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Convergence of some truncated Riesz transforms on predual of generalized Campanato spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations (The geometrical structure of Banach spaces and Function spaces and its applications)

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Convergence of some truncated Riesz transforms on predual of generalized Campanato spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations.

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1. INTRODUCTION

This is an announcement of our recent work [8]. In [6] the first author introduced predual of generalized Campanato spaces. In this report, we show convergence of some truncated Riesz transforms on the function spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations. Our uniqueness theorem is an extension of Kato’s [3].

2. GENERALIZED CAMPANATO SPACE $L_{p,\phi}(\mathbb{R}^{n})$

Let $1 \leq p < \infty$ and $\phi : (0, \infty) \rightarrow (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(r)$. The function spaces $L_{p,\phi} = L_{p,\phi}(\mathbb{R}^{n})$ is defined to be the sets of all $f$ such that $\|f\|_{L_{p,\phi}} < \infty$, where

$$\|f\|_{L_{p,\phi}} = \sup_{B} \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_{B} |f(x) - f_{B}|^{p} dx \right)^{1/p},$$

$$f_{B} = \frac{1}{|B|} \int_{B} f(x) dx.$$

Then $L_{p,\phi}$ is a Banach space modulo constants with the norm $\|f\|_{L_{p,\phi}}$. If $p = 1$ and $\phi \equiv 1$, then $L_{1,\phi} =$ BMO. It is known that if $\phi(r) = r^{\alpha}$, $0 < \alpha \leq 1$, then $L_{p,\phi} = \text{Lip}_{\alpha}$, and, if $\phi(r) = r^{-n/p}$, $1 \leq p < \infty$, then $L_{p,\phi} = L^{p}$.

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A function $\phi : (0, \infty) \to (0, \infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that
\[
C^{-1} \leq \frac{\phi(r)}{\phi(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.
\]
A function $\phi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that
\[
\phi(r) \leq C\phi(s) \quad (\phi(r) \geq C\phi(s)) \quad \text{for} \quad r \leq s.
\]

**Lemma 2.1.** Assume that $\phi(r)r^{n/p}$ is almost increasing and that $\phi(r)/r$ is almost decreasing. Then $\phi$ satisfies the doubling condition and
\[
\|f\|_{\mathcal{L}_{p,\phi}} \leq C\left(\|(1 + |x|^{n+1})f\|_{\infty} + \|\nabla f\|_{\infty}\right).
\]
That is $S \subset \mathcal{L}_{p,\phi}$.

**Proof.** Let $B = B(z, r)$.

**Case 1:** $r < 1$: In this case $r \lesssim \phi(r)$. Then
\[
|f(x) - f(y)| \lesssim r\|\nabla f\|_{\infty} \lesssim \phi(r)\|\nabla f\|_{\infty}, \quad x, y \in B.
\]

\[
\left(\frac{1}{|B|} \int_{B} |f(x) - f_B|^p\, dx\right)^{1/p} \lesssim \sup_{x, y \in B} |f(x) - f(y)| \lesssim \phi(r)\|\nabla f\|_{\infty}.
\]

**Case 2:** $1 \leq r$: In this case $1 \lesssim \phi(r)r^{n/p}$ and
\[
|f(x)| \leq \frac{\|(1 + |x|^{n+1})f\|_{\infty}}{1 + |x|^{n+1}}, \quad \left(\int |f(x)|^p\, dx\right)^{1/p} \lesssim \|(1 + |x|^{n+1})f\|_{\infty}.
\]
Then
\[
\left(\frac{1}{|B|} \int_{B} |f(x) - f_B|^p\, dx\right)^{1/p} \leq 2\left(\frac{1}{|B|} \int_{B} |f(x)|^p\, dx\right)^{1/p} \lesssim \frac{\|(1 + |x|^{n+1})f\|_{\infty}}{|B|^{1/p}} \lesssim \phi(r)\|(1 + |x|^{n+1})f\|_{\infty}.
\]

3. $H_{I}^{[\phi,\infty]}(\mathbb{R}^n)$, Predual of $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$

The space $H_{U}^{[\phi,q]}$ was introduced in [6], which is a generalization of Hardy space. The duality $\left(H_{U}^{[\phi,q]}\right)^{*} = \mathcal{L}_{q',\phi}$ also proved in [6].

In this talk we recall the definition of $H_{I}^{[\phi,\infty]}(\mathbb{R}^n)$, which is a special case of $H_{U}^{[\phi,\varphi]}$.

In what follows, we always assume that $\phi(r)r^n$ is almost increasing and that $\phi(r)/r$ is almost decreasing.
Definition 3.1 ([φ, ∞]-atom). A function a on \(\mathbb{R}^n\) is called a [φ, ∞]-atom if there exists a ball B such that

(i) \(\text{supp} \ a \subset B\),
(ii) \(\|a\|_\infty \leq \frac{1}{|B|\phi(B)}\),
(iii) \(\int_{\mathbb{R}^n} a(x) \, dx = 0\).

where \(\|a\|_\infty\) is the \(L^\infty\) norm of a. We denote by \(A[\phi, \infty]\) the set of all [φ, ∞]-atoms.

If a is a [φ, ∞]-atom and a ball B satisfies (i)-(iii), then, for \(g \in \mathcal{L}_{1,\phi}\),

\[
\left| \int_{\mathbb{R}^n} a(x)g(x) \, dx \right| = \left| \int_B a(x)(g(x) - g_B) \, dx \right| \\
\leq \|a\|_\infty \int_B |g(x) - g_B| \, dx \\
\leq \frac{1}{\phi(B)} \frac{1}{|B|} \int_B |g(x) - g_B| \, dx \\
\leq \|g\|_{\mathcal{L}_{1,\phi}}.
\]

That is, the mapping \(g \mapsto \int_{\mathbb{R}^n} ag \, dx\) is a bounded linear functional on \(\mathcal{L}_{1,\phi}\) with norm not exceeding 1. Hence a is also in \(S'\), since \(S \subset \mathcal{L}_{1,\phi}\).

Definition 3.2 \(H_{I}^{\{\phi, \infty\}}\). The space \(H_{I}^{\{\phi, \infty\}} \subset (\mathcal{L}_{1,\phi})^*\) is defined as follows:

\(f \in H_{I}^{\{\phi, \infty\}}\) if and only if there exist sequences \(\{a_j\} \subset A[\phi, \infty]\) and positive numbers \(\{\lambda_j\}\) such that

\[
f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{1,\phi})^* \text{ and } \sum_j \lambda_j < \infty.
\]

In general, the expression (3.1) is not unique. Let

\[
\|f\|_{H_{I}^{\{\phi, \infty\}}} = \inf \left\{ \sum_j \lambda_j \right\},
\]

where the infimum is taken over all expressions as in (3.1). Then \(H_{I}^{\{\phi, \infty\}}\) is a Banach space equipped with the norm \(\|f\|_{H_{I}^{\{\phi, \infty\}}}\) and \((H_{I}^{\{\phi, \infty\}})^* = \mathcal{L}_{1,\phi}\).

4. TRUNCATED RIESZ TRANSFORMS ON \(H_{I}^{\{\phi, \infty\}}(\mathbb{R}^n)\) AND MAIN RESULT

The Riesz transforms of \(f\) are defined by

\[
R_j f(x) = c_n \text{ p.v.} \int \frac{y_j}{|y|^{n+1}} f(x - y) \, dy, \quad j = 1, \ldots, n,
\]
where \[ c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}. \]

Let \[ k(x) = \begin{cases} \frac{C_n}{|x|^{n-2}} & n \geq 3, \\ C_2 \log \frac{1}{|x|} & n = 2, \end{cases} \]

where \[ C_n = \Gamma(n/2)(2(n-2)\pi^{n/2})^{-1}, \quad C_2 = (2\pi)^{-1}. \]

Then \(-\Delta k = \delta\).

It is known that \[ R_j R_k f(x) = \text{p.v.} \int (\partial_j \partial_k k)(y)f(x-y)dy - \delta_{j,k} \frac{1}{r\iota}f(x), \]

for \(j, k = 1, \ldots, n\), and \[ \sum_j R_j^2 f = -f. \]

Let \(\psi \in C^\infty(\mathbb{R}^n)\) be a radial function with \(0 \leq \psi \leq 1\), \(\psi(x) = 0\) for \(|x| \leq 1\), and \(\psi(x) = 1\) for \(|x| \geq 2\). We set \(\lambda = 1 - \psi\). For \(0 < \epsilon < 1/2\) we define \(\psi_\epsilon(x) = \psi(x/\epsilon)\), \(\lambda_\epsilon(x) = \lambda(\epsilon x)\), and \(k_\epsilon = \psi_\epsilon \lambda_\epsilon k\) so that \(\text{supp} k_\epsilon \subset \{x : \epsilon \leq |x| \leq 2/\epsilon\}\).

**Definition 4.1 \((R_{i,j}^\epsilon)\).** Let \(1 \leq i, j \leq n\). For \(0 < \epsilon < 1/4\), the operators \(R_{i,j}^\epsilon\) are defined by \(R_{i,j}^\epsilon f = \partial_i \partial_j k_\epsilon * f\) for \(f \in S'\).

We consider the following condition.

\[
\left\{ \begin{array}{ll}
\int_1^\infty \frac{\phi(t)}{t^2} \, dt < \infty, & \text{if } n \geq 3, \\
\int_1^\infty \frac{\phi(t) \log(1+t)}{t^2} \, dt < \infty, & \text{if } n = 2.
\end{array} \right.
\] (4.1)

**Theorem 4.1.** Assume that \(\phi\) satisfies (4.1). If \(\varphi \in S\) and \(\int \varphi = 0\), then \(\lim_{\epsilon \to 0} R_{i,j}^\epsilon \varphi = R_i R_j \varphi\) in \(H^1_{[\phi,\infty]}\).

In particular, \(\lim_{\epsilon \to 0} (-\Delta) k_\epsilon * \varphi = \varphi\) in \(L^1_{[\phi,\infty]}\).

Using the duality \(\left(H^1_{[\phi,\infty]}\right)^* = \mathcal{L}_{1,\phi}\) and the equality

\[
\lim_{\epsilon \to 0} \left( \sum_{j=1}^n R_{i,j}^\epsilon \partial_j f, \varphi \right) = \lim_{\epsilon \to 0} \langle f, (-\Delta) k_\epsilon \partial_i \varphi \rangle = \langle f, \partial_i \varphi \rangle
\]

for all \(\varphi \in S\), we have the following.
Corollary 4.2. Assume that $\phi$ satisfies (4.1). For $f \in L_{1,\phi}$,
\[
\lim_{\epsilon \to 0} \sum_{j=1}^{n} R_{t,\epsilon} \partial_{j} f = -\partial_{i} f \quad \text{in} \quad S'.
\]

5. PROOF OF THE MAIN RESULT

To prove Theorem 4.1 we state two lemmas.

Lemma 5.1. Let $\ell$ be a continuous decreasing function from $[0, \infty)$ to $(0, \infty)$ such that $\ell(r)r^\theta$ is almost increasing for some $\theta < 1$ and that
\[
\int_{1}^{\infty} \frac{\phi(t)}{t^2 \ell(t)} \, dt < \infty.
\]
Define
\[
w(x) = (1 + |x|)^{n+1} \ell(|x|) \quad \text{for} \quad x \in \mathbb{R}^n.
\]
If a function $f$ satisfies
\[
w f \in L^\infty \quad \text{and} \quad \int f = 0,
\]
then $f \in H^{[\phi,\infty]}_{1}$. Moreover, there exist a constant $C > 0$ such that
\[
\|f\|_{H^{[\phi,\infty]}_{1}} \leq C \|wf\|_{\infty},
\]
where $C$ is independent of $f$.

Lemma 5.2. Let $\ell$ be a continuous decreasing function from $[0, \infty)$ to $(0, \infty)$ such that $\ell(r) \geq (1 + r)^{-n-1}$ and that
\[
\lim_{r \to \infty} \ell(r) = 0 \quad \text{if} \quad n \geq 3, \quad \lim_{r \to \infty} \ell(r) \log r = 0 \quad \text{if} \quad n = 2.
\]
Define
\[
w(x) = (1 + |x|)^{n+1} \ell(|x|) \quad \text{for} \quad x \in \mathbb{R}^n.
\]
If $\varphi \in S$ and $\int \varphi = 0$, then
\[
\lim_{\epsilon \to 0} \|(R_{t,\epsilon} \varphi - R_{t} R_{\epsilon} \varphi)w\|_{\infty} = 0.
\]

Proof of Theorem 4.1. If (4.1) holds, then there exists a continuous decreasing function $m$ such that $\lim_{r \to \infty} m(r) = 0$ and that
\[
\left\{
\begin{array}{ll}
\int_{1}^{\infty} \frac{\phi(t)}{t^2 m(t)} \, dt < \infty, & \text{if} \quad n \geq 3, \\
\int_{1}^{\infty} \frac{\phi(t) \log(1 + t)}{t^2 m(t)} \, dt < \infty, & \text{if} \quad n = 2.
\end{array}
\right.
\]
Actually, if $\int_{1}^{\infty} F(t) dt < \infty$, $F(t) = \phi(t)/t^2$ or $\phi(t) \log(1 + r)/t^2$, then we can take a positive increasing sequence $\{r_j\}$ and a continuous decreasing function $m$ such that
\[
\int_{r_j}^{\infty} F(t) dt \leq \frac{1}{j^3}, \quad \text{for } j = 1, 2, \ldots ,
\]
and
\[
m(t) \geq \frac{1}{j} \quad \text{for } r_j \leq t \leq r_{j+1}.
\]
Then
\[
\int_{r_1}^{\infty} \frac{F(t)}{m(t)} dt = \sum_{j=1}^{\infty} \int_{r_j}^{r_{j+1}} \frac{F(t)}{m(t)} dt \leq \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.
\]
We may assume that $m(r)r^\nu$ is almost increasing for some small $\nu > 0$. Let $\ell$ be a continuous decreasing function from $[0, \infty)$ to $(0, \infty)$ such that, for $r \geq 1$,
\[
\ell(r) = \begin{cases} 
m(r), & \text{if } n \geq 3, \\
m(r)/\log(1 + r), & \text{if } n = 2. 
\end{cases}
\]
Then $\ell$ satisfies the assumption of both Lemmas 5.1 and 5.2.

Using the following relations,
\[
w f \in L^\infty \quad \text{and} \quad \int f = 0, \quad \text{Lemma4.1} \quad \|f\|_{H_\ell^1, \infty} \leq C\|wf\|_\infty; \\
\varphi \in S \quad \text{and} \quad \int \varphi = 0 \quad \text{Lemma4.2} \quad \lim_{\epsilon \to 0} \|\ell^{\epsilon}(t)\varphi - R_t e^{\epsilon(t)} \varphi\|_\infty = 0;
\]
we have that, if $\varphi \in S$ and $\int \varphi = 0$, then
\[
\|\ell^{\epsilon}(t)\varphi - R_t e^{\epsilon(t)} \varphi\|_{\ell^{\infty}, \infty} \leq C\|\ell^{\epsilon}(t)\varphi - R_t e^{\epsilon(t)} \varphi\|_\infty \to 0,
\]
as $\epsilon \to 0$. \qed

6. Application

Let $n \geq 2$. We are concerned with the uniqueness of solutions for the Navier-Stokes equation,
\[
(6.1) \quad u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad \text{in } (0, T) \times \mathbb{R}^n,
\]
\[
(6.2) \quad \text{div } u = 0 \quad \text{in } (0, T) \times \mathbb{R}^n,
\]
with initial data $u|_{t=0} = u_0$, where $u = u(t, x) = (u_1(t, x), \cdots , u_n(t, x))$ and $p = p(t, x)$ stand for the unknown velocity vector field of the fluid and its pressure field respectively, while $u_0 = u_0(x) = (u_0^1(x), \cdots , u_0^n(x))$ is the given initial velocity vector field.
It is well known (see [2]) that for initial data $u_0 \in L^\infty(\mathbb{R}^n)$ the equations (6.1), (6.2) admit a unique time-local (regular) solution $u$ with

$$p = \sum_{i,j=1}^{n} R_i R_j u_i u_j.$$

In this report, following J. Kato [3], by “a solution in the distribution sense” we mean a weak solution in the following sense.

**Definition 6.1.** We call $(u, p)$ the solution of the Navier-Stokes equations (6.1), (6.2) on $(0, T) \times \mathbb{R}^n$ with initial data $u_0$ in the distribution sense if $(u, p)$ satisfy

$$\text{div} u = 0 \quad \text{in} \quad S' \quad \text{for a.e.} \quad t \quad \text{and}$$

$$\int_0^T \left\{ \langle u(s), \partial_s \Phi(s) \rangle + \langle u(s), \Delta \Phi(s) \rangle + \langle (u \times u)(s), \nabla \Phi(s) \rangle + \langle p(s), \text{div} \Phi(s) \rangle \right\} ds = -\langle u_0, \Phi(0) \rangle$$

for $\Phi \in C^1([0, T] \times \mathbb{R}^n)$ satisfying $\Phi(s, \cdot) \in S(\mathbb{R}^n)$ for $0 \leq s \leq T$, and $\Phi(T, \cdot) \equiv 0$, where $\langle (u \times u), \nabla \Phi \rangle = \sum_{i,j=1}^{n} \langle u_i u_j, \partial_i \Phi_j \rangle$. Here $S$ denotes the space of rapidly decreasing functions in $\mathbb{R}^n$ and $S'$ denotes the space of tempered distributions in the sense of Schwartz. The space $S'$ is the topological dual of $S$ and its canonical pairing is denoted by $\langle ., . \rangle$.

J. Kato [3] proved the following uniqueness theorem.

**Theorem 6.1 (J. Kato [3]).** Let $u_0 \in L^\infty$ with $\text{div} u_0 = 0$. Suppose that $(u, p)$ is the solution in the distribution sense satisfying

$$u \in L^\infty((0, T) \times \mathbb{R}^n), \quad p \in L^1_{\text{loc}}((0, T); \text{BMO}).$$

Then $(u, \nabla p)$ is uniquely determined by the initial data $u_0$. Moreover, $\nabla p = \sum_{i,j=1}^{n} \nabla R_i R_j u_i u_j$ in $S'$ for a.e. $t$.

On the other hand, Galdi and Maremonti [1] showed that if $u$ and $\nabla u$ are bounded in $(0, T) \times \mathbb{R}^3$, then the uniqueness of classical solutions holds provided that for some $C > 0$ and some $\epsilon > 0$ the inequality

$$|p(t, x)| \leq C(1 + |x|)^{1-\epsilon}$$

holds. See also [9] and [4]. The assumption (6.4) does not imply (6.5).
To prove Theorem 6.1, Kato [3] used the duality \((H^1)^* = \text{BMO}\) and the following fact: If \(\varphi \in S\) and \(\int \varphi = 0\), then
\[
\lim_{\epsilon \to 0} R^\epsilon_{i,j} \varphi = R_i R_j \varphi \quad \text{in} \quad H^1.
\]
The duality \((H^1_{I}^{\phi,\infty})^* = \mathcal{L}_{1,\phi}\) is known and we have proved in Theorem 4.1 that if \(\varphi \in S\) and \(\int \varphi = 0\), then
\[
\lim_{\epsilon \to 0} R^\epsilon_{i,j} \varphi = R_i R_j \varphi \quad \text{in} \quad H^1_{I}^{\phi,\infty}.
\]
Then we have the following.

**Theorem 6.2.** Assume that \(\phi \in \mathcal{G}\) satisfies (4.1). Let \(u_0 \in L^\infty\) with \(\text{div } u_0 = 0\). Suppose that \((u, p)\) is the solution of (6.1), (6.2) in the distribution sense satisfying
\[
(6.6) \quad u \in L^\infty((0, T) \times \mathbb{R}^n), \quad p \in L^1_{\text{loc}}((0, T); \mathcal{L}_{1,\phi}).
\]
Then \((u, \nabla p)\) is uniquely determined by the initial data \(u_0\). Moreover, \(\nabla p = \sum_{i,j=1}^n \nabla R_i R_j u^i u^j\) in \(S'\) for a.e. \(t\).

For example, let
\[
(6.7) \quad \phi(r) = \begin{cases} 
  r^{-n} & \text{for } 0 < r < 1, \\
  r(\log(1 + r))^{-\beta} & \text{for } r \geq 1,
\end{cases}
\]
where \(\beta > 1\) if \(n \geq 3\) and \(\beta > 2\) if \(n = 2\). In this case
\[
\mathcal{L}_{1,\phi} \supset L^1 \cup \text{BMO}
\]
and \(\mathcal{L}_{1,\phi}\) contains functions \(f\) such that
\[
|f(x)| \leq C \phi(1 + |x|) = C(1 + |x|)(\log(2 + |x|))^{-\beta} \quad \text{for } x \in \mathbb{R}^n.
\]
Therefore, our result is an extension of both Kato’s theorem and the result of Galdi and Maremonti. Note that, if \(\beta = 0\), then the uniqueness fails (see [2]).

**REFERENCES**


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