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Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients

(joint work with Minoru Murata)

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1 Results

The main purpose of this paper is to establish asymptotics at infinity of Green functions for elliptic equations with periodic coefficients on $\mathbf{R}^d$ and to determin the Martin boundary for the elliptic operators.

Let

$$\begin{aligned} L &= -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k}(a_{jk}(x)\frac{\partial}{\partial x_j}) - \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + c(x) \\ &= -\nabla \cdot a(x)\nabla - b(x)\cdot\nabla + c(x), \end{aligned}$$

be a second-order elliptic operator on $\mathbf{R}^d$ with periodic coefficients, where $d \geq 2$, $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$, $a(x) = (a_{jk}(x))_{j,k=1}^{d}$, and $b(x) = (b_j(x))_{j=1}^{d}$. We assume that the coefficients are $\mathbf{Z}^d$-periodic, real-valued smooth functions on $\mathbf{R}^d$. We assume that a is a symmetric matrix-valued function satisfying for some $\alpha > 0$

$$\alpha|\xi|^2 \leq \sum_{j,k=1}^{d} a_{jk}(x)\xi_j\xi_k \leq \alpha^{-1}|\xi|^2, \quad x, \xi \in \mathbf{R}^d.$$

For $\zeta \in \mathbf{C}^d$, define an operator $L(\zeta)$ on the d-dimensional torus by

$$\begin{aligned} L(\zeta) &= e^{-i\zeta\cdot x} L e^{i\zeta\cdot x} \\ &= -(\nabla + i\zeta)\cdot a(x)(\nabla + i\zeta) - b(x)\cdot(\nabla + i\zeta) + c(x). \end{aligned}$$

We regard $L(\zeta)$ as a closed operator in $L^2(\mathbf{T}^d)$ with domain $H^2(\mathbf{T}^d)$.

By the Krein-Rutman theorem, for each $\beta \in \mathbf{R}^d$, $L(i\beta) = e^{\beta\cdot x} L e^{-\beta\cdot x}$ has the principal eigenvalue $E(\beta)$, i.e. $L(i\beta)$ has an eigenvalue $E(\beta) \in \mathbf{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function $u_\beta \in H^2(\mathbf{T}^d)$; $E(\beta)$ is also an eigenvalue of $L(i\beta)^*$ of multiplicity one such that the eigenspace is generated by a positive function $v_\beta \in H^2(\mathbf{T}^d)$.

Put

$$C_L = \{u \in H^1_{loc}(\mathbf{R}^d);\, Lu = 0 \text{ and } u > 0\}.$$

When a positive Green function for L on $\mathbf{R}^d$ exists, L is called subcritical; in this case $C_L \neq \emptyset$. When a positive Green function for L on $\mathbf{R}^d$ dose not exist but $C_L \neq \emptyset$, L is called critical. Let λ_c be the generalized principal eigenvalue of L on $\mathbf{R}^d$:

$$\lambda_c := \sup\{\lambda \in \mathbf{R};\, L - \lambda \text{ is subcritical}\}.$$

Then it is known that $-\infty < \lambda_c < \infty$, $L - \lambda$ is subcritical for $\lambda < \lambda_c$, and $L - \lambda_c$ is subcritical or critical. The formal adjoint operator L^* of L is subcritical (or critical) if and only if L is subcritical (or critical), and the generalized principal eigenvalue of L and L^* coincide.

For $\lambda \in \mathbf{R}$, put

$$\begin{aligned} \Gamma_\lambda &:= \{\beta \in \mathbf{R}^d;\, \exists\psi(x) = e^{-\beta\cdot x}u(x) \in C_{L-\lambda} \text{ where } u \text{ is periodic}\} \\ K_\lambda &:= \{\beta \in \mathbf{R}^d;\, \exists\psi = e^{-\beta\cdot x}u(x) > 0 \text{ s.t. } (L-\lambda)\psi \geq 0 \text{ and } u \text{ is periodic}\}. \end{aligned}$$

Define K_λ^* and Γ_λ^* for $L^* - \lambda$ analogously to K_λ and Γ_λ for $L - \lambda$. Agmon, Pinsky and Kuchment-Pinchover proved the following theorem.

Theorem AP([A], [P], [KP])

(i) If $\lambda < \lambda_c$, then K_λ is a d-dimensional strictry convex compact set with smooth boundary $\Gamma_\lambda = \partial K_\lambda$.

(ii) If $\lambda = \lambda_c$, then $\Gamma_\lambda = K_\lambda = \{\beta_0\}$ for some $\beta_0 \in \mathbf{R}^d$.

(iii) If $\lambda > \lambda_c$, then $\Gamma_\lambda = K_\lambda = \emptyset$.

(iv) $K_\lambda^* = -K_\lambda$, and $\beta_0 = 0$ if $L^* = L$

(v) $E(\beta)$ is an algebraically simple eigenvalue and it is a real analytic. Hess $E(\beta)$ is neg. def. for $\beta \in \mathbf{R}^d$. The equality $\lambda_c = \sup_{\beta \in \mathbf{R}^d} E(\beta)$ holds, and the sup is attained uniquely at β_0 in (ii). $\nabla E(\beta) = 0$ if and only if $\beta = \beta_0$.

(vi) $\Gamma_\lambda = \{\beta \in \mathbf{R}^d; E(\beta) = \lambda\}$ and $K_\lambda = \{\beta \in \mathbf{R}^d; E(\beta) \geq \lambda\}$.

Let $B_R = \{|x| < R\}$. Let L_R be the Dirichlet realization of L in $L^2(B_R)$: $D(L_R) = H_0^1(B_R) \cap H^2(B_R)$. If L is subcritical, then $\exists$the resolvent L_R^{-1}, and its integral kernel (the Green function) $G_R(x,y) > 0$, and $\exists$the limit $G(x,y) = \lim_{R\to\infty} G_R(x,y)$ which is called the minimal Green function of L on $\mathbf{R}^d$.

First, suppose that $\lambda_c > 0$. Then L is subcritical, and for any $s \in \mathbf{S}^{d-1}$, take $\beta_s \in \Gamma_0$ s.t. $\sup_{\beta\in\Gamma_0} \beta \cdot s = \beta_s \cdot s$.

Theorem 1 *Suppose that $\lambda_c > 0$. Then the minimal Green function G of L admits the following asymptotics as $|x-y| \to \infty$:*

$$G(x,y) = \frac{1}{|\nabla E(\beta_s)|\sqrt{C(\beta_s)}} \frac{e^{-(x-y)\cdot\beta_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})_{L^2(\mathbf{T}^d)}} \times (1 + O(|x-y|^{-1})),$$

where $s = (x-y)/|x-y|$, and $C(\beta_s)$ is the Gauss-Kronecker curvature of Γ_0 at β_s.

Schroeder [S] gave a lower and upper bounds.

Let us determine explicitly the Martin compactification of $\mathbf{R}^d$ with respect to L in the case $\lambda_c > 0$. Fix a reference point x_0 in $\mathbf{R}^d$. Then the following proposition is a consequence of Theorem 1.

Proposition 2 *Suppose that $\lambda_c > 0$. Then for any sequence $\{y_n\}$ in $\mathbf{R}^d$ such that $|y_n| \to \infty$ and $y_n/|y_n| \to \nu$ as $n \to \infty$,*

$$\lim_{n\to\infty} \frac{G(x,y_n)}{G(x_0,y_n)} = e^{-(x-x_0)\cdot\beta_{-\nu}} \frac{u_{\beta_{-\nu}}(x)}{u_{\beta_{-\nu}}(x_0)} =: K(x,\nu).$$

$\psi \in C_L$ is minimal (If $\varphi \in C_L$ satisfies $\varphi(x) \leq \psi(x)$, then $\varphi(x) = c\psi(x)$) if and only if $\psi = e^{\beta x}u(x) \in C_L$ where u is periodic (see [A]). Thus $\Gamma_0 \simeq$ the minimal Martin boundary. On the other hand $K(x,\nu) \in C_L$, $K(x_0,\nu) = 1$, $K(x,\nu) \neq K(x,\nu')$ if $\nu \neq \nu'$. $K(x,\nu)$ is minimal. Hence we can determine the Martin boundary and Martin compactification of $\mathbf{R}^d$ for L as follows.

Theorem 3 *Suppose that* $\lambda_c > 0$. *Then the Martin boundary and the minimal Martin boundary of* $\mathbf{R}^d$ *for* L *are both equal to the sphere* $\mathbf{S}^{d-1}$ *at infinity which is homeomorphic to* Γ_0*; the Martin kernel at* $\nu \in \mathbf{S}^{d-1}$ *is equal to* $K(\cdot, \nu)$*; and the Martin compactification of* $\mathbf{R}^d$ *for* L *is equal to*

$$\{x \in \mathbf{R}^d; |x| < 1\} \cup [1, \infty] \times \mathbf{S}^{d-1}$$

equipped with the standard topology.

Next, suppose that $\lambda_c = E(\beta_0) = 0$. Then Pinsky [P] proved that L is critical if $d \leq 2$, and subcritical if $d \geq 3$.

Theorem 4 *Let* $d \geq 3$. *Suppose* $\lambda_c = 0$. *Put* $H = -\text{Hess } E(\beta_0)$. *Then the minimal Green function* G *of* L *admits the following asymptotics as* $|x - y| \to \infty$*:*

$$G(x, y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}(\det H)^{1/2}} \frac{e^{-(x-y)\cdot\beta_0}}{|H^{-1/2}(x-y)|^{d-2}} \frac{u_{\beta_0}(x) v_{\beta_0}(y)}{(u_{\beta_0}, v_{\beta_0})} \times (1 + O(|x-y|^{-1})).$$

We determine directly from Theorem 4 the Martin boundary. These results, however, are also simple consequences of the known result that C_L is one dimensional in this case.

Theorem 5 *Let* $d \geq 3$. *Suppose that* $\lambda_c = E(\beta_0) = 0$. *Then for any sequence* $\{y_n\}$ *in* $\mathbf{R}^d$ *with* $|y_n| \to \infty$ *as* $n \to \infty$,

$$\lim_{n\to\infty} \frac{G(x, y_n)}{G(x_0, y_n)} = e^{-(x-x_0)\cdot\beta_0} \frac{u_{\beta_0}(x)}{u_{\beta_0}(x_0)}, \quad x \in \mathbf{R}^d.$$

The Martin boundary and the minimal Martin boundary are both equal to one point ∞ *at infinity; the Martin kernel at* ∞ *is equal to the right hand side; and the Martin compactification of* $\mathbf{R}^d$ *for* L *is equal to the one point compactification* $\mathbf{R}^d \cup \{\infty\}$ *of* $\mathbf{R}^d$.

2 Proof of Theorem 1

Assume $\lambda_c = E(\beta_0) > 0$. Put $L_0 = e^{\beta_0 \cdot x} L e^{-\beta_0 \cdot x}$. Then the principal eigenvalue $E_0(0)$ of $\beta = 0$ of L_0 is positive, and the minimal Green function $G_0(x, y)$ of L_0 satisfies $G_0(x, y) = e^{\beta_0 \cdot x} G(x, y) e^{-\beta_0 \cdot y}$. Regard L as a closed operator in $L^2(\mathbf{R}^d)$ with domain $H^2(\mathbf{R}^d)$. We have only to show the following.

Theorem 6 *Assume* $E(0) > 0$. *Then there exists the resolvent* L^{-1}*; and the integral kernel of* L^{-1} *equals the minimal Green function and admits the same asymptotics as in Theorem 1.*

Let

$$\mathcal{H} = L^2((-\pi, \pi)^d, (2\pi)^{-d} d\zeta; L^2(\mathbf{T}^d)).$$

Define an operator $\mathcal{F} : L^2(\mathbf{R}^d) \to \mathcal{H}$ by

$$(\mathcal{F}f)(\zeta, x) = \sum_{l \in \mathbf{Z}^d} f(x - l) e^{-i(x-l)\cdot\zeta}, \quad \zeta \in (-\pi, \pi)^d, \quad x \in \mathbf{T}^d$$

(Bloch transformation). Then $\mathcal{F}$ is a unitary operator, and an isometric isomorphism from $H^1(\mathbf{R}^d)$ to $L^2((-\pi, \pi)^d, (2\pi)^{-d} d\zeta; H^1(\mathbf{T}^d))$. The adjoint $\mathcal{F}^*$ is given by, for $g \in \mathcal{H}$,

$$(\mathcal{F}^* g)(x - l) = (2\pi)^{-d} \int_{(-\pi,\pi)^d} e^{i(x-l)\cdot\zeta} g(\zeta, x)\, d\zeta, \quad x \in \mathbf{T}^d,\, l \in \mathbf{Z}^d.$$

We have

$$\left.\begin{array}{r}\mathcal{F}(\nabla f) = (\nabla_x + i\zeta)\mathcal{F}f \\ \mathcal{F}(af) = a\mathcal{F}f \text{ if } a : \text{periodic}\end{array}\right\} \Rightarrow \mathcal{F}L = L(\zeta)\mathcal{F}.$$

Proposition 7 *Let $E(0) > 0$. Then there exists the resolvent $L^{-1}(\zeta)$, $\zeta \in \mathbf{R}^d$; and $L^{-1} = \mathcal{F}^* L(\zeta)^{-1}\mathcal{F}$, i.e., for $x \in \mathbf{T}^d$, $l \in \mathbf{Z}^d$, and $f \in L^2(\mathbf{R}^d)$,*

$$L^{-1}f(x-l) = (2\pi)^{-d}\int_{(-\pi,\pi)^d} F(\zeta)d\zeta,$$

where

$$F(\zeta) = e^{i(x-l)\cdot\zeta}L(\zeta)^{-1}(\sum_{m\in\mathbf{Z}^d} f(\cdot - m)e^{-i(\cdot-m)\cdot\zeta})(x).$$

Moreover, $F(\zeta)$ is $2\pi\mathbf{Z}^d$-periodic.

$\{L(\zeta)\}_{\zeta\in\mathbf{C}^d}$ is an analytic family of type (B). By the analytic perturbation theory, $E(\beta)$ has an analytic continuation $\Lambda(\zeta)$, $\zeta = \alpha + i\beta$, near $\zeta = i\beta_s$; note that $E(\beta) = \Lambda(i\beta)$. Mreover $\Lambda(\zeta)$ is also an algebraically simple eigenvalue of $L(\zeta)$ with eigenfunction u_ζ: $(L(\zeta) - \Lambda(\zeta))u_\zeta = 0$. $\overline{\Lambda(\zeta)}$ is an algebraically simple eigenvalue of $L(\zeta)^*$ with eigenfunction v_ζ: $(L(\zeta)^* - \overline{\Lambda(\zeta)})v_\zeta = 0$.

Put $\eta_s := \beta_s/|\beta_s|$, and let $\{e_{s,1}, \dots, e_{s,d-1}, s\}$ be an orthonormal basis of $\mathbf{R}^d$. Put $e_s := (e_{s,1}, \dots, e_{s,d-1})$. We introduce new coordinates (w, z) near $i\beta_s$ such that $\zeta = w\eta_s + z\cdot e_s = w\eta_s + \sum_{j=1}^{d-1} z_j e_{s,j}$, $w \in \mathbf{C}$, $z = (z_1, \dots, z_{d-1}) \in \mathbf{R}^{d-1}$.

Proposition 8 *For $z \in \mathbf{R}^{d-1}$ with $|z| \ll 1$, the resolvent $L(w\eta_s + z\cdot e_s)^{-1}$ has a simple pole $w_s(z)$ as a function of w, and has the following asymptotics at the pole*

$$L(w\eta_s + z\cdot e_s)^{-1} = \frac{A_{s,z}}{w - w_s(z)} + O(1).$$

Here $A_{s,z}$ is a rank one operator-valued function with

$$A_{s,z} = \frac{1}{\eta_s\cdot\nabla\Lambda(\zeta(z))}\frac{(\cdot, v_{\zeta(z)})u_{\zeta(z)}}{(u_{\zeta(z)}, v_{\zeta(z)})}, \quad \zeta(z) = w_s(z)\eta_s + z\cdot e_s$$

and $w_s(z)$ satisfies $w_s(0) = i|\beta_s|$, for $1 \le j,k \le d-1$,

$$\frac{\partial w_s}{\partial z_j}(0) = 0,$$

$$\frac{\partial^2 w_s}{\partial z_j \partial z_k}(0) = i\frac{\partial^2 \mathrm{Im}\, w_s}{\partial z_j\partial z_k}(0) = i\frac{e_{s,j}\cdot \mathrm{Hess}\, E(\beta_s)e_{s,k}}{\eta_s\cdot\nabla E(\beta_s)},$$

$$\mathrm{Hess}\,\mathrm{Im}\, w_s(0) = \left(\frac{\partial^2 \mathrm{Im}\, w_s}{\partial z_j\partial z_k}(0)\right)_{1\le j,k\le d-1} : \textit{positive definite}.$$

Here the function $\zeta(z) = w_s(z)\eta_s + z\cdot e_s$ is the zeros of $\Lambda(\zeta)$.

Proof. $\Lambda(\zeta)$ is an algebraically simple eigenvalue, so

$$(L(\zeta) - \lambda)^{-1} = \frac{P(\zeta)}{\Lambda(\zeta) - \lambda} + O(1), \quad P(\zeta) = \frac{(\cdot, v_\zeta)u_\zeta}{(u_\zeta, v_\zeta)}.$$

Putting $\lambda = 0$, we have

$$L(\zeta)^{-1} = \frac{P(\zeta)}{\Lambda(\zeta)} + O(1).$$

Noting that

$$\begin{aligned}\Lambda(\zeta) &= \Lambda(w\eta_s + z \cdot e_s) \\ &= (w - w_s(z))\eta_s \cdot \nabla\Lambda(w_s(z)\eta_s + z \cdot e_s) + O((w - w_s(z))^2),\end{aligned}$$

we have the proposition. □

Let $P : t\eta_s + z \cdot e_s \to z$ be a projection, and $Q = P(-\pi, \pi)^d$. We have $(-\pi, \pi)^d = \{t\eta_s + z \cdot e_s;\ z \in Q, \exists t_1(z) < t < \exists t_2(z)\}$. We change the integral variables from ζ to $(t, z) \in \mathbf{R} \times \mathbf{R}^{d-1}$ such that $\zeta = t\eta_s + z \cdot e_s$ to obtain that

$$\begin{aligned}(L^{-1}f)(x-l) &= (2\pi)^{-d} \int_{(-\pi,\pi)^d} F(\zeta)d\zeta \\ &= \frac{|D_s|}{(2\pi)^d} \int_Q dz \int_{t_1(z)}^{t_2(z)} dt\, F(t\eta_s + z \cdot e_s),\end{aligned}$$

where $D_s = \det(\eta_s, e_{s,1}, \cdots, e_{s,d-1})$, and

$$F(\zeta) = e^{i(x-l)\cdot\zeta} L(\zeta)^{-1} \Big(\sum_{m \in \mathbf{Z}^d} f(\cdot - m) e^{-i(\cdot - m)\cdot\zeta} \Big)(x).$$

For $0 < \delta \ll 1$, put

$$U_\delta = \{z \in \mathbf{R}^{d-1}; \operatorname{Im} w_s(z) < |\beta_s| + \delta\}.$$

For $z \in Q$ let $C(z) = C_1(z) \cup C_2(z)$ be a closed contour in $\mathbf{C}$:

$$\begin{aligned}C_1(z) &= \{t : t_1(z) \to t_2(z)\}, \\ C_2(z) &= \{t_2(z) + it;\ t : 0 \to |\beta_s| + h\} \\ &\quad \cup \{t + i(|\beta_s| + h);\ t : t_2(z) \to t_1(z)\} \\ &\quad \cup \{t_1(z) + it; t : |\beta_s| + h \to 0\}\end{aligned}$$

where $h = 2\delta$ if $z \in U_\delta$, $h = \delta/2$ if $z \in Q \setminus U_\delta$. For $z \in U_\delta$ the integrand has only a simple pole $w_s(z)$ near and inside $C(z)$, and for $z \in Q \setminus U_\delta$ the integrand is holomorphic near and inside $C(z)$. Hence by the residue theorem we have

$$(L^{-1}f)(x-l) = I_1 f(x-l) + I_2 f(x-l)$$

with $\zeta(z) = w_s(z)\eta_s + z \cdot e_s$, where

$$\begin{aligned}I_1 f(x-l) &= \frac{2\pi i |D_s|}{(2\pi)^d} \int_{U_\delta} dz \exp[i(x-l)\cdot\zeta(z)] \\ &\quad \times \frac{(\sum_m f(\cdot - m)\exp[-i(\cdot - m)\cdot\zeta(z)], v_{\zeta(z)}) u_{\zeta(z)}(x)}{\eta_s \cdot \nabla\Lambda(\zeta(z))(u_{\zeta(z)}, v_{\zeta(z)})}, \\ I_2 f(x-l) &= -\frac{|D_s|}{(2\pi)^d} \int_Q dz \int_{C_2(z)} dw F(w\eta_s + z \cdot e_s).\end{aligned}$$

The integral kernel $I_1(x,y)$, $x,y \in \mathbf{R}^d$, of I_1 is equal to

$$I_1(x,y) = \frac{i|D_s|}{(2\pi)^{d-1}} \int_{U_\delta} dz \exp[i(x-y)\cdot(w_s(z)\eta_s + z\cdot e_s)]a(z;x,y),$$

$$a(z;x,y) := \frac{1}{\eta_s \cdot \nabla\Lambda(\zeta(z))} \frac{u_{\zeta(z)}(x)\overline{v_{\zeta(z)}(y)}}{(u_{\zeta(z)}, v_{\zeta(z)})}.$$

Take $s = (x-y)/|x-y|$. We regard $(x-y)\cdot\eta_s \gg 1$ as a large parameter, and note that $(x-y)\cdot(z\cdot e_s) = 0$. We have shown that the critical point of $w_s(z)$ is $z = 0$. By the saddle point method

$$I_1(x,y) = \frac{-|D_s|}{(2\pi)^{d-1}} \left(\frac{2\pi}{(x-y)\cdot\eta_s}\right)^{(d-1)/2} \frac{e^{-(x-y)\cdot\beta_s}}{(\det \mathrm{Hess}\, \mathrm{Im}\, w_s(0))^{1/2}} \times\left(\frac{1}{\eta_s\cdot\nabla E(\beta_s)} \frac{u_{\beta_s}(x)\overline{v_{\beta_s}(y)}}{(u_{\beta_s}, v_{\beta_s})} + O(|x-y|^{-1})\right).$$

This leads to the main term of the asymptotics.

We can show that the integral kernel of I_2 satisfies

$$|I_2(x,y)| \le Ce^{-(x-y)\cdot\beta_s}e^{-c|x-y|},$$

using the $2\pi\mathbf{Z}^d$-periodicity of $F(\zeta)$. These are an outline of the proof of Theorem 1. □

Remark. We can get the following asymptotic expansion. Assume that $\lambda_c > 0$. There exist bounded functions $g_j(x,y)$, $j = 1, 2, \cdots$, s.t. for any natural number n

$$G(x,y) = \frac{1}{|\nabla E(\beta_s)|\sqrt{C(\beta_s)}} \frac{e^{-(x-y)\cdot\beta_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})_{L^2(\mathbf{T}^d)}} \times\left(1 + \sum_{j=1}^{n} \frac{g_j(x,y)}{|x-y|^j} + O(|x-y|^{-n-1})\right).$$

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