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Kyoto University
Asymptotics of Green functions and Martin boundaries
for elliptic operators with periodic coefficients

(joint work with Minoru Murata)

Tetsuo Tsuchida (Meijo University)

1 Results

The main purpose of this paper is to establish asymptotics at infinity of Green functions for elliptic equations with periodic coefficients on $\mathbb{R}^d$ and to determine the Martin boundary for the elliptic operators.

Let

$$
L = - \sum_{j,k=1}^{d} \frac{\partial}{\partial x_k} (a_{jk}(x) \frac{\partial}{\partial x_j}) - \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x)
$$

be a second-order elliptic operator on $\mathbb{R}^d$ with periodic coefficients, where $d \geq 2$, $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$, $a(x) = (a_{jk}(x))_{j,k=1}^{d}$, and $b(x) = (b_j(x))_{j=1}^{d}$. We assume that the coefficients are $\mathbb{Z}^d$-periodic, real-valued smooth functions on $\mathbb{R}^d$. We assume that $a$ is a symmetric matrix-valued function satisfying for some $\alpha > 0$

$$
\alpha |\xi|^2 \leq \sum_{j,k=1}^{d} a_{jk}(x) \xi_j \xi_k \leq \alpha^{-1} |\xi|^2, \quad x, \xi \in \mathbb{R}^d.
$$

For $\zeta \in \mathbb{C}^d$, define an operator $L(\zeta)$ on the $d$-dimensional torus by

$$
L(\zeta) = e^{-i\zeta \cdot x} L e^{i\zeta \cdot x}
$$

$$
= -(\nabla + i\zeta) \cdot a(x)(\nabla + i\zeta) - b(x) \cdot (\nabla + i\zeta) + c(x).
$$

We regard $L(\zeta)$ as a closed operator in $L^2(\mathbb{T}^d)$ with domain $H^2(\mathbb{T}^d)$.

By the Krein-Rutman theorem, for each $\beta \in \mathbb{R}^d$, $L(i\beta) = e^{\beta \cdot x} L e^{-\beta \cdot x}$ has the principal eigenvalue $E(\beta)$, i.e. $L(i\beta)$ has an eigenvalue $E(\beta) \in \mathbb{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function $u_\beta \in H^2(\mathbb{T}^d)$; $E(\beta)$ is also an eigenvalue of $L(i\beta)^*$ of multiplicity one such that the eigenspace is generated by a positive function $v_\beta \in H^2(\mathbb{T}^d)$.

Put

$$
C_L = \{ u \in H^1_{\text{loc}}(\mathbb{R}^d); Lu = 0 \text{ and } u > 0 \}.
$$

When a positive Green function for $L$ on $\mathbb{R}^d$ exists, $L$ is called subcritical; in this case $C_L \neq \emptyset$. When a positive Green function for $L$ on $\mathbb{R}^d$ dose not exist but $C_L \neq \emptyset$, $L$ is called critical. Let $\lambda_c$ be the generalized principal eigenvalue of $L$ on $\mathbb{R}^d$.

$$
\lambda_c := \sup \{ \lambda \in \mathbb{R}; L - \lambda \text{ is subcritical} \}.
$$

Then it is known that $-\infty < \lambda_c < \infty$, $L - \lambda$ is subcritical for $\lambda < \lambda_c$, and $L - \lambda$ is subcritical or critical. The formal adjoint operator $L^*$ of $L$ is subcritical (or critical) if and only if $L$ is subcritical (or critical), and the generalized principal eigenvalue of $L$ and $L^*$ coincide.

For $\lambda \in \mathbb{R}$, put

$$
\Gamma_\lambda := \{ \beta \in \mathbb{R}^d; \exists \psi(x) = e^{-\beta \cdot x} u(x) \in C_{L-\lambda} \text{ where } u \text{ is periodic} \}
$$

$$
K_\lambda := \{ \beta \in \mathbb{R}^d; \exists \psi = e^{-\beta \cdot x} u(x) > 0 \text{ s.t. } (L - \lambda)\psi \geq 0 \text{ and } u \text{ is periodic} \}.
$$
Define $K^*_\lambda$ and $\Gamma^*_\lambda$ for $L^* - \lambda$ analogously to $K_\lambda$ and $\Gamma_\lambda$ for $L - \lambda$. Agmon, Pinsky and Kuchment-Pinchover proved the following theorem.

**Theorem AP** ([A], [P], [KP])

(i) If $\lambda < \lambda_c$, then $K_\lambda$ is a $d$-dimensional stricry convex compact set with smooth boundary $\Gamma_\lambda = \partial K_\lambda$.

(ii) If $\lambda = \lambda_c$, then $\Gamma_\lambda = K_\lambda = \{\beta_0\}$ for some $\beta_0 \in \mathbb{R}^d$.

(iii) If $\lambda > \lambda_c$, then $\Gamma_\lambda = K_\lambda = \emptyset$.

(iv) $K^*_\lambda = -K_\lambda$, and $\beta_0 = 0$ if $L^* = L$.

(v) $E(\beta)$ is an algebraically simple eigenvalue and it is a real analytic. Hess $E(\beta)$ is neg. def. for $\beta \in \mathbb{R}^d$. The equality $\lambda_c = \sup_{\beta \in \mathbb{R}^d} E(\beta)$ holds, and the sup is attained uniquely at $\beta_0$ in (ii). $\nabla E(\beta) = 0$ if and only if $\beta = \beta_0$. 

(vi) $\Gamma_\lambda = \{\beta \in \mathbb{R}^d; E(\beta) = \lambda\}$ and $K_\lambda = \{\beta \in \mathbb{R}^d; E(\beta) \geq \lambda\}$.

Let $B_R = \{|x| < R\}$. Let $L_R$ be the Dirichlet realization of $L$ in $L^2(B_R)$: $D(L_R) = H^2(B_R) \cap H^2(B_R)$. If $L$ is subcritical, then the resolvent $L^{-1}_R$, and its integral kernel (the Green function) $G_R(x, y) > 0$, and the limit $G(x, y) = \lim_{R \to \infty} G_R(x, y)$ which is called the minimal Green function of $L$ on $\mathbb{R}^d$.

First, suppose that $\lambda_c > 0$. Then $L$ is subcritical, and for any $s \in S^{d-1}$, take $\beta_s \in \Gamma_0$ s.t. $\sup_{\beta \in \Gamma_0} \beta \cdot s = \beta_s \cdot s$.

**Theorem 1** Suppose that $\lambda_c > 0$. Then the minimal Green function $G$ of $L$ admits the following asymptotics as $|x - y| \to \infty$:

$$G(x, y) = \frac{1}{|\nabla E(\beta_s)|\sqrt{C(\beta_s)}} \frac{e^{-(x-y) \cdot \beta_s}}{(2\pi |x-y|)^{d-1/2}} \frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})_{L^2(T^d)}} \times (1 + O(|x-y|^{-1})), $$

where $s = (x-y)/|x-y|$, and $C(\beta_s)$ is the Gauss-Kronecker curvature of $\Gamma_0$ at $\beta_s$.

Schroeder [S] gave a lower and upper bounds.

Let us determine explicitly the Martin compactification of $\mathbb{R}^d$ with respect to $L$ in the case $\lambda_c > 0$. Fix a reference point $x_0$ in $\mathbb{R}^d$. Then the following proposition is a consequence of Theorem 1.

**Proposition 2** Suppose that $\lambda_c > 0$. Then for any sequence $\{y_n\}$ in $\mathbb{R}^d$ such that $|y_n| \to \infty$ and $y_n/|y_n| \to \nu$ as $n \to \infty$,

$$\lim_{n \to \infty} \frac{G(x, y_n)}{G(x_0, y_n)} = e^{-(x-x_0) \cdot \beta_- \nu} \frac{u_{\beta_- \nu}(x)}{u_{\beta_- \nu}(x_0)} =: K(x, \nu). $$

$\psi \in C_L$ is minimal (If $\phi \in C_L$ satisfies $\phi(x) \leq \psi(x)$, then $\phi(x) = c\psi(x)$) if and only if $\psi = e^{\beta_0 u}(x) \in C_L$ where $u$ is periodic (see [A]). Thus $\Gamma_0 \simeq$ the minimal Martin boundary. On the other hand $K(x, \nu) \in C_L$, $K(x_0, \nu) = 1$, $K(x, \nu) \neq K(x, \nu')$ if $\nu \neq \nu'$. $K(x, \nu)$ is minimal. Hence we can determine the Martin boundary and Martin compactification of $\mathbb{R}^d$ for $L$ as follows.
**Theorem 3** Suppose that $\lambda_c > 0$. Then the Martin boundary and the minimal Martin boundary of $\mathbb{R}^d$ for $L$ are both equal to the sphere $S^{d-1}$ at infinity which is homeomorphic to $\Gamma_0$; the Martin kernel at $\nu \in S^{d-1}$ is equal to $K(\cdot, \nu)$; and the Martin compactification of $\mathbb{R}^d$ for $L$ is equal to

$$\{x \in \mathbb{R}^d; |x| < 1\} \cup [1, \infty) \times S^{d-1}$$

equipped with the standard topology.

Next, suppose that $\lambda_c = E(\beta_0) = 0$. Then Pinsky [P] proved that $L$ is critical if $d \leq 2$, and subcritical if $d \geq 3$.

**Theorem 4** Let $d \geq 3$. Suppose $\lambda_c = 0$. Put $H = -\text{Hess} \ E(\beta_0)$. Then the minimal Green function $G$ of $L$ admits the following asymptotics as $|x-y| \to \infty$:

$$G(x, y) = \frac{\Gamma((d-2)/2)}{2\pi^{d/2}(\det H)^{1/2}} \frac{e^{-(x-y) \beta_0}}{|H^{-1/2}(x-y)|^{d-2}} \frac{u_{\beta_0}(x)v_{\beta_0}(y)}{(u_{\beta_0}, v_{\beta_0})} \times (1 + O(|x-y|^{-1})).$$

We determine directly from Theorem 4 the Martin boundary. These results, however, are also simple consequences of the known result that $C_L$ is one dimensional in this case.

**Theorem 5** Let $d \geq 3$. Suppose that $\lambda_c = E(\beta_0) = 0$. Then for any sequence $\{y_n\}$ in $\mathbb{R}^d$ with $|y_n| \to \infty$ as $n \to \infty$,

$$\lim_{n \to \infty} \frac{G(x, y_n)}{G(x_0, y_n)} = e^{-(x-x_0) \beta_0} \frac{u_{\beta_0}(x)}{u_{\beta_0}(x_0)}, \quad x \in \mathbb{R}^d.$$

The Martin boundary and the minimal Martin boundary are both equal to one point $\infty$ at infinity; the Martin kernel at $\infty$ is equal to the right hand side; and the Martin compactification of $\mathbb{R}^d$ for $L$ is equal to the one point compactification $\mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$.

**2 Proof of Theorem 1**

Assume $\lambda_c = E(\beta_0) > 0$. Put $L_0 = e^{\beta_0 x}Le^{-\beta_0 x}$. Then the principal eigenvalue $E_0(0)$ of $\beta = 0$ of $L_0$ is positive, and the minimal Green function $G_0(x, y)$ of $L_0$ satisfies

$$G_0(x, y) = e^{\beta_0 x}G(x, y)e^{-\beta_0 y}.$$  

Regard $L$ as a closed operator in $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. We have only to show the following.

**Theorem 6** Assume $E(0) > 0$. Then there exists the resolvent $L^{-1}$; and the integral kernel of $L^{-1}$ equals the minimal Green function and admits the same asymptotics as in Theorem 1.

Let

$$\mathcal{H} = L^2((-\pi, \pi)^d, (2\pi)^{-d}d\zeta; L^2(\mathbb{T}^d)).$$

Define an operator $\mathcal{F} : L^2(\mathbb{R}^d) \to \mathcal{H}$ by

$$\mathcal{F}f(\zeta, x) = \sum_{l \in \mathbb{Z}^d} f(x-l)e^{-i(x-l)\zeta}, \quad \zeta \in (-\pi, \pi)^d, \quad x \in \mathbb{T}^d.$$

(Bloch transformation). Then $\mathcal{F}$ is a unitary operator, and an isometric isomorphism from $H^1(\mathbb{R}^d)$ to $L^2((-\pi, \pi)^d, (2\pi)^{-d}d\zeta; H^1(\mathbb{T}^d))$. The adjoint $\mathcal{F}^*$ is given by, for $g \in \mathcal{H}$,

$$(\mathcal{F}^*g)(x-l) = (2\pi)^{-d} \int_{(-\pi, \pi)^d} e^{i(x-l)\zeta}g(\zeta, x) d\zeta, \quad x \in \mathbb{T}^d, l \in \mathbb{Z}^d.$$
We have
\[ \mathcal{F}(\nabla f) = (\nabla x + i \zeta) \mathcal{F} f \quad \text{if} \quad \mathcal{F} f \text{ is periodic} \]
\[ \Rightarrow \mathcal{F} L = L(\zeta) \mathcal{F}. \]

**Proposition 7** Let \( E(0) > 0 \). Then there exists the resolvent \( L^{-1}(\zeta) \), \( \zeta \in \mathbb{R}^d \); and \( L^{-1} = \mathcal{F}^* L(\zeta)^{-1} \mathcal{F} \), i.e., for \( x \in \mathbb{T}^d \), \( l \in \mathbb{Z}^d \), and \( f \in L^2(\mathbb{R}^d) \),
\[ L^{-1} f(x-l) = (2\pi)^{-d} \int_{(-\pi,\pi)^d} F(\zeta) d\zeta, \]
where
\[ F(\zeta) = e^{i(x-l) \cdot \zeta} L(\zeta)^{-1} \left( \sum_{m \in \mathbb{Z}^d} f(\cdot - m) e^{-i(\cdot-m) \cdot \zeta} \right)(x). \]
Moreover, \( F(\zeta) \) is \( 2\pi \mathbb{Z}^d \)-periodic.

\( \{L(\zeta)\}_{\zeta \in \mathbb{C}^d} \) is an analytic family of type (B). By the analytic perturbation theory, \( E(\beta) \) has an analytic continuation \( \Lambda(\zeta) \), \( \zeta = \alpha + i \beta \), near \( \zeta = i \beta_z \); note that \( E(\beta) = \Lambda(i \beta) \).

Moreover \( \Lambda(\zeta) \) is also an algebraically simple eigenvalue of \( L(\zeta) \) with eigenfunction \( u_\zeta \): \( (L(\zeta) - \Lambda(\zeta)) u_\zeta = 0 \). \( \overline{\Lambda(\zeta)} \) is an algebraically simple eigenvalue of \( L(\zeta)^* \) with eigenfunction \( v_\zeta \): \( (L(\zeta)^* - \overline{\Lambda(\zeta)}) v_\zeta = 0 \).

Put \( \eta_z := \beta_z / |\beta_z| \), and let \( \{e_{s,1}, \ldots, e_{s,d-1}, s\} \) be an orthonormal basis of \( \mathbb{R}^d \). We introduce new coordinates \((w, z)\) near \( i \beta_z \) such that \( \zeta = w \eta_z + z \cdot e_s = w \eta_z + \sum_{j=1}^{d-1} z_j e_{s,j} \), \( w \in \mathbb{C} \), \( z = (z_1, \ldots, z_{d-1}) \in \mathbb{R}^{d-1} \).

**Proposition 8** For \( z \in \mathbb{R}^{d-1} \) with \( |z| \ll 1 \), the resolvent \( L(w \eta_z + z \cdot e_s)^{-1} \) has a simple pole \( w_s(z) \) as a function of \( w \), and has the following asymptotics at the pole
\[ L(w \eta_z + z \cdot e_s)^{-1} = \frac{A_{s,z}}{w - w_s(z)} + O(1). \]
Here \( A_{s,z} \) is a rank one operator-valued function with
\[ A_{s,z} = \frac{1}{\eta_z \cdot \nabla \Lambda(\zeta(z))} \left( \frac{\langle \cdot, u_\zeta(z) \rangle u_\zeta(z)}{\langle u_\zeta(z), u_\zeta(z) \rangle} \right), \quad \zeta(z) = w_s(z) \eta_z + z \cdot e_s \]
and \( w_s(z) \) satisfies \( w_s(0) = i |\beta_z| \), for \( 1 \leq j, k \leq d-1 \),
\[ \frac{\partial w_s}{\partial z_j}(0) = 0, \]
\[ \frac{\partial^2 w_s}{\partial z_j \partial z_k}(0) = i \frac{\partial^2 \text{Im} w_s}{\partial z_j \partial z_k}(0) = \frac{e_{s,j} \cdot \text{Hess} E(\beta) e_{s,k}}{\eta_z \cdot \nabla E(\beta)}, \]
\[ \text{Hess} \text{Im} w_s(0) = \left( \frac{\partial^2 \text{Im} w_s}{\partial z_j \partial z_k}(0) \right)_{1 \leq j, k \leq d-1} : \text{positive definite}. \]
Here the function \( \zeta(z) = w_s(z) \eta_z + z \cdot e_s \) is the zeros of \( \Lambda(\zeta) \).

**Proof.** \( \Lambda(\zeta) \) is an algebraically simple eigenvalue, so
\[ (L(\zeta) - \lambda)^{-1} = \frac{P(\zeta)}{\Lambda(\zeta) - \lambda} + O(1), \quad P(\zeta) = \frac{(\cdot, u_\zeta) u_\zeta}{(u_\zeta, u_\zeta)}. \]
Putting $\lambda = 0$, we have
\[
L(\zeta)^{-1} = \frac{P(\zeta)}{\Lambda(\zeta)} + O(1).
\]
Noting that
\[
\Lambda(\zeta) = \Lambda(w \eta_s + z \cdot e_s) = (w - w_s(z)) \eta_s \cdot \nabla \Lambda(w_s(z) \eta_s + z \cdot e_s) + O((w - w_s(z))^2),
\]
we have the proposition. $\Box$

Let $P : t \eta_s + z \cdot e_s \to z$ be a projection, and $Q = P(-\pi, \pi)^d$. We have $(-\pi, \pi)^d = \{t \eta_s + z \cdot e_s; z \in Q, \exists t_1(z) < t < \exists t_2(z)\}$. We change the integral variables from $\zeta$ to $(t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ such that $\zeta = t \eta_s + z \cdot e_s$ to obtain that
\[
(L^{-1}f)(x - l) = (2\pi)^{-d} \int_{(-\pi, \pi)^d} F(\zeta) d\zeta = \frac{|D_s|}{(2\pi)^d} \int_Q dt \int_{t_1(z)}^{t_2(z)} F(t \eta_s + z \cdot e_s),
\]
where $D_s = \det(\eta_s, e_{s,1}, \cdots, e_{s,d-1})$, and
\[
F(\zeta) = e^{i(x-l) \cdot \zeta} L(\zeta)^{-1} \left( \sum_{m \in \mathbb{Z}^d} f(\cdot - m) e^{-i(\cdot - m) \cdot \zeta} \right)(x).
\]

For $0 < \delta \ll 1$, put
\[
U_\delta = \{ z \in \mathbb{R}^{d-1}; \text{Im} \ w_s(z) < |\beta_s| + \delta \}.
\]
For $z \in Q$ let $C(z) = C_1(z) \cup C_2(z)$ be a closed contour in $\mathbb{C}$:
\[
C_1(z) = \{ t : t_1(z) \to t_2(z) \}, \quad C_2(z) = \{ t_2(z) + it; t : 0 \to |\beta_s| + h \} \cup \{ t + i(|\beta_s| + h); t : t_2(z) \to t_1(z) \} \cup \{ t_1(z) + it; t : |\beta_s| + h \to 0 \}
\]
where $h = 2\delta$ if $z \in U_\delta$, $h = \delta/2$ if $z \in Q \setminus U_\delta$. For $z \in U_\delta$ the integrand has only a simple pole $w_s(z)$ near and inside $C(z)$, and for $z \in Q \setminus U_\delta$ the integrand is holomorphic near and inside $C(z)$. Hence by the residue theorem we have
\[
(L^{-1}f)(x - l) = I_1 f(x - l) + I_2 f(x - l)
\]
with $\zeta(z) = w_s(z) \eta_s + z \cdot e_s$, where
\[
I_1 f(x - l) = \frac{2\pi i |D_s|}{(2\pi)^d} \int_{U_\delta} dz \exp[i(x - l) \cdot \zeta(z)] \times \frac{\left( \sum_m f(\cdot - m) \exp[-i(\cdot - m) \cdot \zeta(z)] \right) \eta_s \cdot \nabla \Lambda(\zeta(z)) (u_{\zeta(z)}, v_{\zeta(z)})}{\eta_s \cdot \nabla \Lambda(\zeta(z)) (u_{\zeta(z)}, v_{\zeta(z)})} u_{\zeta(z)}(x)
\]
and
\[
I_2 f(x - l) = -\frac{|D_s|}{(2\pi)^d} \int_Q dz \int_{C_2(z)} dw F(w \eta_s + z \cdot e_s).
\]
The integral kernel $I_1(x, y)$, $x, y \in \mathbb{R}^d$, of $I_1$ is equal to

$$I_1(x, y) = \frac{i |D_s|}{(2\pi)^{d-1}} \int_{U_{\delta}} dz \exp[i(x - y) \cdot (w_s(z) \eta_s + z \cdot e_s)] a(z; x, y),$$

$$a(z; x, y) := \frac{1}{\eta_s \cdot \nabla \Lambda(\zeta(z))} \frac{u_{\zeta}(x) v_{\zeta}(y)}{(u_{\zeta(z)}, v_{\zeta(z)})}.$$

Take $s = (x - y)/|x - y|$. We regard $(x - y) \cdot \eta_s \gg 1$ as a large parameter, and note that $(x - y) \cdot (z \cdot e_s) = 0$. We have shown that the critical point of $w_s(z)$ is $z = 0$. By the saddle point method

$$I_1(x, y) = \frac{-|D_s|}{(2\pi)^{d-1}} \left( \frac{2\pi}{(x - y) \cdot \eta_s} \right)^{(d-1)/2} \frac{e^{-(x-y) \cdot \beta_s}}{(\det \Hess \Im w_s(0))^{1/2}} \left( \frac{1}{\eta_s \cdot \nabla E(\beta_s)} \frac{u_{\beta_s}(x) v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})_{L^2(T^d)}} + O(|x - y|^{-1}) \right).$$

This leads to the main term of the asymptotics.

We can show that the integral kernel of $I_2$ satisfies

$$|I_2(x, y)| \leq C e^{-(x-y) \cdot \beta_s} e^{-c|x-y|},$$

using the $2\pi \mathbb{Z}^d$-periodicity of $F(\zeta)$. These are an outline of the proof of Theorem 1. □

**Remark.** We can get the following asymptotic expansion. Assume that $\lambda_c > 0$. There exist bounded functions $g_j(x, y)$, $j = 1, 2, \ldots$, s.t. for any natural number $n$

$$G(x, y) = \frac{1}{|\nabla E(\beta_s)| \sqrt{C(\beta_s)}} \left( 2\pi |x - y| \right)^{(d-1)/2} \frac{u_{\beta_s}(x) v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})_{L^2(T^d)}}$$

$$\times \left( 1 + \sum_{j=1}^{n} \frac{g_j(x, y)}{|x - y|^j} + O(|x - y|^{-n-1}) \right).$$

### References


