ON THE HERZ-TYPE SPACES WITH POWER WEIGHTS
AND THE BOUNDEDNESS OF SOME SUBLINEAR OPERATORS

日本大学・経済学部 松岡勝男 (KATSUO MATSUOKA)
COLLEGE OF ECONOMICS OF NIHON UNIVERSITY

1. INTRODUCTION

First, we state the notation which is used throughout this paper. For a measurable set $E \subset \mathbb{R}^n$, we denote the Lebesgue measure of $E$ by $|E|$ and the characteristic function of the set $E$ by $\chi_E$. Also, let for $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $P_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{P_k}$. And let for $k \in \mathbb{N}$, $\tilde{P}_k = P_k$, $\tilde{\chi}_k = \chi_{\overline{P}_k}$ and $\tilde{P}_0 = B_0$, $\tilde{\chi}_0 = \chi_{\overline{P}_0}$. Further, we denote the open ball in $\mathbb{R}^n$, having center 0 and radius $R > 0$, by $B(0, R)$.

Now, we define the homogeneous and non-homogeneous Herz spaces (see [LiY]).

**Definition 1.** Let $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$.

(a) The homogeneous Herz space $\dot{K}_{p,r}^\alpha(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,

$$\dot{K}_{p,r}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p,r}^\alpha} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha r} \|f \chi_k\|_{L^p}^r \right)^{1/r} < \infty \right\};$$

$$\dot{K}_{p,\infty}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p,\infty}^\alpha} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|f \chi_k\|_{L^p} < \infty \right\}.$$

(b) The non-homogeneous Herz space $K_{p,r}^\alpha(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,

$$K_{p,r}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{K_{p,r}^\alpha} = \left( \sum_{k=0}^{\infty} 2^{k\alpha r} \|f \tilde{\chi}_k\|_{L^p}^r \right)^{1/r} < \infty \right\};$$

$$K_{p,\infty}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{K_{p,\infty}^\alpha} = \sup_{k \geq 0} 2^{k\alpha} \|f \tilde{\chi}_k\|_{L^p} < \infty \right\}.$$

Here, throughout this talk, there are similar definitions and results for the non-homogeneous case as those for the homogeneous case. But, for simplicity, we only state the definitions and results for the homogeneous case.
Next, we recall the definition of the Hardy-Littlewood maximal operator $M$: that is, for any measurable function $f$ on $\mathbb{R}^n$,

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^n$ containing $x$.

Moreover, we define the standard singular integral operator $T$.

**Definition 2.** We say that $T$ is a standard singular integral operator, if there exists a function $K$ which satisfies the following conditions:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

exists almost everywhere, where $f \in L^2(\mathbb{R}^n)$; $|K(x)| \leq \frac{C_K}{|x|^n}$ and $|\nabla K(x)| \leq \frac{C_K}{|x|^{n+1}}$, $x \neq 0$;

$$\int_{\epsilon < |x| < N} K(x)dx = 0 \text{ for all } 0 < \epsilon < N.$$

Then, the following strong-type estimates of the boundedness of the Hardy-Littlewood maximal operator $M$ and a standard singular integral operator $T$ on $L^p(\mathbb{R}^n)$ are well-known:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $1 < p \leq \infty$;

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $1 < p < \infty$.

Furthermore, let $S$ be a sublinear operator satisfying for any integrable function $f$ with a compact support,

$$(*) \quad |Sf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}dy, \quad x \notin \text{supp } f,$$

where $c > 0$ is independent of $f$ and $x$.

We remark that $(*)$ is satisfied by several operators in harmonic analysis, including the Hardy-Littlewood maximal operator $M$ and a standard singular integral operator $T$.

Then, the following theorem was shown.

**Theorem 3 ([LiY]).** Let $1 < p < \infty$, $0 < r \leq \infty$ and $-n/p < \alpha < n/p'$, where $1/p + 1/p' = 1$, and let $T$ be a sublinear operator satisfying $(*)$. If $T$ is bounded on $L^p(\mathbb{R}^n)$, then $T : K^\alpha_{p,r}(\mathbb{R}^n) \to K^\alpha_{p,r}(\mathbb{R}^n)$.

Second, we define the weighted Herz spaces $K^\alpha_{p,r}(w_1,w_2)(\mathbb{R}^n)$ (see [K], [LuY] and [LYY]).

Now, for a nonnegative locally integrable function on $\mathbb{R}^n$, i.e. a weight (or a weight function), $w$, we write $w(E) = \int_E w(x)dx$ ($E \subset \mathbb{R}^n$) and define

$$L^p(w)(\mathbb{R}^n) = \left\{ f : \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{1/p} < \infty \right\}. $$
Definition 4. For $0 < \alpha < \infty$, $1 \leq p < \infty$, $0 < r \leq \infty$ and the weights $w_1$ and $w_2$,
\[ \dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(w_2)(\mathbb{R}^n \setminus \{0\}) : \| f \|_{\dot{K}_{p,r}^\alpha(w_1, w_2)} < \infty \right\}, \]
where
\[ \| f \|_{\dot{K}_{p,r}^\alpha(w_1, w_2)} = \left\{ \sum_{k=-\infty}^{\infty} \left[ w_1(B_k) \right]^{\alpha r/n} \| f \chi_k \|_{L^p(w_2)} \right\}^{1/r}. \]
In particular, when $w_1 = w_2 = w$, we put
\[ \dot{K}_{p,r}^\alpha(w)(\mathbb{R}^n) = \dot{K}_{p,r}^\alpha(w, w)(\mathbb{R}^n). \]

Also, the following theorem was proved.

Theorem 5 ([LiY]). Let $1 < p < \infty$, $0 < r < \infty$, $0 < \alpha < n/p'$, where $1/p + 1/p' = 1$, $w_1(x) = 1$, $w_2(x) = |x|^{-a}$ ($0 \leq a < n$), and let $T$ be a sublinear operator satisfying (*). If $T$ is bounded on $L^p(\mathbb{R}^n)$, then
\[ T : \dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n) \to \dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n). \]

In this talk, we will introduce some weighted Herz-type space, $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which is a weighted Herz space $\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ with the critical index $\alpha = n/p'$, where $1/p + 1/p' = 1$, and show the boundedness of the sublinear operator $T$ satisfying (*) at the critical index $\alpha = n/p'$.

2. THE BOUNDEDNESS ON SOME WEIGHTED HERZ-TYPE SPACES

First, we define the particular cases of the Herz spaces $\dot{K}_{p,r}^\alpha(\mathbb{R}^n)$ and the weighted Herz spaces $\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ (see [CL], [FW], [G], [GH], [LS1], [LS2] and [M]).

Definition 6. For $1 \leq p < \infty$
\[ \dot{A}^p(\mathbb{R}^n) = \dot{K}_{p,1}^{n/p'}(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \| f \|_{\dot{A}^p} = \sum_{k=-\infty}^{\infty} 2^{kn/p'} \| f \chi_k \|_p < \infty \right\}, \]
where $1/p + 1/p' = 1$.

Definition 7. Let $w_1$ and $w_2$ be the weights. For $1 \leq p < \infty$
\[ \dot{A}^p(w_1, w_2)(\mathbb{R}^n) = \dot{K}_{p,1}^{n/p'}(w_1, w_2)(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \| f \|_{\dot{A}^p(w_1, w_2)} < \infty \right\}, \]
where $1/p + 1/p' = 1$ and
\[ \| f \|_{\dot{A}^p(w_1, w_2)} = \sum_{k=-\infty}^{\infty} \left[ w_1(B_k) \right]^{1/p'} \| f \chi_k \|_{L^p(w_2)}. \]
In particular, when $w_1 = w_2 = w$, we put
\[ \dot{A}^p(w)(\mathbb{R}^n) = \dot{A}^p(w, w)(\mathbb{R}^n). \]
Next, we define the central $(\alpha, p; w_1, w_2)$-block, and observe the block decomposition of $K_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ (see \cite{LS1}, \cite{LS2} and \cite{LuY}).

**Definition 8.** Let $0 < \alpha < \infty$ and $1 \leq p < \infty$, and let $w_1, w_2$ be a weights. Then, we state that a measurable function $b(x)$ is a central $(\alpha, p; w_1, w_2)$-block, if the support of $b$ is contained in a ball $B = B(0, R)$ ($R > 0$), and so that

$$\|b\|_{L^p(w_2)} \leq [w_1(B)]^{-\alpha/n}.$$

**Theorem 9.** Let $0 < \alpha < \infty, 1 \leq p < \infty$, and let $w_1, w_2$ be a weight. Then, the following are equivalent:

(i) $f \in K_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$;

(ii) $f = \sum_{k=-\infty}^{\infty} \lambda_k b_k$ where the $b_k$'s are central $(\alpha, p; w_1, w_2)$-blocks and $\sum_{k=-\infty}^{\infty} |\lambda_k|^r < \infty$.

Besides,

$$\|f\|_{K_{p,r}^\alpha} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^r\right)^{1/r},$$

where the infimum is taken over all such decompositions.

Then, using the block decomposition of $\dot{A}^p(w)(\mathbb{R}^n)$, the boundedness of the sublinear operator satisfying $(\ast)$ on $\dot{A}^p(w)(\mathbb{R}^n)$ was shown.

**Theorem 10** (\cite{LS1} and \cite{LS2}). Let $1 < p < \infty, w(x) = |x|^{-a}$ ($0 < a < n$), and let $T$ be a sublinear operator satisfying $(\ast)$. If $T$ is bounded on $L^p(\mathbb{R}^n)$, then

$$T : K_{p,1}^{n/p'}(w)(\mathbb{R}^n) \rightarrow K_{p,1}^{n/p'}(w)(\mathbb{R}^n),$$

where $1/p + 1/p' = 1$, i.e.

$$T : \dot{A}^p(w)(\mathbb{R}^n) \rightarrow \dot{A}^p(w)(\mathbb{R}^n).$$

Now, we are in a position to show the result of our purpose, i.e. the boundedness of the sublinear operator satisfying $(\ast)$ on $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which extends the above results.

**Theorem 11.** Let $1 < p < \infty, w_i(x) = |x|^{-a_i}$ such that $0 < a_i < n$ ($i = 1, 2$), and let $T$ be a sublinear operator satisfying $(\ast)$. If $T$ is bounded on $L^p(\mathbb{R}^n)$, then

$$T : K_{p,1}^{n/p'}(w_1, w_2)(\mathbb{R}^n) \rightarrow K_{p,1}^{n/p'}(w_1, w_2)(\mathbb{R}^n),$$

where $1/p + 1/p' = 1$, i.e.

$$T : \dot{A}^p(w_1, w_2)(\mathbb{R}^n) \rightarrow \dot{A}^p(w_1, w_2)(\mathbb{R}^n).$$

**Proof.** The proof of this theorem is similar to that of Theorem 2 of \cite{LS2}.

By Theorem 9, it suffices to show that for any central $(n/p', p; w_1, w_2)$-block $b$,

$$\|Tb\|_{\dot{A}^p(w_1, w_2)} \leq C,$$

where $C$ is independent of $b$. 
Now, let $B = B(0, R)$ be the supporting ball of $b$. Then, since we can choose a $j \in \mathbb{N}$ such that $2^{j-2} < R \leq 2^{j-1}$. Therefore,

$$
\|Tb\|_{A^{p}(w_{1},w_{2})} = \left( \sum_{k \leq j} + \sum_{k > j} \right) [w_{1}(B_k)]^{1/p'} \|(Tb)\chi_{k}\|_{L^{p}(w_{2})}
= S_{1} + S_{2}, \text{ say.}
$$

First, we estimate $S_{1}$. By the assumption, it follows that $T$ maps $L^{p}(w_{2})(\mathbb{R}^{n})$ into $L^{p}(w_{2})(\mathbb{R}^{n})$ (see [SW]). Consequently,

$$
\|(Tb)\chi_{k}\|_{L^{p}(w_{2})} \leq C \left( \int_{B} |b(x)|^{p}w_{2}(x)dx \right)^{1/p}
\leq C[w_{1}(B_{j})]^{1/p'} .
$$

Thus,

$$
S_{1} \leq C \sum_{k \leq j} \left[ \frac{w_{1}(B_{k})}{w_{1}(B_{j})} \right]^{1/p'} \leq C \sum_{k \leq j} 2^{(k-j)(n-a_{1})/p'} < \infty .
$$

Next, in order to estimate $S_{2}$, note that if $x \in P_{k}$, $y \in B$ and $j < k$, then $|x - y| \sim |x|$. Hence, using the size condition of $T$, it follows that

$$
\|(Tb)\chi_{k}\|_{L^{p}(w_{2})}^{p} \leq C \int_{P_{k}} \left( \int_{B} |b(y)| \left| \frac{1}{|x - y|^{n}} \right| dy \right)^{p} w_{2}(x)dx
\leq C \int_{P_{k}} \frac{1}{|x|^{np}} \left( \int_{B} |b(y)|^{p}dy \right) |B|^{p-1} w_{2}(x)dx
\leq C \int_{P_{k}} \frac{1}{|x|^{np}} ess\inf_{y \in B} w_{2}(y) \left( \int_{B} |b(y)|^{p}w_{2}(y)dy \right) |B|^{p-1} w_{2}(x)dx .
$$

Since $w_{2} \in A_{1}$,

$$
\frac{w_{2}(B)}{|B|} \leq C ess\inf_{y \in B} w_{2}(y) ,
$$

and therefore we have

$$
\|(Tb)\chi_{k}\|_{L^{p}(w_{2})} \leq C[w_{1}(B)]^{-1/p'}[w_{2}(B)]^{-1/p} |B| \left( \int_{P_{k}} \frac{1}{|x|^{np}} w_{2}(x)dx \right)^{1/p} .
$$

Thus, by the assumption,

$$
S_{2} \leq C \sum_{k > j} \left[ \frac{w_{1}(B_{k})}{w_{1}(B)} \right]^{1/p'} \left[ \frac{w_{2}(B_{k})}{w_{2}(B)} \right]^{1/p} |B| 2^{-kn}
\leq C \sum_{k > j} 2^{(k-j)(n-a_{1})/p'} 2^{(k-j)(n-a_{2})/p} 2^{j-k} 2^{(j-k)n}
= C \sum_{k > j} 2^{j-k}(a_{1}/p' + a_{2}/p)
< \infty .
$$

$\square$
REFERENCES


