MAPS BETWEEN UNIFORM ALGEBRAS WHICH PRESERVE THE NORMS OF MONOMIALS NON-SYMMETRICALLY (The geometrical structure of Banach spaces and Function spaces and its applications)

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1. Introduction

Let $C(X)$ be the set of all complex-valued continuous functions on a compact Hausdorff space $X$ and $\|f\|_\infty = \sup_{x \in X} |f(x)|$, the supremum norm on $X$ for $f \in C(X)$. Then $C(X)$ is a Banach algebra with pointwise multiplication and the supremum norm. The subset $A$ of $C(X)$ is said to be a uniform algebra on $X$ if $A$ is a closed subalgebra of $C(X)$ which separates the points of $X$ and contains the constant functions. Let $A$ and $B$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. For $f \in A$, let $\sigma(f)$ be the spectrum of $f$. Recall that $f(X)$ is a subset of $\sigma(f)$ and $\|f\|_\infty$ equals the spectral radius of $f$.

Molnár [10] showed the following:

**Theorem 1.** (Molnár [10]) If $X$ is first-countable and $T$ is a surjection from $C(X)$ onto itself with $\sigma(T(f)T(g)) = \sigma(fg)$ for all $f, g \in C(X)$, then $T/T(1)$ is an algebra isomorphism.

Rao and Roy [11] extended this result (see also [2, 3, 6, 7]). Most recently, Hatori, Hino, Miura and Oka [4] generalized their results. In particular, they showed the following:

**Theorem 2.** (Hatori, Hino, Miura and Oka [4, Theorem 1.1]) Let $\sigma_\pi(f) = \{f(x) : x \in X, |f(x)| = \|f\|_\infty\}$ for $f \in A$. If a surjection $T : A \rightarrow B$ satisfies $\sigma_\pi(T(f)^mT(g)^n) \subset \sigma_\pi(f^m g^n)$ for some fixed positive integers $m, n$ and all $f, g \in A$, then there exists a real-algebra isomorphism $\tilde{T}$ such that $\tilde{T}(f)^d = (T(f)/T(1))^d$ for every $f \in A$, where $d$ is the greatest common divisor of $m$ and $n$.

Hatori, Miura and Takagi [3, Corollary 7.5], and Luttman and Lambert [8] independently showed the following:

**Theorem 3.** (Hatori, Miura and Takagi [3, Corollary 7.5], and Luttman and Lambert [8]) If a surjection $T : A \rightarrow B$ satisfies $\|T(f)T(g) - \alpha\|_\infty = \|fg - \alpha\|_\infty$ for some fixed non-zero complex number $\alpha$ and all $f, g \in A$, then $T/T(1)$ is a real-algebra isomorphism.

Note that, for some fixed complex number $\alpha$ and $f, g \in A$, $\sigma(f) = \sigma(g)$ if and only if $\sigma(f - \alpha) = \sigma(g - \alpha)$, which implies $\|f - \alpha\|_\infty = \|g - \alpha\|_\infty$. Hence their result is a generalization of Theorem 1 (see also [5, 6, 9]). We denote by $A^{-1}$ the set of invertible
elements of $A$. Let $\hat{f}$ be the Gelfand transform of $f \in A$, $M_A$ the maximal ideal space of $A$ and $\overline{\cdot}$ the complex conjugate. Our main result is the following:

**Theorem 4.** [12, Theorem 1.2] Let $m, n$ be positive integers and $\alpha$ a non-zero complex number. Suppose that $S_A, S_B$ are subsets of $A, B$ that contain $A^{-1}, B^{-1}$ respectively. If $T : S_A \to S_B$ is a surjection such that

\[ \|T(f)^m T(g)^n - \alpha\|_{\infty} = \|f^m g^n - \alpha\|_{\infty} \]

for all $f, g \in S_A$, then there exist a real-algebra isomorphism $\overline{T} : A \to B$, a clopen subset $\mathcal{K}$ of $M_B$ and a homeomorphism $\Phi : M_B \to M_A$ such that

\[ \overline{T}(f) = \begin{cases} \hat{f} \circ \Phi & \text{on } \mathcal{K} \\ \overline{\hat{f} \circ \Phi} & \text{on } M_B \setminus \mathcal{K} \end{cases} \]

for every $f \in A$ and $\overline{T}(f)^d = (T(f)/T(1))^d$ for every $f \in S_A$, where $d$ is the greatest common divisor of $m$ and $n$.

**2. A proof of Main result**

We denote by $\exp A$ the range of the exponential map on $A$. Let $\sigma_{\pi}(f) = \{f(x) : x \in X, |f(x)| = \|f\|_{\infty}\}$ for $f \in A$ and $P_{\exp A}(x) = \{u \in \exp A : \sigma_{\pi}(u) = \{1\}, u(x) = 1\}$ for $x \in X$. If $\sigma_{\pi}(p) = 1$ for $p \in A$, then $p$ is called a peaking function of $A$. For a peaking function $p$, the set of points on which $p$ takes the value 1 is called the peak set of $p$. A point $x \in X$ is called a weak peak point of $A$ if the set $\{x\}$ equals the intersection of a family of peak sets of $A$. The set $\text{Ch}(A)$ of all weak peak points of $A$ coincides with the Choquet boundary of $A$. It is known that $\text{Ch}(A)$ is a boundary for $A$. In order to prove the main theorem, we will need Lemma 5, 6 and Proposition 7.

**Lemma 5.** (cf. [4, Proposition 2.2]. See also [1, 2, 3, 5, 6, 8, 9, 11].) Let $v \in A^{-1}$ and $x_0 \in \text{Ch}(A)$. If $F$ is a closed subset in $X$ with $x_0 \notin F$, there exists a $u \in P_{\exp A}(x_0)$ such that $\sigma_{\pi}(uv) = \{v(x_0)\}$ and $|uv| < |v(x_0)|$ on $F$.

**Lemma 6.** (cf. [8, Lemma 2.1].) Let $f_1, f_2 \in A$. If $\|f_1 g - 1\|_{\infty} = \|f_2 g - 1\|_{\infty}$ for all $g \in \exp A$, then $f_1 = f_2$.

**Proposition 7.** [12, Proposition 2.6 and 3.2] Suppose that $A_0, B_0$ are subgroups of $A^{-1}, B^{-1}$ that contain $\exp A, \exp B$ respectively. If $S : A_0 \to B_0$ is a surjection such that $S(1) = 1$ and

\[ \|S(f)S(g)^{-1} - 1\|_{\infty} = \|fg^{-1} - 1\|_{\infty} \]

for all $f, g \in A_0$, then there exist a real-algebra isomorphism $\overline{T} : A \to B$, a clopen subset $\mathcal{K}$ of $M_B$ and a homeomorphism $\Phi : M_B \to M_A$ such that

\[ \overline{T}(f) = \begin{cases} \hat{f} \circ \Phi & \text{on } \mathcal{K} \\ \overline{\hat{f} \circ \Phi} & \text{on } M_B \setminus \mathcal{K} \end{cases} \]

for every $f \in A$ and $\overline{T}(f) = S(f)$ for every $f \in A_0$.

**Proof.** We begin by showing that there exists a homeomorphism $\phi$ from $\text{Ch}(B)$ onto $\text{Ch}(A)$ such that

\[ |S(f)(y)| = |f(\phi(y))| \]
for every \( f \in A_0 \) and \( y \in \text{Ch}(B) \) (cf. [3, 4, 5, 9]). For \( y \in \text{Ch}(B) \), let
\[
W_y = \{ f \in B_0 : |f(i)| = 1 = \|f\|_{\infty} \}.
\]
Then, \( \exp B(y) \) is a subset of \( W_y \). For every \( y \in \text{Ch}(B) \), the set \( \cap_{f \in S^{-1}(W_y)}|f|^{-1}(\{1\}) \) is a singleton that belongs to \( \text{Ch}(A) \). If \( \phi(y) \) is the single element, i.e.
\[
\{\phi(y)\} = \cap_{f \in S^{-1}(W_y)}|f|^{-1}(\{1\}),
\]
we can define the mapping \( \phi : y \mapsto \phi(y) \) from \( \text{Ch}(B) \) into \( \text{Ch}(A) \). Then \( \phi : \text{Ch}(B) \to \text{Ch}(A) \) is bijective and satisfies (3). This implies the continuities of \( \phi \) and \( \phi^{-1} \).

Let \( y \in \text{Ch}(B) \) and \( S^1 = \{ z; \text{a complex number with } |z| = 1 \} \). We will show that
\[
S(f) = \begin{cases} \frac{f \circ \phi}{f \circ \phi} & \text{if } y \in K \\ \frac{f \circ \phi}{f \circ \phi} & \text{if } y \in \text{Ch}(B) \setminus K \end{cases}
\]
for every \( f \in A_0 \) (cf. [3, 5, 8, 9]). For every \( \beta \in S^1 \) and \( u \in \exp B(y) \), there exists a \( u \in A_0 \) such that \( S(u) = S(\beta)u \). By (3), we have \( |u(\phi(y))| = 1 \). We also have \( \|S(\beta)u/S(-u(\phi(y)))\|_{\infty} = 1 \). Equation (2) shows that
\[
\left\| \frac{S(\beta)u}{S(-u(\phi(y)))} - \frac{1}{u(\phi(y))} \right\|_{\infty} = 2,
\]
which implies that there exists a \( y' \in \text{Ch}(B) \) with \( S(-u(\phi(y)))(y') = S(\beta)(y')u(y') \). Since \( |u(y')| = 1 \) and \( u \in \exp B(y) \), we obtain \( u(y') = 1 \), so by (3),
\[
2 = \left| \frac{S(-u(\phi(y')))(y')}{S(\beta)(y')} - 1 \right| \leq \left\| \frac{S(-u(\phi(y'))}{S(\beta)} - 1 \right\|_{\infty} \leq \left\| \frac{S(-u(\phi(y'))}{S(\beta)} \right\|_{\infty} + 1 = 2.
\]
Thus, by (2), \( |-u(\phi(y))|^2 - 1 = 2 \), which shows that
\[
S(\phi(y)) = \beta.
\]
Since, by (2) and (3), \( \|S(\beta)uS(-\beta)^{-1} - 1\|_{\infty} = \|S(\beta)uS(-\beta)^{-1}\|_{\infty} + 1 = 2 \), there exists a \( y_{\beta} \in \text{Ch}(B) \) such that \( S(-\beta)(y_{\beta}) = S(\beta)(y_{\beta})u(y_{\beta}) \). Notice that \( |u(y_{\beta})| = 1 \) and \( u \in \exp B(y) \), which implies that
\[
u(y_{\beta}) = 1 \text{ and } S(-\beta)(y_{\beta}) = S(\beta)(y_{\beta}).
\]
Applying Lemma 5 for \( S(1)^{-1} \in B^{-1} \) and equation (6) for \( \beta = 1 \), we obtain \( S(-1)(y) = -1 \) for every \( y \in \text{Ch}(A) \). Thus, by (2), we have \( \|S(\beta) - 1\|_{\infty} = |\beta - 1| \) and \( \|S(\beta) + 1\|_{\infty} = |\beta + 1| \) for every \( \beta \in S^1 \). Since \( |S(\beta)| = 1 \) on \( \text{Ch}(B) \), we obtain \( S(\beta)(\text{Ch}(B)) = \{ \beta, \overline{\beta} \} \) for every \( \beta \in S^1 \). Define
\[
K = \{ y \in \text{Ch}(B) : S(i)(y) = i \}.
\]
Then \( K \) is a clopen subset of \( \text{Ch}(B) \) and the closures in \( Y \) of \( K \) and \( \text{Ch}(B) \setminus K \) are disjoint. Let \( F_0 \) be the closure in \( Y \) of \( K \) or \( \text{Ch}(B) \setminus K \) with \( y \notin F_0 \). Applying Lemma 5 for \( S(i)^{-1} \in B^{-1} \), \( F_0 \subset Y \) and equation (6) for \( \beta = i \), we obtain \( S(-i)(y) = -S(i)(y) \) for every \( y \in \text{Ch}(B) \). Together with equations (2) and (3), this shows that \( \|S(\beta) - S(i)\|_{\infty} = |\beta - i| \) and \( \|S(\beta) + S(i)\|_{\infty} = |\beta + i| \). Hence,
\[
S(\beta)(y) = \begin{cases} \beta & \text{if } y \in K \\ \overline{\beta} & \text{if } y \in \text{Ch}(B) \setminus K \end{cases}
\]
for every \( \beta \in S^1 \). Given \( f \in A_0 \), set \( \beta_0 = -f(\phi(y))|S(f)(y)|^{-1} \). Then \( \beta_0 \in S^1 \). By Lemma 5, there exists a \( u_0 \in \exp B(y) \) such that
\[
\sigma_{\pi}(u_0S(f)^{-1}) = \{ S(f)(y)^{-1} \} \text{ and } |u_0S(f)^{-1}| < |S(f)(y)|^{-1} \text{ on } \text{Ch}(B) \setminus F_0.
Applying (5) for \( \beta = \beta_0 \), there exists a \( u_0 \in A_0 \) such that \( S(u_0) = S(\beta)u_0 \) and \( u_0(\phi(y)) = \beta_0 \). This shows that, by (2),
\[
\left\| \frac{S(\beta_0)u_0}{S(f)} - 1 \right\|_\infty = \left\| \frac{u_0}{f} - 1 \right\|_\infty \geq \left| \frac{\beta_0}{f(\phi(y))} - 1 \right| = |S(f)(y)|^{-1} + 1.
\]
By (3), we have \( \|S(\beta_0)u_0S(f)^{-1} - 1\|_\infty = |S(f)(y)|^{-1} \), that is
\[
\|S(\beta_0)u_0S(f)^{-1} - 1\|_\infty = |S(f)(y)|^{-1} + 1.
\]
Hence there exists a \( y_0 \in \text{Ch}(B) \) such that
\[
(S(\beta_0)u_0S(f)^{-1})(y_0) = -|S(f)(y)|^{-1}.
\]
The hypotheses of \( u_0 \) and equation (7) imply that
\[
(u_0S(f)^{-1})(y_0) = S(f)(y)^{-1} \quad \text{and} \quad S(\beta_0)(y_0) = S(\beta_0)(y),
\]
which shows (4).

We will show that there exists a real-algebra isomorphism \( \tilde{T} : A \to B \) (cf. [3, 4, 9]). For each \( f \in A \), there exist a complex number \( \lambda_0 \) and an \( f_0 \in A_0 \) such that the imaginary part of \( \lambda_0 \) is not zero, the real part of \( f_0 \) is positive and \( f = f_0 + \lambda_0 \). Notice that \( f_0 \in \exp A \). Thus \( f_0, \lambda_0 \in A_0 \).

Define a map \( \tilde{T} \) on \( A \) by
\[
\tilde{T}(f) = S(f_0) + S(\lambda_0).
\]
Then, by (4), \( \tilde{T} \) is a real-algebra isomorphism such that
\[
(8) \quad \tilde{T}(f) = \begin{cases} f \circ \phi & \text{on } K \\ f \circ \phi & \text{on } \text{Ch}(B) \setminus K \end{cases}
\]
for every \( f \in A \) and \( \tilde{T} = S \) on \( A_0 \).

Finally, we will construct a homeomorphism \( \Phi \) from \( M_B \) onto \( M_A \) (cf. [8, The proof of Theorem 2.1]). By (8), we obtain \( \tilde{T}(i)(M_B) \subset \{i, -i\} \). Define a subset \( \mathcal{K} \) of \( M_B \) by
\[
\mathcal{K} = \{y \in M_B : \tilde{T}(i)(y) = i\}.
\]
Then \( \mathcal{K} \) is a clopen subset of \( M_B \) with \( \text{Ch}(B) \cap \mathcal{K} = K \). Let \( e = (\tilde{T}(i) + i)/(2i) \), then \( e \) is an idempotent such that
\[
\hat{e} = \begin{cases} 1 & \text{on } \mathcal{K} \\ 0 & \text{on } M_B \setminus \mathcal{K} \end{cases}
\]
For \( y \in M_B \), let \( \Phi(y) \) be defined as
\[
\Phi(y)(f) = \tilde{T}(f)(y)\hat{e}(y) + \tilde{T}(f)(y)(1 - \hat{e})(y)
\]
for every \( f \in A \). Then, by (8) and (9), the mapping \( \Phi : y \mapsto \Phi(y) \) is a homeomorphism from \( M_B \) onto \( M_A \). By the definition of \( \Phi \) and equation (9), we obtain the conclusion. \( \square \)

Here we prove Theorem 4, stated in the first section. Below we make use of subsets of \( A^{-1} \) defined as follows: Let \( k, l \) be positive integers and \( \mathcal{X} \) a subset of a uniform algebra \( A \). Define a subset \( (\mathcal{X})^k_l \) of \( A \) by
\[
(\mathcal{X})^k_l = \{f' \in \mathcal{X} : \text{there exists an } f' \in \mathcal{X} \text{ with } f^k(f')^l = 1\}.
\]
Then \( (\mathcal{X})^k_l \) is a subset of \( (A^{-1})^k_l \).
Proof of Theorem 4. Recall that $S_A, S_B$ are subsets of $A, B$ that contain $A^{-1}, B^{-1}$ respectively and $T : S_A \to S_B$ is a surjection such that

\[ \|T(f)^mT(g)^n - \alpha\|_\infty = \|f^m g^n - \alpha\|_\infty \]

for all $f, g \in S_A$. By a simple calculation, we obtain $(S_A)^m = (A^{-1})^m_m$ and $(S_B)^m = (B^{-1})^m_m$, since $S_A, S_B$ contain $A^{-1}, B^{-1}$ respectively. We will show that $T((A^{-1})^m_m) = (B^{-1})^m_m$. Suppose that $\nu_\alpha$ is a complex number with $(\nu_\alpha)^m = \alpha$. For every $g \in (A^{-1})^m_m$, let $g' \in A^{-1}$ with $g^m(g')^n = 1$. Since, by (1),

\[ \|T(g)^mT(\nu_\alpha g')^n - \alpha\|_\infty = \|g^m(\nu_\alpha g')^n - \alpha\|_\infty = \|g^m\alpha(g')^n - \alpha\|_\infty = 0, \]

we obtain

\[ T(g)^mT(\nu_\alpha g')^n = \alpha. \]

This shows that $T(g)^m(\nu_\alpha^{-1}T(\nu_\alpha g'))^n = 1$, that is $T(g) \in (B^{-1})^m_m$. Together with the surjectivity of $\tilde{T}$, similar arguments show the opposite inclusion. Consequently, $T((A^{-1})^m_m) = (B^{-1})^m_m$. Furthermore, we have

\[
\left\| \frac{T(f)^m}{T(g)^m} - 1 \right\|_\infty = \frac{1}{|\alpha|} \left\| T(f)^mT(\nu_\alpha g')^n - \alpha \right\|_\infty = \frac{\|f^m\|_\infty}{\|g^m\|_\infty} - 1
\]

for every $f \in S_A$ and $g \in (A^{-1})^m_m$. Define a map $T_m$ on $((A^{-1})^m_m)^m = \{ f^m; f \in (A^{-1})^m_m \}$ by

\[ T_m(f^m) = T(f)^m/T(1)^m \]

for $f^m \in ((A^{-1})^m_m)^m$. Then, by (10), $T_m$ is well-defined in the sense that $T(f)^m = T(g)^m$ for every $f, g \in (A^{-1})^m_m$ with $f^m = g^m$, and $T_m(1) = 1$. Since $T((A^{-1})^m_m) = (B^{-1})^m_m$, we have $T_m(((A^{-1})^m_m)^m) = ((B^{-1})^m_m)^m$. By (10), we also have

\[ \left\| T_m(f^m)T_m(g^m)^{-1} - 1 \right\|_\infty = \left\| f^m(g^m)^{-1} - 1 \right\|_\infty \]

for all $f^m, g^m \in ((A^{-1})^m_m)^m$. Notice that $((A^{-1})^m_m)^m$, $(B^{-1})^m_m$ are subgroups that contain exp $A, \exp B$ respectively. Proposition 7 shows that there exists a real-algebra isomorphism $\tilde{T} : A \to B$, a clopen subset $\mathcal{K}$ of $M_B$ and a homeomorphism $\Phi : M_B \to M_A$ such that

\[ \tilde{T}(f) = \begin{cases} \hat{f} \circ \Phi & \text{on } \mathcal{K} \\ \hat{f} \circ \Phi & \text{on } M_B \setminus \mathcal{K} \end{cases} \]

for every $f \in A$ and $\tilde{T}(f^m) = T_m(f^m)$ for every $f^m \in ((A^{-1})^m_m)^m$. By the definition of $T_m$ and equation (11), we have $\tilde{T}(f)^m = (T(f)/T(1))^m$ for every $f \in (A^{-1})^m_m$. By (10) and (11), we also have

\[ \left\| \frac{(T(f)/T(1))^m - 1}{(T(g)/T(1))^m} \right\|_\infty = \left\| \frac{T(f)^m}{T(g)^m} - 1 \right\|_\infty = \left\| \frac{f^m}{g^m} - 1 \right\|_\infty = \left\| \frac{\tilde{T}(f)^m}{\tilde{T}(g)^m} - 1 \right\|_\infty \]

for every $f \in S_A$ and $g \in (A^{-1})^m_m$. Since $(B^{-1})^m_m$ contains exp $B$, we obtain

\[ \left\| (T(f)/T(1))^m g - 1 \right\|_\infty = \|\tilde{T}(f)^m g - 1\|_\infty \]

for every $f \in S_A$ and all $g \in \exp B$. By Lemma 6, we obtain

\[ \tilde{T}(f)^m = (T(f)/T(1))^m \]

for every $f \in S_A$. 


Finally, we will show that $\overline{T}(f)^{d} = (T(f)/T(1))^{d}$ for every $f \in S_{A}$, where $d$ is the greatest common divisor of $m$ and $n$. By raising both sides of equation (12) to the $n$-th power, we have $\overline{T}(f)^{mn} = (T(f)/T(1))^{mn}$ for every $f \in S_{A}$. By (11), we also have
\begin{equation}
\|\overline{T}(f)(T(g)/T(1))^{-mn} - 1\|_{\infty} = \|fg^{-mn} - 1\|_{\infty}
\end{equation}
for every $f \in A$ and $g \in T^{-1}((B^{-1})_{m}^{n})$. If we consider the map $T_{n}$ on $((A^{-1})_{m}^{n})^{n} = \{ f^{n} : f \in (A^{-1})_{m}^{n} \}$ defined as $T_{n}(f^{n}) = T(f)^{n}/T(1)^{n}$ for $f^{n} \in ((A^{-1})_{m}^{n})^{n}$, similar arguments show that there exists a real-algebra isomorphism $\overline{T}' : A \rightarrow B$ such that $\overline{T}'(f)^{n} = (T(f)/T(1))^{n}$ for every $f \in S_{A}$ and
\begin{equation}
\|\overline{T}'(f)(T(g)/T(1))^{-mn} - 1\|_{\infty} = \|fg^{-mn} - 1\|_{\infty}
\end{equation}
for every $f \in A$ and $g \in T^{-1}((B^{-1})_{m}^{n})$. Together with (13), this shows that
\begin{equation}
\|\overline{T}(f)(T(g)/T(1))^{-mn} - 1\|_{\infty} = \|\overline{T}'(f)(T(g)/T(1))^{-mn} - 1\|_{\infty}
\end{equation}
for every $f \in A$ and $g \in T^{-1}((B^{-1})_{m}^{n} \cap (B^{-1})_{m}^{n})$. Since $(B^{-1})_{m}^{n}$ and $(B^{-1})_{m}^{n}$ contain $\exp B$, we obtain
\begin{equation}
\|\overline{T}(f)g - 1\|_{\infty} = \|\overline{T}'(f)g - 1\|_{\infty}
\end{equation}
for every $f \in A$ and all $g \in \exp B$. It follows from Lemma 6 that $\overline{T} = \overline{T}'$ on $A$. Consequently, $\overline{T}(f)^{n} = (T(f)/T(1))^{n}$ for every $f \in S_{A}$, which implies that $\overline{T}(f)^{d} = (T(f)/T(1))^{d}$ for every $f \in S_{A}$.

\[\square\]

\textbf{References}


