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Liouville type theorem associate with the wave equation

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1 Introduction

The well-known Liouville's theorem states that every conformal mapping in the n-dimensional Euclidean space ($n \geqq 3$) is a similarity or an inversion with respect to a sphere. The conformal mapping associates with the Laplace equation in the following sense. Let $U, V \subset \mathbb{R}^n$ be domains and $f = (f_1, f_2, \ldots, f_n) : U \to V$ a C^2-mapping, and φ be a positive C^2-function on U. Assume that $\varphi(x) \cdot (u \circ f)(x)$ satisfies the Laplace equation on U for every solution u of the Laplace equation on V. This is possible only if f is a conformal mapping.

In this note, we consider Liouville type theorem associate with the wave equation instead of the Laplace equation. We note that Sugimoto considered this problem in [4] and partially solved the problem.

2 Transformation which preserves the solution of the wave equation

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space ($n \geqq 2$), and denote its point by $x = (x_0, x_1, \ldots, x_n)$. We define the $\langle \cdot, \cdot \rangle$ quadratic form in $\mathbb{R}^{n+1}$ by

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

Let

$$J = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \in GL(n+1, \mathbb{R})$$

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and put $O_J = \{R \in GL(n+1,\mathbb{R}); {}^tRJR = J\}$. Then $\langle Rx, Ry\rangle = \langle x, y\rangle$ holds for all $x, y \in \mathbb{R}^{n+1}$ if and only if $R \in O_J$.

In the following, we consider the wave equation

$$Wu := \Big(\frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\Big)u = 0$$

on $\mathbb{R}^{n+1}$.

Let $U, V \subset \mathbb{R}^{n+1}$ are domains, $f = (f_0, f_1, \ldots, f_n) : U \to V$ a C^2-mapping, and φ a positive C^2-function on U. The pair (f, φ) is called a transformation which preserves the solution of the wave equation, if the function $\varphi(x) \cdot (u \circ f)(x)$ satisfies the wave equation on U for every solution u of the wave equation on V. This is possible if and only if f and φ satisfy the following equations on U ([4]):

$$W\varphi = 0, \tag{1}$$

$$\varphi\, Wf_j - 2\langle\nabla\varphi, \nabla f_j\rangle = 0, \qquad (j = 0, 1, \ldots, n) \tag{2}$$

$$\langle\nabla f_j, \nabla f_k\rangle = 0, \qquad (0 \leqq j < k \leqq n) \tag{3}$$

$$\langle\nabla f_j, \nabla f_j\rangle = -\langle\nabla f_0, \nabla f_0\rangle, \qquad (1 \leqq j \leqq n) \tag{4}$$

where $\nabla f_j = (\frac{\partial f_j}{\partial x_0}, \frac{\partial f_j}{\partial x_1}, \ldots, \frac{\partial f_j}{\partial x_n})$ $(j = 0, 1, \ldots, n)$.

It is easy to see that if (f, φ) and (g, ψ) are transformations which preserve the solution of the wave equation such that the image of f is contained in the domain of g, then $(g \circ f, \varphi \cdot \psi \circ f)$ is also a transformation which preserves the solution of the wave equation. We call this new transformation $(g \circ f, \varphi \cdot \psi \circ f)$ the composition of the transformations (f, φ) and (g, ψ)

In the following, we list fundamental transformations which preserves the solution of the wave equation.

Example 1 (J-similarity). The pair of a mapping

$$f(x) = \alpha Rx + a \quad (\alpha \in \mathbb{R},\ \alpha > 0,\ R \in O_J,\ a \in \mathbb{R}^{n+1})$$

and a positive constant function $\varphi(x) = C$ $(C \in \mathbb{R},\ C > 0)$ is a transformation which preserves the solution of the wave equation. We call such transformation J-similarity.

Example 2 (J-inversion). The pair of the mapping j and the function φ

$$j(x) = \frac{1}{\langle x, x\rangle}, \qquad \varphi(x) = \frac{1}{|\langle x, x\rangle|^{\frac{n-1}{2}}}$$

is a transformation which preserves the solution of the wave equation defined on each connected component of $\{x \in \mathbb{R}^{n+1}; \langle x, x\rangle \neq 0\}$. We call this transformation J-inversion. By simple calculation, we have

$$\langle \nabla j_i(x), \nabla j_i(x)\rangle = -\langle \nabla j_0(x), \nabla j_0(x)\rangle = \frac{1}{\langle x, x\rangle^2}, \quad (i = 1, \dots, n),$$

where we put $j(x) = (j_0(x), j_1(x), \dots, j_n(x))$.

Example 3 (Bateman transformation [2]). The pair of the mapping B and φ

$$B(x) = (\frac{\langle x, x\rangle + 1}{2(x_0 + x_1)}, \frac{\langle x, x\rangle - 1}{2(x_0 + x_1)}, \frac{x_2}{x_0 + x_1}, \dots, \frac{x_n}{x_0 + x_1}),$$
$$\varphi(x) = |x_0 + x_1|^{\frac{n-1}{2}}$$

is a transformation which preserves the solution of the wave equation defined on each connected component half space of $\{x \in \mathbb{R}^{n+1}; x_0 + x_1 \neq 0\}$. We call this transformation Bateman transformation. By easy calculation, we have

$$\langle \nabla B_i(x), \nabla B_i(x)\rangle = -\langle \nabla B_0(x), \nabla B_0(x)\rangle = \frac{1}{(x_0 + x_1)^2}, \quad (i = 1, \dots, n),$$

where we put $B(x) = (B_0(x), B_1(x), \dots, B_n(x))$.

3 J-conformal mapping

We can write the quadratic form $\langle \cdot, \cdot \rangle$ as

$$\langle x, y\rangle = (Jx, y) = (x, Jy), \quad x, y \in \mathbb{R}^{n+1},$$

where $(\cdot, \cdot)$ is the usual Euclidean inner product.

Let $U, V \subset \mathbb{R}^{n+1}$ be domains and $f = (f_0, f_1, \dots, f_n) : U \to V$ a C^2-mapping and let $\frac{\partial f}{\partial x}(x)$ be the Jacobian matrix of f :

$$\frac{\partial f}{\partial x}(x) := \begin{pmatrix} \frac{\partial f_0}{\partial x_0}(x) & \frac{\partial f_0}{\partial x_1}(x) & \dots & \frac{\partial f_0}{\partial x_n}(x) \\ \frac{\partial f_1}{\partial x_0}(x) & \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_0}(x) & \frac{\partial f_n}{\partial x_1}(x) & \dots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix}.$$

The mapping f is said to be J-conformal if there exists a function $\lambda_f(x) > 0$ defined on U such that

$$\langle \frac{\partial f}{\partial x}(x)u, \frac{\partial f}{\partial x}(x)v\rangle = \lambda(x)^2\langle u, v\rangle \quad \forall x \in U, \forall u, v \in \mathbb{R}^{n+1}.$$

If $f : V \longrightarrow W$ and $g : U \longrightarrow V$ are J-conformal mappings, then the composition mapping $f \circ g : U \longrightarrow W$ is also a J-conformal mapping. In fact, by the chain rule

$$\langle \frac{\partial(f\circ g)}{\partial x}(x)u, \frac{\partial(f\circ g)}{\partial x}(x)v\rangle = \langle \frac{\partial f}{\partial y}(g(x))\frac{\partial g}{\partial x}(x)u, \frac{\partial f}{\partial y}(g(x))\frac{\partial g}{\partial x}(x)v\rangle$$

$$= \lambda_f(g(x))^2\langle \frac{\partial g}{\partial x}(x)u, \frac{\partial g}{\partial x}(x)v\rangle = \lambda_f(g(x))^2\lambda_g(x)^2\langle u, v\rangle$$

holds for all $x \in U$, and all $u, v \in \mathbb{R}^{n+1}$. Hence $f \circ g$ is a J-conformal mapping and

$$\lambda_{f\circ g}(x) = \lambda_f(g(x))\lambda_g(x), \quad x \in U \tag{5}$$

holds.

It is easily seen that the combination of the conditions (3) and (4) is equivalent to the condition that f is a J-conformal mapping and

$$\lambda_f(x)^2 = \langle \nabla f_j, \nabla f_j\rangle = -\langle \nabla f_0, \nabla f_0\rangle.$$

Example 4. The mapping

$$f(x) = \alpha Rx + a \quad (\alpha \in \mathbb{R},\ \alpha > 0,\ R \in O_J,\ a \in \mathbb{R}^{n+1})$$

is a J-conformal mapping defined on $\mathbb{R}^{n+1}$ satisfying $\lambda_f(x) = \alpha$. We call such mapping J-similarity (mapping).

Example 5. The mapping

$$j(x) = \frac{1}{\langle x, x\rangle}$$

is a J-conformal mapping defined on each connected component of $\{x \in \mathbb{R}^{n+1}; \langle x, x\rangle \neq 0\}$. We call j J-inversion (mapping). By simple calculation, we have

$$j^{-1} = j$$

and

$$\lambda_j(x) = \frac{1}{|\langle x, x\rangle|}.$$

Example 6. The mapping

$$B(x) = (\frac{\langle x,x\rangle + 1}{2(x_0+x_1)}, \frac{\langle x,x\rangle - 1}{2(x_0+x_1)}, \frac{x_2}{x_0+x_1}, \dots, \frac{x_n}{x_0+x_1})$$

is a J-conformal mapping defined on each connected component half space of $\{x \in \mathbb{R}^{n+1}; x_0 + x_1 \neq 0\}$. we call B the mapping of the Bateman transformation. By easy calculation, we have

$$B^{-1}(x) = (\frac{\langle x,x\rangle + 1}{2(x_0-x_1)}, \frac{1-\langle x,x\rangle}{2(x_0-x_1)}, \frac{x_2}{x_0-x_1}, \dots, \frac{x_n}{x_0-x_1}),$$

and

$$\lambda_B(x) = \frac{1}{|x_0+x_1|}, \quad \lambda_{B^{-1}}(x) = \frac{1}{|x_0-x_1|}.$$

4 Liouville type theorem for J-conformal mapping

In this section, we determine the J-conformal C^4-mapping on $\mathbb{R}^{n+1}$ $(n \geq 2)$. Our main theorem is the following.

Theorem 1 ([3], cf.[4]). *Let f be a J-conformal C^4-mapping defined on a domain $U \subset \mathbb{R}^{n+1}$. Then f has one of the following forms :*

(a) $f(x) = (h \circ j \circ g)(x)$,

(b) $f(x) = (h \circ B \circ g)(x)$,

(c) $f(x) = g(x)$,

where g and h are J-similarities.

First part of the proof, up to the following proposition, follows from the same argument as the proof Liouville's theorem for conformal mapping in [1].

Proposition 1. *If f is a J-conformal C^4-mapping, then $\rho(x) = \dfrac{1}{\lambda_f(x)}$ satisfies*

$$\sum_{i,j=0}^{n} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) u_i v_j = c\langle u,v\rangle,$$

where c is a constant.

Therefore, $\rho(x)$ is a polynomial of degree at most 2. Integrating the both sides of the above equation, we have the following proposition.

Proposition 2. *If f is J-conformal and C^4, then $\lambda_f(x)$ has one of the following forms :*

$$\lambda_f(x) = \frac{1}{c\langle x-a, x-a\rangle + b}, \quad \text{with } c, b \in \mathbb{R},\ c \neq 0,\ a \in \mathbb{R}^{n+1},$$
$$\lambda_f(x) = \frac{1}{\langle d, x-a\rangle}, \quad \text{with } a, d \in \mathbb{R}^{n+1},\ d \neq 0,$$
$$\lambda_f(x) = l, \quad \text{with } l \in \mathbb{R},\ l > 0.$$

In contrast to the conformal mapping case, there exists a J-conformal mapping which has the second form. The mapping of the Bateman transformation on $\{x; x_0 + x_1 > 0\}$ satisfies

$$\lambda_B(x) = \frac{1}{x_0 + x_1},$$

which is of second form with $d = (-1, 1, 0, \ldots, 0)$ and $a = 0$. Note that every J-conformal mapping has local inverse and the local inverse is also a J-conformal mapping. With a help of some geometric argument, we obtain the following proposition.

Proposition 3. *If f is a J-conformal C^4-mapping, then $\lambda_f(x)$ has one of the following forms :*

$$\lambda_f(x) = \frac{1}{c\langle x-a, x-a\rangle}, \quad \text{with } c \in \mathbb{R},\ c \neq 0,\ a \in \mathbb{R}^{n+1}, \tag{6}$$
$$\lambda_f(x) = \frac{1}{\langle d, x-a\rangle}, \quad \text{with } a, d \in \mathbb{R}^{n+1},\ d \neq 0,\ \langle d, d\rangle = 0, \tag{7}$$
$$\lambda_f(x) = l, \quad \text{with } l \in \mathbb{R},\ l > 0. \tag{8}$$

In the following, we shall show that the above cases (6), (7) and (8) correspond to the cases (a), (b) and (c) of Theorem 1, respectively.

First we study the case (8):

$$\lambda_f(x) = l, \quad l \in \mathbb{R},\ l > 0.$$

It can be shown that if a J-conformal mapping has constant λ_f with $\lambda_f(x) = l$ then f is equal to a J-similarity

$$f(x) = lRx + a, \quad (R \in O_J,\ a \in \mathbb{R}^{n+1}).$$

This is the case (c) of Theorem 1.

Next we study the case (6):

$$\lambda_f(x) = \frac{1}{c\langle x-a, x-a\rangle}, \quad c \in \mathbb{R},\ c \neq 0,\ a \in \mathbb{R}^{n+1}.$$

We put

$$g(x) = x - a,$$

then $g^{-1}(j(x)) = j(x) + a$ and

$$\begin{aligned}\lambda_{f\circ g^{-1}\circ j} &= \lambda_f(j(x)+a)\lambda_{g^{-1}}(j(x))\lambda_j(x) \\ &= \frac{1}{c\langle j(x), j(x)\rangle} \cdot 1 \cdot \frac{1}{|\langle x,x\rangle|} = \frac{1}{\frac{c\langle x,x\rangle}{\langle x,x\rangle^2}} \frac{1}{|\langle x,x\rangle|} \\ &= \frac{1}{|c|}\end{aligned}$$

which imply that $(f \circ g^{-1} \circ j)(x)$ is equal to a J-similarity

$$h(x) = \frac{1}{|c|}Rx + b, \quad (R \in O_J,\ b \in \mathbb{R}^{n+1}).$$

Thus we have $(f \circ g^{-1} \circ j)(y) = h(y)$ and

$$f(x) = (h \circ j \circ g)(x), \quad x \in U,$$

because $j^{-1}(x) = j(x)$. This is the case (a) of Theorem 1.

Finally we study the case (7):

$$\lambda_f(x) = \frac{1}{\langle d, x-a\rangle}, \quad a, d \in \mathbb{R}^{n+1},\ d = (d_0, d_1, \dots, d_n) \neq 0,\ \langle d, d\rangle = 0.$$

Since $\langle d, d\rangle = 0$ and $d \neq 0$, $|(d_1, \dots, d_n)| = |d_0| \neq 0$ and there exists a matrix $R_0 \in O(n)$ whose first column is equal to the vector $\frac{{}^t(d_1, \dots, d_n)}{d_0}$. We put $v_0 = (-1, 1, 0, \dots, 0)$. Then

$$d = d_0 \begin{pmatrix} -1 & 0 \\ 0 & R_0 \end{pmatrix} v_0 =: d_0 R_1 v_0,$$

where $R_1 \in O_J$, and

$$\langle d, x-a\rangle = \langle d_0 R_1 v_0, x-a\rangle = d_0\langle v_0, J^t R_1 J(x-a)\rangle$$

holds. Define the J-similarity g by

$$g(x) = J^t R_1 J(x-a),$$

so that

$$\langle d, x-a\rangle = d_0\langle v_0, g(x)\rangle$$

and

$$\begin{aligned}\lambda_f(g^{-1}(x)) &= \frac{1}{\langle d, g^{-1}(x)-a\rangle} = \frac{1}{d_0\langle v_0, g(g^{-1}(x))\rangle} = \frac{1}{d_0\langle v_0, x\rangle}\\ &= \frac{1}{d_0(x_0+x_1)}.\end{aligned}$$

Now consider the mapping $f\circ g^{-1}\circ B^{-1}$ defined on $B(g(U))$. We have

$$\begin{aligned}\lambda_{f\circ g^{-1}\circ B^{-1}}(x) &= \lambda_f(g^{-1}(B^{-1}(x)))\lambda_{g^{-1}}(B^{-1}(x))\lambda_{B^{-1}}(x)\\ &= \frac{1}{d_0\left(\dfrac{\langle x,x\rangle+1}{2(x_0-x_1)}+\dfrac{1-\langle x,x\rangle}{2(x_0-x_1)}\right)}\cdot 1\cdot\frac{1}{|x_0-x_1|}\\ &= \frac{1}{|d_0|},\end{aligned}$$

which implies that $(f\circ g^{-1}\circ B^{-1})(x)$ is equal to a J-similarity

$$h(x) = \frac{1}{|d_0|}Rx+b, \quad (R\in O_J,\ b\in\mathbb{R}^{n+1}).$$

Thus we have $(f\circ g^{-1}\circ B^{-1})(y) = h(y)$ and

$$f(x) = (h\circ B\circ g)(x), \quad x\in U.$$

This is the case (b) of Theorem 1.

By Theorem 1, we can determine the transformation which preserves the wave equation as follows.

Theorem 2. *Every transformation which preserves the solution of the wave equation is one of the following :*

(a) J-inversion composed with J-similarities,

(b) Bateman transformation composed with J-similarities,

(c) J-similarity.

References

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