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Convergence of some truncated Riesz transforms on predual of generalized Campanato spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations.

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1. Introduction

This is an announcement of our recent work [8]. In [6] the first author introduced predual of generalized Campanato spaces. In this report, we show convergence of some truncated Riesz transforms on the function spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations. Our uniqueness theorem is an extension of Kato's [3].

2. Generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$

Let $1 \le p < \infty$ and $\phi : (0,\infty) \to (0,\infty)$. For a ball $B = B(x,r)$, we shall write $\phi(B)$ in place of $\phi(r)$. The function spaces $\mathcal{L}_{p,\phi} = \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right)^{1/p},$$

$$f_B = \frac{1}{|B|} \int_B f(x) \, dx.$$

Then $\mathcal{L}_{p,\phi}$ is a Banach space modulo constants with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{1,\phi} = \mathrm{BMO}$. It is known that if $\phi(r) = r^\alpha$, $0 < \alpha \le 1$, then $\mathcal{L}_{p,\phi} = \mathrm{Lip}_\alpha$, and, if $\phi(r) = r^{-n/p}$, $1 \le p < \infty$, then $\mathcal{L}_{p,\phi} = L^p$.

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A function $\phi : (0,\infty) \to (0,\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\phi(r)}{\phi(s)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2.$$

A function $\phi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\phi(r) \le C\phi(s) \quad (\phi(r) \ge C\phi(s)) \quad \text{for} \quad r \le s.$$

Lemma 2.1. *Assume that $\phi(r)r^{n/p}$ is almost increasing and that $\phi(r)/r$ is almost decreasing. Then ϕ satisfies the doubling condition and*

$$\|f\|_{\mathcal{L}_{p,\phi}} \le C\big(\|(1+|x|^{n+1})f\|_\infty + \|\nabla f\|_\infty\big).$$

That is $\mathcal{S} \subset \mathcal{L}_{p,\phi}$.

Proof. Let $B = B(z,r)$.
Case 1: $r < 1$: In this case $r \lesssim \phi(r)$. Then

$$|f(x) - f(y)| \lesssim r\|\nabla f\|_\infty \lesssim \phi(r)\|\nabla f\|_\infty, \quad x, y \in B.$$

$$\left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\,dx\right)^{1/p} \lesssim \sup_{x,y\in B} |f(x) - f(y)| \lesssim \phi(r)\|\nabla f\|_\infty.$$

Case 2: $1 \le r$: In this case $1 \lesssim \phi(r)r^{n/p}$ and

$$|f(x)| \le \frac{\|(1+|x|^{n+1})f\|_\infty}{1+|x|^{n+1}}, \quad \left(\int |f(x)|^p\,dx\right)^{1/p} \lesssim \|(1+|x|^{n+1})f\|_\infty.$$

Then

$$\begin{aligned}\left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\,dx\right)^{1/p} &\le 2\left(\frac{1}{|B|}\int_B |f(x)|^p\,dx\right)^{1/p} \\ &\lesssim \frac{\|(1+|x|^{n+1})f\|_\infty}{|B|^{1/p}} \lesssim \phi(r)\|(1+|x|^{n+1})f\|_\infty. \quad \square\end{aligned}$$

3. $H_I^{[\phi,\infty]}(\mathbb{R}^n)$, PREDUAL OF $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$

The space $H_U^{[\phi,q]}$ was introduced in [6], which is a generalization of Hardy space. The duality $\left(H_U^{[\phi,q]}\right)^* = \mathcal{L}_{q',\phi}$ also proved in [6].

In this talk we recall the definition of $H_I^{[\phi,\infty]}(\mathbb{R}^n)$, which is a special case of $H_U^{[\phi,q]}$.

In what follows, we always assume that $\phi(r)r^n$ is almost increasing and that $\phi(r)/r$ is almost decreasing.

Definition 3.1 ($[\phi,\infty]$-atom). A function a on $\mathbb{R}^n$ is called a $[\phi,\infty]$-atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_\infty \le \dfrac{1}{|B|\phi(B)}$,

(iii) $\displaystyle\int_{\mathbb{R}^n} a(x)\,dx = 0$.

where $\|a\|_\infty$ is the L^∞ norm of a. We denote by $A[\phi,\infty]$ the set of all $[\phi,\infty]$-atoms.

If a is a $[\phi,\infty]$-atom and a ball B satisfies (i)–(iii), then, for $g \in \mathcal{L}_{1,\phi}$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a(x)g(x)\,dx\right| &= \left|\int_B a(x)(g(x)-g_B)\,dx\right| \\ &\le \|a\|_\infty \int_B |g(x)-g_B|\,dx \\ &\le \frac{1}{\phi(B)}\frac{1}{|B|}\int_B |g(x)-g_B|\,dx \\ &\le \|g\|_{\mathcal{L}_{1,\phi}}.\end{aligned}$$

That is, the mapping $g \mapsto \int_{\mathbb{R}^n} ag\,dx$ is a bounded linear functional on $\mathcal{L}_{1,\phi}$ with norm not exceeding 1. Hence a is also in $\mathcal{S}'$, since $\mathcal{S} \subset \mathcal{L}_{1,\phi}$.

Definition 3.2 ($H_I^{[\phi,\infty]}$). The space $H_I^{[\phi,\infty]} \subset (\mathcal{L}_{1,\phi})^*$ is defined as follows:

$f \in H_I^{[\phi,\infty]}$ if and only if there exist sequences $\{a_j\} \subset A[\phi,\infty]$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{1,\phi})^* \quad \text{and} \quad \sum_j \lambda_j < \infty. \tag{3.1}$$

In general, the expression (3.1) is not unique. Let

$$\|f\|_{H_I^{[\phi,\infty]}} = \inf\left\{\sum_j \lambda_j\right\},$$

where the infimum is taken over all expressions as in (3.1). Then $H_I^{[\phi,\infty]}$ is a Banach space equipped with the norm $\|f\|_{H_I^{[\phi,\infty]}}$ and $\left(H_I^{[\phi,\infty]}\right)^* = \mathcal{L}_{1,\phi}$.

4. Truncated Riesz transforms on $H_I^{[\phi,\infty]}(\mathbb{R}^n)$ and main result

The Riesz transforms of f are defined by

$$R_j f(x) = c_n \,\text{p.v.}\int \frac{y_j}{|y|^{n+1}} f(x-y)\,dy, \quad j = 1,\cdots,n,$$

where

$$c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}.$$

Let

$$k(x) = \begin{cases} C_n \dfrac{1}{|x|^{n-2}} & n \geq 3, \\ C_2 \log \dfrac{1}{|x|}, & n = 2, \end{cases}$$

where

$$C_n = \Gamma(n/2)(2(n-2)\pi^{n/2})^{-1}, \quad C_2 = (2\pi)^{-1}.$$

Then $-\Delta k = \delta$.

It is known that

$$R_j R_k f(x) = \text{p.v.} \int (\partial_j \partial_k k)(y) f(x-y)\, dy - \delta_{j,k} \frac{1}{n} f(x),$$

for $j, k = 1, \cdots, n$, and

$$\sum_j {R_j}^2 f = -f.$$

Let $\psi \in C^\infty(\mathbb{R}^n)$ be a radial function with $0 \leq \psi \leq 1$, $\psi(x) = 0$ for $|x| \leq 1$, and $\psi(x) = 1$ for $|x| \geq 2$. We set $\lambda = 1 - \psi$. For $0 < \epsilon < 1/2$ we define $\psi_\epsilon(x) = \psi(x/\epsilon)$, $\lambda_\epsilon(x) = \lambda(\epsilon x)$, and $k_\epsilon = \psi_\epsilon \lambda_\epsilon k$ so that $\operatorname{supp} k_\epsilon \subset \{x : \epsilon \leq |x| \leq 2/\epsilon\}$.

Definition 4.1 ($R^\epsilon_{i,j}$). Let $1 \leq i, j \leq n$. For $0 < \epsilon < 1/4$, the operators $R^\epsilon_{i,j}$ are defined by $R^\epsilon_{i,j} f = \partial_i \partial_j k_\epsilon * f$ for $f \in \mathcal{S}'$.

We consider the following condition.

$$\text{(4.1)} \qquad \begin{cases} \displaystyle\int_1^\infty \frac{\phi(t)}{t^2}\, dt < \infty, & \text{if } n \geq 3, \\ \displaystyle\int_1^\infty \frac{\phi(t)\log(1+t)}{t^2}\, dt < \infty, & \text{if } n = 2. \end{cases}$$

Theorem 4.1. *Assume that ϕ satisfies* (4.1). *If $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then*

$$\lim_{\epsilon \to 0} R^\epsilon_{i,j} \varphi = R_i R_j \varphi \quad \text{in } H_I^{[\phi,\infty]}.$$

In particular, $\lim_{\epsilon \to 0} (-\Delta) k_\epsilon * \varphi = \varphi$ *in* $H_I^{[\phi,\infty]}$.

Using the duality $\left(H_I^{[\phi,\infty]}\right)^* = \mathcal{L}_{1,\phi}$ and the equality

$$\lim_{\epsilon \to 0} \left\langle \sum_{j=1}^n R^\epsilon_{i,j} \partial_j f, \varphi \right\rangle = \lim_{\epsilon \to 0} \langle f, (-\Delta) k_\epsilon * \partial_i \varphi \rangle = \langle f, \partial_i \varphi \rangle$$

for all $\varphi \in \mathcal{S}$, we have the following.

Corollary 4.2. *Assume that ϕ satisfies* (4.1). *For $f \in \mathcal{L}_{1,\phi}$,*

$$\lim_{\epsilon \to 0} \sum_{j=1}^{n} R_{i,j}^{\epsilon} \partial_j f = -\partial_i f \quad in \quad \mathcal{S}'.$$

5. Proof of the main result

To prove Theorem 4.1 we state two lemmas.

Lemma 5.1. *Let ℓ be a continuous decreasing function from $[0,\infty)$ to $(0,\infty)$ such that $\ell(r)r^{\theta}$ is almost increasing for some $\theta < 1$ and that*

$$\int_1^{\infty} \frac{\phi(t)}{t^2 \ell(t)}\, dt < \infty.$$

Define

$$w(x) = (1+|x|)^{n+1} \ell(|x|) \quad for \quad x \in \mathbb{R}^n.$$

If a function f satisfies

$$wf \in L^{\infty} \quad and \quad \int f = 0, \tag{5.1}$$

then $f \in H_I^{[\phi,\infty]}$. Moreover, there exist a constant $C > 0$ such that

$$\|f\|_{H_I^{[\phi,\infty]}} \le C \|wf\|_{\infty}, \tag{5.2}$$

where C is independent of f.

Lemma 5.2. *Let ℓ be a continuous decreasing function from $[0,\infty)$ to $(0,\infty)$ such that $\ell(r) \ge (1+r)^{-n-1}$ and that*

$$\lim_{r \to \infty} \ell(r) = 0 \text{ if } n \ge 3, \quad \lim_{r \to \infty} \ell(r) \log r = 0 \text{ if } n = 2.$$

Define

$$w(x) = (1+|x|)^{n+1} \ell(|x|) \quad for \quad x \in \mathbb{R}^n.$$

If $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then

$$\lim_{\epsilon \to 0} \|(R_{i,j}^{\epsilon} \varphi - R_i R_j \varphi) w\|_{\infty} = 0.$$

Proof of Theorem 4.1. If (4.1) holds, then there exists a continuous decreasing function m such that $\lim_{r\to\infty} m(r) = 0$ and that

$$\begin{cases} \displaystyle\int_1^{\infty} \frac{\phi(t)}{t^2 m(t)}\, dt < \infty, & \text{if } n \ge 3, \\ \displaystyle\int_1^{\infty} \frac{\phi(t)\log(1+t)}{t^2 m(t)}\, dt < \infty, & \text{if } n = 2. \end{cases}$$

Actually, if $\int_1^\infty F(t)\,dt < \infty$, $F(t) = \phi(t)/t^2$ or $\phi(t)\log(1+r)/t^2$, then we can take a positive increasing sequence $\{r_j\}$ and a continuous decreasing function m such that

$$\int_{r_j}^\infty F(t)\,dt \le \frac{1}{j^3}, \quad \text{for} \quad j = 1, 2, \cdots,$$

and

$$m(t) \ge \frac{1}{j} \quad \text{for} \quad r_j \le t \le r_{j+1}.$$

Then

$$\int_{r_1}^\infty \frac{F(t)}{m(t)}\,dt = \sum_{j=1}^\infty \int_{r_j}^{r_{j+1}} \frac{F(t)}{m(t)}\,dt \le \sum_{j=1}^\infty \frac{1}{j^2} < \infty.$$

We may assume that $m(r)r^\nu$ is almost increasing for some small $\nu > 0$. Let ℓ be a continuous decreasing function from $[0,\infty)$ to $(0,\infty)$ such that, for $r \ge 1$,

$$\ell(r) = \begin{cases} m(r), & \text{if } n \ge 3, \\ m(r)/\log(1+r), & \text{if } n = 2. \end{cases}$$

Then ℓ satisfies the assumption of both Lemmas 5.1 and 5.2.

Using the following relations,

$$wf \in L^\infty \quad \text{and} \quad \int f = 0, \quad \overset{\text{Lemma4.1}}{\Longrightarrow} \quad \|f\|_{H_I^{[\phi,\infty]}} \le C\|wf\|_\infty;$$

$$\varphi \in \mathcal{S} \text{ and } \int \varphi = 0 \quad \overset{\text{Lemma4.2}}{\Longrightarrow} \quad \lim_{\epsilon \to 0} \|(R^\epsilon_{i,j}\varphi - R_iR_j\varphi)w\|_\infty = 0;$$

we have that, if $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then

$$\|R^\epsilon_{i,j}\varphi - R_iR_j\varphi\|_{H_I^{[\phi,\infty]}} \le C\|(R^\epsilon_{i,j}\varphi - R_iR_j\varphi)w\|_\infty \to 0,$$

as $\epsilon \to 0$. □

6. Application

Let $n \ge 2$. We are concerned with the uniqueness of solutions for the Navier-Stokes equation,

$$u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \tag{6.1}$$

$$\operatorname{div} u = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \tag{6.2}$$

with initial data $u|_{t=0} = u_0$, where $u = u(t,x) = (u_1(t,x), \cdots, u_n(t,x))$ and $p = p(t,x)$ stand for the unknown velocity vector field of the fluid and its pressure field respectively, while $u_0 = u_0(x) = (u_0^1(x), \cdots, u_0^n(x))$ is the given initial velocity vector field.

It is well known (see [2]) that for initial data $u_0 \in L^\infty(\mathbb{R}^n)$ the equations (6.1), (6.2) admit a unique time-local (regular) solution u with

$$p = \sum_{i,j=1}^{n} R_i R_j u_i u_j.$$

In this report, following J. Kato [3], by "a solution in the distribution sense" we mean a weak solution in the following sense.

Definition 6.1. We call (u,p) the solution of the Navier-Stokes equations (6.1), (6.2) on $(0,T)\times\mathbb{R}^n$ with initial data u_0 in the distribution sense if (u,p) satisfy $\operatorname{div} u = 0$ in $\mathcal{S}'$ for a.e. t and

$$\begin{aligned}(6.3)\quad \int_0^T \Big\{ &\langle u(s), \partial_s \Phi(s)\rangle + \langle u(s), \Delta\Phi(s)\rangle + \langle (u\times u)(s), \nabla\Phi(s)\rangle \\ &+ \langle p(s), \operatorname{div}\Phi(s)\rangle \Big\}\, ds = -\langle u_0, \Phi(0)\rangle\end{aligned}$$

for $\Phi \in C^1([0,T]\times\mathbb{R}^n)$ satisfying $\Phi(s,\cdot)\in\mathcal{S}(\mathbb{R}^n)$ for $0 \le s \le T$, and $\Phi(T,\cdot)\equiv 0$, where $\langle (u\times u), \nabla\Phi\rangle = \sum_{i,j=1}^n \langle u_i u_j, \partial_i \Phi_j\rangle$. Here $\mathcal{S}$ denotes the space of rapidly decreasing functions in $\mathbb{R}^n$ and $\mathcal{S}'$ denotes the space of tempered distributions in the sense of Schwartz. The space $\mathcal{S}'$ is the topological dual of $\mathcal{S}$ and its canonical pairing is denoted by $\langle , \rangle$.

J. Kato [3] proved the following uniqueness theorem.

Theorem 6.1 (J. Kato [3]). *Let $u_0 \in L^\infty$ with $\operatorname{div} u_0 = 0$. Suppose that (u,p) is the solution in the distribution sense satisfying*

$$(6.4)\qquad u \in L^\infty((0,T)\times\mathbb{R}^n), \quad p \in L^1_{\mathrm{loc}}((0,T); \mathrm{BMO}).$$

Then $(u,\nabla p)$ is uniquely determined by the initial data u_0. Moreover, $\nabla p = \sum_{i,j=1}^n \nabla R_i R_j u^i u^j$ in S' for a.e. t.

On the other hand, Galdi and Maremonti [1] showed that if u and ∇u are bounded in $(0,T)\times\mathbb{R}^3$, then the uniqueness of classical solutions holds provided that for some $C > 0$ and some $\epsilon > 0$ the inequality

$$(6.5)\qquad |p(t,x)| \le C(1+|x|)^{1-\epsilon}$$

holds. See also [9] and [4]. The assumption (6.4) does not imply (6.5).

To prove Theorem 6.1, Kato [3] used the duality $(H^1)^* = \mathrm{BMO}$ and the following fact: If $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then

$$\lim_{\epsilon \to 0} R^{\epsilon}_{i,j}\varphi = R_i R_j \varphi \quad \textit{in } H^1.$$

The duality $\left(H_I^{[\phi,\infty]}\right)^* = \mathcal{L}_{1,\phi}$ is known and we have proved in Theorem 4.1 that if $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then

$$\lim_{\epsilon \to 0} R^{\epsilon}_{i,j}\varphi = R_i R_j \varphi \quad \textit{in } H_I^{[\phi,\infty]}.$$

Then we have the following.

Theorem 6.2. *Assume that* $\phi \in \mathcal{G}$ *satisfies* (4.1). *Let* $u_0 \in L^\infty$ *with* $\operatorname{div} u_0 = 0$. *Suppose that* (u,p) *is the solution of* (6.1), (6.2) *in the distribution sense satisfying*

$$u \in L^\infty((0,T) \times \mathbb{R}^n), \quad p \in L^1_{\mathrm{loc}}((0,T); \mathcal{L}_{1,\phi}). \tag{6.6}$$

Then $(u, \nabla p)$ *is uniquely determined by the initial data* u_0. *Moreover,* $\nabla p = \sum_{i,j=1}^n \nabla R_i R_j u^i u^j$ *in* $\mathcal{S}'$ *for a.e.* t.

For example, let

$$\phi(r) = \begin{cases} r^{-n} & \text{for} \quad 0 < r < 1, \\ r(\log(1+r))^{-\beta} & \text{for} \quad r \geq 1, \end{cases} \tag{6.7}$$

where $\beta > 1$ if $n \geq 3$ and $\beta > 2$ if $n = 2$. In this case

$$\mathcal{L}_{1,\phi} \supset L^1 \cup \mathrm{BMO}$$

and $\mathcal{L}_{1,\phi}$ contains functions f such that

$$|f(x)| \leq C\phi(1+|x|) = C(1+|x|)(\log(2+|x|))^{-\beta} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Therefore, our result is an extension of both Kato's theorem and the result of Galdi and Maremonti. Note that, if $\beta = 0$, then the uniqueness fails (see [2]).

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