Title
Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients: joint work with Minoru Murata (Potential Theory and its related Fields)

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Citation
数理解析研究所講究録 (2009), 1669: 157-162

Issue Date
2009-11

URL
http://hdl.handle.net/2433/141122

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients

(joint work with Minoru Murata)

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1 Results

The main purpose of this paper is to establish asymptotics at infinity of Green functions for elliptic equations with periodic coefficients on $\mathbb{R}^d$ and to determine the Martin boundary for the elliptic operators.

Let

$$L = -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k} (a_{jk}(x) \frac{\partial}{\partial x_j}) - \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

be a second-order elliptic operator on $\mathbb{R}^d$ with periodic coefficients, where $d \geq 2$, $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$, $a(x) = (a_{jk}(x))_{j,k=1}^{d}$, and $b(x) = (b_j(x))_{j=1}^{d}$. We assume that the coefficients are $\mathbb{Z}^d$-periodic, real-valued smooth functions on $\mathbb{R}^d$. We assume that $a$ is a symmetric matrix-valued function satisfying for some $\alpha > 0$

$$\alpha|\xi|^2 \leq \sum_{j,k=1}^{d} a_{jk}(x) \xi_j \xi_k \leq \alpha^{-1}|\xi|^2, \quad x, \xi \in \mathbb{R}^d.$$

For $\zeta \in \mathbb{C}^d$, define an operator $L(\zeta)$ on the $d$-dimensional torus by

$$L(\zeta) = e^{-i\zeta \cdot x} L e^{i\zeta \cdot x}$$

$$= -(\nabla + i\zeta) \cdot a(x)(\nabla + i\zeta) - b(x) \cdot (\nabla + i\zeta) + c(x).$$

We regard $L(\zeta)$ as a closed operator in $L^2(T^d)$ with domain $H^2(T^d)$.

By the Krein-Rutman theorem, for each $\beta \in \mathbb{R}^d$, $L(i\beta) = e^{\beta \cdot x} L e^{-\beta \cdot x}$ has the principal eigenvalue $E(\beta)$, i.e., $L(i\beta)$ has an eigenvalue $E(\beta) \in \mathbb{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function $u_\beta \in H^2(T^d)$; $E(\beta)$ is also an eigenvalue of $L(i\beta)^*$ of multiplicity one such that the eigenspace is generated by a positive function $v_\beta \in H^2(T^d)$.

Put

$$C_L = \{ u \in H^1_{loc}(\mathbb{R}^d); Lu = 0 \text{ and } u > 0 \}. $$

When a positive Green function for $L$ on $\mathbb{R}^d$ exists, $L$ is called subcritical; in this case $C_L \neq \emptyset$. When a positive Green function for $L$ on $\mathbb{R}^d$ dose not exist but $C_L \neq \emptyset$, $L$ is called critical. Let $\lambda_c$ be the generalized principal eigenvalue of $L$ on $\mathbb{R}^d$.

$$\lambda_c := \sup \{ \lambda \in \mathbb{R}; L - \lambda \text{ is subcritical} \}. $$

Then it is known that $-\infty < \lambda_c < \infty$, $L - \lambda$ is subcritical for $\lambda < \lambda_c$, and $L - \lambda_c$ is subcritical or critical. The formal adjoint operator $L^*$ of $L$ is subcritical (or critical) if and only if $L$ is subcritical (or critical), and the generalized principal eigenvalue of $L$ and $L^*$ coincide.

For $\lambda \in \mathbb{R}$, put

$$\Gamma_\lambda := \{ \beta \in \mathbb{R}^d; \exists \psi(x) = e^{-\beta \cdot x} u(x) \in C_L-\lambda \text{ where } u \text{ is periodic} \}$$

$$K_\lambda := \{ \beta \in \mathbb{R}^d; \exists \psi = e^{-\beta \cdot x} u(x) > 0 \text{ s.t. } (L-\lambda)\psi \geq 0 \text{ and } u \text{ is periodic} \}. $$
Define $K^{*}_{\lambda}$ and $\Gamma^{*}_{\lambda}$ for $L^{*} - \lambda$ analogously to $K_{\lambda}$ and $\Gamma_{\lambda}$ for $L - \lambda$. Agmon, Pinsky and Kuchment-Pinchover proved the following theorem.

**Theorem AP** ([A], [P], [KP])

(i) If $\lambda < \lambda_{c}$, then $K_{\lambda}$ is a $d$-dimensional strictly convex compact set with smooth boundary $\Gamma_{\lambda} = \partial K_{\lambda}$.

(ii) If $\lambda = \lambda_{c}$, then $\Gamma_{\lambda} = K_{\lambda} = \{\beta_{0}\}$ for some $\beta_{0} \in \mathbb{R}^{d}$.

(iii) If $\lambda > \lambda_{c}$, then $\Gamma_{\lambda} = K_{\lambda} = \emptyset$.

(iv) $K^{*}_{\lambda} = -K_{\lambda}$, and $\beta_{0} = 0$ if $L^{*} = L$

(v) $E(\beta)$ is an algebraically simple eigenvalue and it is a real analytic. $\text{Hess } E(\beta)$ is neg. def. for $\beta \in \mathbb{R}^{d}$. The equality $\lambda_{c} = \sup_{\beta \in \mathbb{R}^{d}} E(\beta)$ holds, and the sup is attained uniquely at $\beta_{0}$ in (ii). $\nabla E(\beta) = 0$ if and only if $\beta = \beta_{0}$.

(vi) $\Gamma_{\lambda} = \{\beta \in \mathbb{R}^{d}; E(\beta) = \lambda\}$ and $K_{\lambda} = \{\beta \in \mathbb{R}^{d}; E(\beta) \geq \lambda\}$.

Let $B_{R} = \{|x| < R\}$. Let $L_{R}$ be the Dirichlet realization of $L$ in $L^{2}(B_{R})$: $D(L_{R}) = H_{0}^{1}(B_{R}) \cap H^{2}(B_{R})$. If $L$ is subcritical, then there is a unique solution $U_{R}^{1}$, and its integral kernel (the Green function) $G_{R}(x, y) > 0$, and the limit $G(x, y) = \lim_{R \to \infty} G_{R}(x, y)$ which is called the minimal Green function of $L$ on $\mathbb{R}^{d}$.

First, suppose that $\lambda_{c} > 0$. Then $L$ is subcritical, and for any $\tau \in S^{d-1}$, take $\beta_{s} \in \Gamma_{0}$ s.t. $\sup_{\beta \in \Gamma_{0}} \beta \cdot s = \beta_{s} \cdot s$.

**Theorem 1** Suppose that $\lambda_{c} > 0$. Then the minimal Green function $G$ of $L$ admits the following asymptotics as $|x - y| \to \infty$:

$$G(x, y) = \frac{1}{|\nabla E(\beta_{s})| \sqrt{C(\beta_{s})}} \frac{e^{-(x-y) \beta_{s}} u_{\beta_{s}}(x)v_{\beta_{s}}(y)}{(u_{\beta_{s}}, v_{\beta_{s}})_{L^{2}(T^{d})}} \times (1 + O(|x-y|^{-1})), $$

where $s = (x-y)/(x-y)$, and $C(\beta_{s})$ is the Gauss-Kronecker curvature of $\Gamma_{0}$ at $\beta_{s}$.

Schroeder [S] gave a lower and upper bounds.

Let us determine explicitly the Martin compactification of $\mathbb{R}^{d}$ with respect to $L$ in the case $\lambda_{c} > 0$. Fix a reference point $x_{0}$ in $\mathbb{R}^{d}$. Then the following proposition is a consequence of Theorem 1.

**Proposition 2** Suppose that $\lambda_{c} > 0$. Then for any sequence $\{y_{n}\}$ in $\mathbb{R}^{d}$ such that $|y_{n}| \to \infty$ and $y_{n}/|y_{n}| \to \nu$ as $n \to \infty$,

$$\lim_{n \to \infty} G(x, y_{n}) = e^{-(x-x_{0}) \beta_{\nu}} \frac{u_{\beta_{\nu}}(x)}{u_{\beta_{\nu}}(x_{0})} =: K(x, \nu).$$

$\psi \in C_{L}$ is minimal (If $\varphi \in C_{L}$ satisfies $\varphi(x) \leq \psi(x)$, then $\varphi(x) = c\psi(x)$) if and only if $\psi = e^{\beta_{\nu}u(x)} \in C_{L}$ where $u$ is periodic (see [A]). Thus $\Gamma_{0} \simeq$ the minimal Martin boundary. On the other hand $K(x, \nu) \in C_{L}$, $K(x_{0}, \nu) = 1$, $K(x, \nu) \neq K(x, \nu')$ if $\nu \neq \nu'$. $K(x, \nu)$ is minimal. Hence we can determine the Martin boundary and Martin compactification of $\mathbb{R}^{d}$ for $L$ as follows.
Theorem 3 Suppose that $\lambda_c > 0$. Then the Martin boundary and the minimal Martin boundary of $\mathbb{R}^d$ for $L$ are both equal to the sphere $S^{d-1}$ at infinity which is homeomorphic to $\Gamma_0$; the Martin kernel at $\nu \in S^{d-1}$ is equal to $K(\cdot, \nu)$; and the Martin compactification of $\mathbb{R}^d$ for $L$ is equal to

$$\{x \in \mathbb{R}^d; |x| < 1\} \cup [1, \infty] \times S^{d-1}$$

equipped with the standard topology.

Next, suppose that $\lambda_c = E(\beta_0) = 0$. Then Pinsky [P] proved that $L$ is critical if $d \leq 2$, and subcritical if $d \geq 3$.

Theorem 4 Let $d \geq 3$. Suppose $\lambda_c = 0$. Put $H = -\text{Hess } E(\beta_0)$. Then the minimal Green function $G$ of $L$ admits the following asymptotics as $|x - y| \to \infty$:

$$G(x, y) = \frac{\Gamma(d-2)}{2\pi^{d/2}(\det H)^{1/2}} \frac{e^{-(x-y)\beta_0}}{|H^{-1/2}(x-y)|^{d-2}} \frac{u_{\beta_0}(x)v_{\beta_0}(y)}{(u_{\beta_0}, v_{\beta_0})} \times (1 + O(|x - y|^{-1})).$$

We determine directly from Theorem 4 the Martin boundary. These results, however, are also simple consequences of the known result that $C_L$ is one dimensional in this case.

Theorem 5 Let $d \geq 3$. Suppose that $\lambda_c = E(\beta_0) = 0$. Then for any sequence $\{y_n\}$ in $\mathbb{R}^d$ with $|y_n| \to \infty$ as $n \to \infty$,

$$\lim_{n \to \infty} \frac{G(x, y_n)}{G(x_0, y_n)} = e^{-(x-x_0)\beta_0} \frac{u_{\beta_0}(x)}{u_{\beta_0}(x_0)}, \quad x \in \mathbb{R}^d.$$ 

The Martin boundary and the minimal Martin boundary are both equal to one point $\infty$ at infinity; the Martin kernel at $\infty$ is equal to the right hand side; and the Martin compactification of $\mathbb{R}^d$ for $L$ is equal to the one point compactification $\mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$.

2 Proof of Theorem 1

Assume $\lambda_c = E(\beta_0) > 0$. Put $L_0 = e^{\beta_0 x}Le^{-\beta_0 x}$. Then the principal eigenvalue $E_0(0)$ of $\beta = 0$ of $L_0$ is positive, and the minimal Green function $G_0(x, y)$ of $L_0$ satisfies $G_0(x, y) = e^{\beta_0 x}G(x, y)e^{-\beta_0 y}$. Regard $L$ as a closed operator in $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. We have only to show the following.

Theorem 6 Assume $E(0) > 0$. Then there exists the resolvent $L^{-1}$; and the integral kernel of $L^{-1}$ equals the minimal Green function and admits the same asymptotics as in Theorem 1.

Let

$$\mathcal{H} = L^2((-\pi, \pi)^d, (2\pi)^{-d} d\zeta; L^2(\mathbb{T}^d)).$$

Define an operator $\mathcal{F} : L^2(\mathbb{R}^d) \to \mathcal{H}$ by

$$(\mathcal{F}f)(\zeta, x) = \sum_{l \in \mathbb{Z}^d} f(x-l)e^{-i(x-l)\zeta}, \quad \zeta \in (-\pi, \pi)^d, \quad x \in \mathbb{T}^d$$

(Bloch transformation). Then $\mathcal{F}$ is a unitary operator, and an isometric isomorphism from $H^1(\mathbb{R}^d)$ to $L^2((-\pi, \pi)^d, (2\pi)^{-d} d\zeta; H^1(\mathbb{T}^d))$. The adjoint $\mathcal{F}^* \mathcal{F}$ is given by, for $g \in \mathcal{H}$,

$$(\mathcal{F}^*g)(x-l) = (2\pi)^{-d} \int_{(-\pi, \pi)^d} e^{i(x-l)\zeta} g(\zeta, x) d\zeta, \quad x \in \mathbb{T}^d, \quad l \in \mathbb{Z}^d.$$
We have
\[ \mathcal{F}(\nabla f) = (\nabla_x + i(\mathcal{F}f \right) \Rightarrow \mathcal{F}L = L(\xi)\mathcal{F}. \]

**Proposition 7** Let \( E(0) > 0 \). Then there exists the resolvent \( L^{-1}(\xi) \), \( \xi \in \mathbb{R}^d \); and \( L^{-1} = \mathcal{F}^*L(\xi)^{-1}\mathcal{F} \), i.e., for \( x \in \mathbb{T}^d \), \( l \in \mathbb{Z}^d \), and \( f \in L^2(\mathbb{R}^d) \),
\[ L^{-1}f(x-l) = (2\pi)^{-d}\int_{(-\pi,\pi)^d} F(\xi)d\xi, \]
where
\[ F(\xi) = e^{i(x_l-\xi)l}L(\xi)^{-1}\left( \sum_{m \in \mathbb{Z}^d} f(\cdot-m)e^{-i(\cdot-m)\xi} \right). \]

Moreover, \( F(\xi) \) is \( 2\pi \mathbb{Z}^d \)-periodic.

\( \{L(\xi)\}_{\xi \in \mathbb{C}^d} \) is an analytic family of type (B). By the analytic perturbation theory, \( E(\beta) \) has an analytic continuation \( \Lambda(\xi) \), \( \xi = \alpha + i\beta \), near \( \xi = i\beta_s \); note that \( E(\beta) = \Lambda(i\beta) \). Moreover \( \Lambda(\xi) \) is also an algebraically simple eigenvalue of \( L(\xi) \) with eigenfunction \( u_\xi: \)
\[ (L(\xi) - \Lambda(\xi))u_\xi = 0. \]

Put \( \eta_s := \beta_s/|\beta_s| \), and let \( \{e_{s,1}, \ldots, e_{s,d-1}, s\} \) be an orthonormal basis of \( \mathbb{R}^d \). Put \( e_s := (e_{s,1}, \ldots, e_{s,d-1}) \). We introduce new coordinates \((w, z)\) near \( i\beta_s \) such that \( \zeta = w\eta_s + z \cdot e_s = w\eta_s + \sum_{j=1}^{d-1} z_j e_{s,j}, w \in \mathbb{C}, \ z = (z_1, \ldots, z_{d-1}) \in \mathbb{R}^{d-1} \).

**Proposition 8** For \( z \in \mathbb{R}^{d-1} \) with \( |z| \ll 1 \), the resolvent \( L(w\eta_s + z \cdot e_s)^{-1} \) has a simple pole \( w_s(z) \) as a function of \( w \), and has the following asymptotics at the pole
\[ L(w\eta_s + z \cdot e_s)^{-1} = \frac{A_{s,z}}{w - w_s(z)} + O(1). \]
Here \( A_{s,z} \) is a rank one operator-valued function with
\[ A_{s,z} = \frac{1}{\eta_s \cdot \nabla \Lambda(\zeta(z))} \left( \frac{1}{(u_{\zeta(z)}, v_{\zeta(z)})} \right), \quad \zeta(z) = w_s(z)\eta_s + z \cdot e_s \]
and \( w_s(z) \) satisfies \( w_s(0) = i|\beta_s|, \) for \( 1 \leq j, k \leq d-1 \),
\[ \frac{\partial w_s}{\partial z_j}(0) = 0, \quad \frac{\partial^2 w_s}{\partial z_j \partial z_k}(0) = \frac{i e_{s,j} \cdot \text{Hess} E(\beta_s)e_{s,k}}{\eta_s \cdot \nabla E(\beta_s)}, \]
\[ \text{Hess Im } w_s(0) = \left( \frac{\partial^2 \text{Im } w_s}{\partial z_j \partial z_k}(0) \right)_{1 \leq j, k \leq d-1} : \text{ positive definite.} \]
Here the function \( \zeta(z) = w_s(z)\eta_s + z \cdot e_s \) is the zeros of \( \Lambda(\zeta) \).

**Proof.** \( \Lambda(\xi) \) is an algebraically simple eigenvalue, so
\[ (L(\xi) - \lambda)^{-1} = \frac{P(\xi)}{\Lambda(\xi) - \lambda} + O(1), \quad P(\xi) = \left( \frac{1}{(u_{\zeta}, v_{\zeta})} \right). \]
Putting $\lambda = 0$, we have
\[ L(\zeta)^{-1} = \frac{P(\zeta)}{\Lambda(\zeta)} + O(1). \]
Noting that
\[ \Lambda(\zeta) = \Lambda(w\eta_s + z \cdot e_s) \]
\[ = (w - w_s(z))\eta_s \cdot \nabla \Lambda(w_s(z)\eta_s + z \cdot e_s) + O((w - w_s(z))^2), \]
we have the proposition. $\square$

Let $P : t\eta_s + z \cdot e_s \rightarrow z$ be a projection, and $Q = P(-\pi, \pi)^d$. We have $(-\pi, \pi)^d = \{t\eta_s + z \cdot e_s; z \in Q, \exists t_1(z) < t < \exists t_2(z)\}$. We change the integral variables from $\zeta$ to $(t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ such that $\zeta = t\eta_s + z \cdot e_s$ to obtain that
\[ (L^{-1}f)(x-l) = (2\pi)^{-d} \int_{(-\pi,\pi)^d} F(\zeta) d\zeta \]
\[ = \frac{|D_s|}{(2\pi)^d} \int_Q dz \int_{t_1(z)}^{t_2(z)} dt F(t\eta_s + z \cdot e_s), \]
where $D_s = \det(\eta_s, e_{s,1}, \cdots, e_{s,d-1})$, and
\[ F(\zeta) = e^{i(x-l) \cdot \zeta} L(\zeta)^{-1} \left( \sum_{m \in \mathbb{Z}^d} f(\cdot - m) e^{-i(\cdot - m) \cdot \zeta} \right)(x). \]
For $0 < \delta \ll 1$, put
\[ U_\delta = \{z \in \mathbb{R}^{d-1}; \text{Im } w_s(z) < |\beta_s| + \delta\}. \]
For $z \in Q$ let $C(z) = C_1(z) \cup C_2(z)$ be a closed contour in $\mathbb{C}$:
\[ C_1(z) = \{t : t_1(z) \rightarrow t_2(z)\}, \]
\[ C_2(z) = \{t_2(z) + it; t : 0 \rightarrow |\beta_s| + h\} \]
\[ \cup \{t + i(|\beta_s| + h); t : t_2(z) \rightarrow t_1(z)\} \]
\[ \cup \{t_1(z) + it; t : |\beta_s| + h \rightarrow 0\} \]
where $h = 2\delta$ if $z \in U_\delta$, $h = \delta/2$ if $z \in Q \setminus U_\delta$. For $z \in U_\delta$ the integrand has only a simple pole $w_s(z)$ near and inside $C(z)$, and for $z \in Q \setminus U_\delta$ the integrand is holomorphic near and inside $C(z)$. Hence by the residue theorem we have
\[ (L^{-1}f)(x-l) = I_1 f(x-l) + I_2 f(x-l) \]
with $\zeta(z) = w_s(z)\eta_s + z \cdot e_s$, where
\[ I_1 f(x-l) = \frac{2\pi i |D_s|}{(2\pi)^d} \int_{U_\delta} dz \exp[i(x-l) \cdot \zeta(z)] \]
\[ \times \left( \sum_m f(\cdot - m) \exp[-i(\cdot - m) \cdot \zeta(z)], v_{\zeta(z)} \right) u_{\zeta(z)}(x) \]
\[ \eta_s \cdot \nabla \Lambda(\zeta(z))(u_{\zeta(z)}, v_{\zeta(z)}) \]
\[ I_2 f(x-l) = -\frac{|D_s|}{(2\pi)^d} \int_Q dz \int_{C_2(z)} dw F(w\eta_s + z \cdot e_s). \]
The integral kernel $I_1(x, y)$, $x, y \in \mathbb{R}^d$, of $I_1$ is equal to

$$I_1(x, y) = \frac{|D_s|}{(2\pi)^{d-1}} \int_{U_z} dz \exp[i(x - y) \cdot (w_s(z)\eta_s + z \cdot e_s)]a(z; x, y),$$

$$a(z; x, y) := \frac{1}{\eta_s \cdot \nabla \Lambda(z)} \frac{u_{\zeta(z)}(x), \overline{v_{\zeta(z)}(y)}}{(u_{\zeta(z)}, v_{\zeta(z)})}.$$

Take $s = (x - y)/|x - y|$. We regard $(x - y) \cdot \eta_s \gg 1$ as a large parameter, and note that $(x - y) \cdot (z \cdot e_s) = 0$. We have shown that the critical point of $w_s(z)$ is $z = 0$. By the saddle point method

$$I_1(x, y) = \frac{-|D_s|}{(2\pi)^{d-1}} \left( \frac{2\pi}{(x - y) \cdot \eta_s} \right)^{(d-1)/2} e^{-(x-y) \cdot \beta_s} \left( \frac{1}{\eta_s \cdot \nabla E(\beta_s)} \frac{u_{\beta_s}(x), \overline{v_{\beta_s}(y)}}{(u_{\beta_s}, v_{\beta_s})_{L^2(T^d)}} + O(|x - y|^{-1}) \right).$$

This leads to the main term of the asymptotics.

We can show that the integral kernel of $I_2$ satisfies

$$|I_2(x, y)| \leq Ce^{-(x-y) \cdot \beta_s} e^{-c|x-y|},$$

using the $2\pi \mathbb{Z}^d$-periodicity of $F(\zeta)$. These are an outline of the proof of Theorem 1. \( \square \)

**Remark.** We can get the following asymptotic expansion. Assume that $\lambda_c > 0$. There exist bounded functions $g_j(x, y)$, $j = 1, 2, \ldots$, s.t. for any natural number $n$

$$G(x, y) = \frac{1}{|\nabla E(\beta_s)|} \left( \frac{2\pi}{(x - y) \cdot \eta_s} \right)^{(d-1)/2} \frac{e^{-(x-y) \cdot \beta_s} u_{\beta_s}(x), \overline{v_{\beta_s}(y)}}{(u_{\beta_s}, v_{\beta_s})_{L^2(T^d)}} \times \left( 1 + \sum_{j=1}^{n} \frac{g_j(x, y)}{|x - y|^j} + O(|x - y|^{-n-1}) \right).$$

**References**


