{\overline{P}_k}$ and $\tilde{P}_0 = B_0$, $\tilde{\chi}_0 = \chi_{\overline{P}_0}$. Further, we denote the open ball in $\mathbb{R}^n$, having center 0 and radius $R > 0$, by $B(0, R)$.

Now, we define the homogeneous and non-homogeneous Herz spaces (see [LiY]).

Definition 1. Let $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$.

(a) The homogeneous Herz space $K_{p,r}^\alpha(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,

$$
K_{p,r}^\alpha(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K_{p,r}^\alpha} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha r} \|f \chi_k\|_{L^p}^r \right)^{1/r} < \infty \right\};
$$

$$
K_{p,\infty}^\alpha(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K_{p,\infty}^\alpha} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|f \chi_k\|_{L^p} < \infty \right\}.
$$

(b) The non-homogeneous Herz space $K_{p,r}^\alpha(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,

$$
K_{p,r}^\alpha(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K_{p,r}^\alpha} = \left( \sum_{k=0}^{\infty} 2^{k\alpha r} \|f \tilde{\chi}_k\|_{L^p}^r \right)^{1/r} < \infty \right\};
$$

$$
K_{p,\infty}^\alpha(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K_{p,\infty}^\alpha} = \sup_{k \geq 0} 2^{k\alpha} \|f \tilde{\chi}_k\|_{L^p} < \infty \right\}.
$$

Here, throughout this talk, there are similar definitions and results for the non-homogeneous case as those for the homogeneous case. But, for simplicity, we only state the definitions and results for the homogeneous case.
Next, we recall the definition of the Hardy-Littlewood maximal operator $M$: that is, for any measurable function $f$ on $\mathbb{R}^n$,

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^n$ containing $x$.

Moreover, we define the standard singular integral operator $T$.

**Definition 2.** We say that $T$ is a standard singular integral operator, if there exists a function $K$ which satisfies the following conditions:

$$Tf(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) dy$$

exists almost everywhere, where $f \in L^2(\mathbb{R}^n)$;

$$|K(x)| \leq \frac{C_K}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \leq \frac{C_K}{|x|^{n+1}}, \quad x \neq 0;$$

$$\int_{\epsilon < |x| < N} K(x) dx = 0 \quad \text{for all} \quad 0 < \epsilon < N.$$

Then, the following strong-type estimates of the boundedness of the Hardy-Littlewood maximal operator $M$ and a standard singular integral operator $T$ on $L^p(\mathbb{R}^n)$ are well-known:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $1 < p \leq \infty$;

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $1 < p < \infty$.

Furthermore, let $S$ be a sublinear operator satisfying for any integrable function $f$ with a compact support,

$$|Sf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \text{supp } f,$$

where $c > 0$ is independent of $f$ and $x$.

We remark that (*) is satisfied by several operators in harmonic analysis, including the Hardy-Littlewood maximal operator $M$ and a standard singular integral operator $T$.

Then, the following theorem was shown.

**Theorem 3 ([LiY]).** Let $1 < p < \infty$, $0 < r \leq \infty$ and $-n/p < \alpha < n/p'$, where $1/p + 1/p' = 1$, and let $T$ be a sublinear operator satisfying (*). If $T$ is bounded on $L^p(\mathbb{R}^n)$, then

$$T : \dot{K}^\alpha_{p,r}(\mathbb{R}^n) \to \dot{K}^\alpha_{p,r}(\mathbb{R}^n).$$

Second, we define the weighted Herz spaces $\dot{K}^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n)$ (see [K], [LuY] and [LYY]).

Now, for a nonnegative locally integrable function on $\mathbb{R}^n$, i.e. a weight (or a weight function), $w$, we write $w(E) = \int_E w(x) dx$ ($E \subset \mathbb{R}^n$) and define

$$L^p(w)(\mathbb{R}^n) = \left\{ f : \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty \right\}.$$
Definition 4. For $0 < \alpha < \infty$, $1 \leq p < \infty$, $0 < r \leq \infty$ and the weights $w_1$ and $w_2$, 

\[
\dot{K}^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(w_2)(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K^\alpha_{p,r}(w_1, w_2)} < \infty \right\},
\]

where

\[
\|f\|_{K^\alpha_{p,r}(w_1, w_2)} = \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha r/n} \|f \chi_k\|_{L^r_{\text{loc}}(w_2)} \right\}^{1/r}.
\]

In particular, when $w_1 = w_2 = w$, we put

\[
\dot{K}^\alpha_{p,r}(w)(\mathbb{R}^n) = \dot{K}^\alpha_{p,r}(w, w)(\mathbb{R}^n).
\]

Also, the following theorem was proved.

Theorem 5 ([LiY]). Let $1 < p < \infty$, $0 < r < \infty$, $0 < \alpha < n/p'$, where $1/p + 1/p' = 1$, $w_1(x) = 1$, $w_2(x) = |x|^{-a}$ $(0 \leq a < n)$, and let $T$ be a sublinear operator satisfying $(\ast)$. If $T$ is bounded on $L^p(\mathbb{R}^n)$, then

\[
T : \dot{K}^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n) \rightarrow \dot{K}^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n).
\]

In this talk, we will introduce some weighted Herz-type space, $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which is a weighted Herz space $\dot{K}^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n)$ with the critical index $\alpha = n/p'$, where $1/p + 1/p' = 1$, and show the boundedness of the sublinear operator $T$ satisfying $(\ast)$ at the critical index $\alpha = n/p'$.

2. THE BOUNDEDNESS ON SOME WEIGHTED HERZ-TYPE SPACES

First, we define the particular cases of the Herz spaces $\dot{K}^\alpha_{p,r}(\mathbb{R}^n)$ and the weighted Herz spaces $\dot{K}^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n)$ (see [CL], [FW], [G], [GH], [LS$_1$], [LS$_2$] and [M]).

Definition 6. For $1 \leq p < \infty$

\[
\dot{A}^p(\mathbb{R}^n) = \dot{K}^{n/p'}_{p,1}(\mathbb{R}^n)
\]

\[
= \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{A}^p} = \sum_{k=-\infty}^{\infty} 2^{kn/p'} \|f \chi_k\|_p < \infty \right\},
\]

where $1/p + 1/p' = 1$.

Definition 7. Let $w_1$ and $w_2$ be the weights. For $1 \leq p < \infty$

\[
\dot{A}^p(w_1, w_2)(\mathbb{R}^n) = \dot{K}^{n/p'}_{p,1}(w_1, w_2)(\mathbb{R}^n)
\]

\[
= \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{A}^p(w_1, w_2)} < \infty \right\},
\]

where $1/p + 1/p' = 1$ and

\[
\|f\|_{\dot{A}^p(w_1, w_2)} = \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1/p'} \|f \chi_k\|_{L^p_{\text{loc}}(w_2)}.
\]

In particular, when $w_1 = w_2 = w$, we put

\[
\dot{A}^p(w)(\mathbb{R}^n) = \dot{A}^p(w, w)(\mathbb{R}^n).
\]
Next, we define the central \((\alpha, p; w_1, w_2)\)-block, and observe the block decomposition of \(K^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n)\) (see [LS1], [LS2] and [LuY]).

**Definition 8.** Let \(0 < \alpha < \infty\) and \(1 \leq p < \infty\), and let \(w_1, w_2\) be a weights. Then, we state that a measurable function \(b(x)\) is a central \((\alpha, p; w_1, w_2)\)-block, if the support of \(b\) is contained in a ball \(B = B(0, R)\) \((R > 0)\), and so that
\[
\|b\|_{L^{p}(w_2)} \leq [w_1(B)]^{-\alpha/n}.
\]

**Theorem 9.** Let \(0 < \alpha < \infty\), \(1 \leq p < \infty\), and let \(w_1\) and \(w_2\) be a weights. Then, the following are equivalent:

(i) \(f \in K^\alpha_{p,r}(w_1, w_2)(\mathbb{R}^n)\);

(ii) \(f = \sum_{k=-\infty}^{\infty} \lambda_k b_k\) where the \(b_k\)'s are central \((\alpha, p; w_1, w_2)\)-blocks and \(\sum_{k=-\infty}^{\infty} |\lambda_k|^r < \infty\).

Besides,
\[
\|f\|_{K^\alpha_{p,r}} \approx \inf \left( \sum_{k=-\infty}^{\infty} |\lambda_k|^r \right)^{1/r},
\]
where the infimum is taken over all such decompositions.

Then, using the block decomposition of \(\dot{A}^{p}(w)(\mathbb{R}^n)\), the boundedness of the sublinear operator satisfying \((*)\) on \(\dot{A}^{p}(w)(\mathbb{R}^n)\) was shown.

**Theorem 10** ([LS1] and [LS2]). Let \(1 < p < \infty\), \(w(x) = |x|^{-a}\) \((0 < a < n)\), and let \(T\) be a sublinear operator satisfying \((*)\). If \(T\) is bounded on \(L^{p}(\mathbb{R}^n)\), then
\[T : K_{p,1}^{n/p'}(w)(\mathbb{R}^n) \rightarrow K_{p_1}^{n/p'}(w)(\mathbb{R}^n),\]
where \(1/p + 1/p' = 1\), i.e.
\[T : \dot{A}^{p}(w)(\mathbb{R}^n) \rightarrow \dot{A}^{p}(w)(\mathbb{R}^n).\]

Now, we are in a position to show the result of our purpose, i.e. the boundedness of the sublinear operator satisfying \((*)\) on \(\dot{A}^{p}(w_1, w_2)(\mathbb{R}^n)\), which extends the above results.

**Theorem 11.** Let \(1 < p < \infty\), \(w_i(x) = |x|^{-a_i}\) such that \(0 < a_i < n\) \((i = 1, 2)\), and let \(T\) be a sublinear operator satisfying \((*)\). If \(T\) is bounded on \(L^{p}(\mathbb{R}^n)\), then
\[T : K_{p_1}^{n/p'}(w_1, w_2)(\mathbb{R}^n) \rightarrow K_{p_1}^{n/p'}(w_1, w_2)(\mathbb{R}^n),\]
where \(1/p + 1/p' = 1\), i.e.
\[T : \dot{A}^{p}(w_1, w_2)(\mathbb{R}^n) \rightarrow \dot{A}^{p}(w_1, w_2)(\mathbb{R}^n).\]

**Proof.** The proof of this theorem is similar to that of Theorem 2 of [LS2].

By Theorem 9, it suffices to show that for any central \((n/p', p; w_1, w_2)\)-block \(b\),
\[
\|Tb\|_{\dot{A}^{p}(w_1, w_2)} \leq C,
\]
where \(C\) is independent of \(b\).
Now, let $B = B(O, R)$ be the supporting ball of $b$. Then, since we can choose a $j \in \mathbb{N}$ such that $2^{j-2} < R \leq 2^{j-1}$. Therefore,

$$
||Tb||_{A^p(w_1,w_2)} = \left( \sum_{k \leq j} + \sum_{k > j} \right) \left[ \frac{w_1(B_k)}{w_1(B_j)} \right]^{1/p'} ||(Tb)\chi_k||_{L^p(w_2)} = S_1 + S_2, \text{ say.}
$$

First, we estimate $S_1$. By the assumption, it follows that $T$ maps $L^p(w_2)(\mathbb{R}^n)$ into $L^p(w_2)(\mathbb{R}^n)$ (see [SW]). Consequently,

$$
||(Tb)\chi_k||_{L^p(w_2)} \leq C \left( \int_B |b(x)|^p w_2(x)dx \right)^{1/p} \leq C[w_1(B_j)]^{1/p'}.
$$

Thus,

$$
S_1 \leq C \sum_{k \leq j} \left[ \frac{w_1(B_k)}{w_1(B_j)} \right]^{1/p'} \leq C \sum_{k \leq j} 2^{(k-j)(n-a_1)/p} < \infty.
$$

Next, in order to estimate $S_2$, note that if $x \in P_k$, $y \in B$ and $j < k$, then $|x-y| \sim |x|$. Hence, using the size condition of $T$, it follows that

$$
||(Tb)\chi_k||_{L^p(w_2)} \leq C \int_{P_k} \left( \int_B \frac{|b(y)|}{|x-y|^n} dy \right)^p w_2(x)dx \leq C \int_{P_k} \frac{1}{|x|^p} \left( \int_B |b(y)|^p dy \right) |B|^{1/p-1} w_2(x)dx \leq C \int_{P_k} \frac{1}{|x|^p \text{ess}\inf_{y \in B} w_2(y)} \left( \int_B |b(y)|^p w_2(y)dy \right) |B|^{1/p-1} w_2(x)dx.
$$

Since $w_2 \in A_1$,

$$
\frac{w_2(B)}{|B|} \leq C \text{ess}\inf_{y \in B} w_2(y),
$$

and therefore we have

$$
||(Tb)\chi_k||_{L^p(w_2)} \leq C[w_1(B)]^{-1/p'} [w_2(B)]^{-1/p} |B| \left( \int_{P_k} \frac{1}{|x|^p} w_2(x)dx \right)^{1/p}.
$$

Thus, by the assumption,

$$
S_2 \leq C \sum_{k > j} \left[ \frac{w_1(B_k)}{w_1(B)} \right]^{1/p'} \left[ \frac{w_2(B_k)}{w_2(B)} \right]^{1/p} |B| 2^{-kn} \leq C \sum_{k > j} 2^{(k-j)(n-a_1)/p'} 2^{(k-j)(n-a_2)/p_2} 2^{(j-k)n} = C \sum_{k > j} 2^{(j-k)(a_1/p' + a_2/p)} < \infty.
$$

$\square$
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