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タイトル: 線型関係の構成オペレーター

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LINEAR RELATIONS OF COMPOSITION OPERATORS

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Abstract. We will characterize the compactness of linear combinations of composition operators on the Banach algebra of bounded analytic functions on the open unit disk.

1 Introduction

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk and $\mathcal{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Denote by $S(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Then, for $\varphi \in S(\mathbb{D})$, the composition operator $C_{\varphi}$ is defined by

$$C_{\varphi}f(z) = (f \circ \varphi)(z)$$

for $z \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$. During the past few decades, many authors have investigated operator theoretic properties of composition operator $C_{\varphi}$ on various analytic function spaces using function theoretic properties of symbol $\varphi$. For an overview of the study of composition operators, we refer to the books [2], [14] and [17].

Presently some of the long standing open questions in this field are related to the topological structure of the set of composition operators. For a Banach space $\mathcal{X}$ in $\mathcal{H}(\mathbb{D})$, we write $\mathcal{C}(\mathcal{X})$ for the set of composition operators on $\mathcal{X}$ with the operator norm topology. Berkson [1] focused

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attention on the topological structure with his isolation results on composition operators on the Hardy spaces. In the case of the Hilbert Hardy space, Shapiro and Sundberg [15] gave further progress, obtained results on compact differences and isolation and suggested questions in the case of the Hilbert Hardy space.

The problems are the following in the general case:

1. Characterize the components of $C(X)$.
2. Which composition operators are isolated in $C(X)$?
3. Which composition differences are compact on $X$?

One conjecture was proposed: for $\varphi$ and $\psi \in \mathcal{S}(\mathbb{D})$, $C_\varphi - C_\psi$ is compact on $X$ if and only if $C_\varphi$ and $C_\psi$ are in the same component in $C(X)$. The topological structure of $C(X)$ has been studied on various analytic function spaces $X$. These problems seem quite hard.

In view of the other, for $\varphi$ and $\psi \in \mathcal{S}(\mathbb{D})$, it holds that $C_\varphi C_\psi = C_{\psi \circ \varphi}$, that is, the product of two composition operators becomes a composition operator. But the sum $C_\varphi + C_\psi$ is not necessarily a composition operator. The set of composition operators has no obvious additive or linear structure. Note that Toeplitz-Hankel operators have additive and linear structure but their products are not clear.

Let $\mathcal{B}(X)$ be the set of bounded linear operators on $X$ and $\mathcal{K}$ the set of all compact operators on $X$. Then $\mathcal{B}(X)/\mathcal{K}$ is called the Calkin algebra. The compactness of $C_\varphi - C_\psi$ is that $C_\varphi \equiv C_\psi \pmod{\mathcal{K}}$. Topological structure problem (compact difference problem) implies linear relations problem. That is, if

$\sum_{i=1}^{N} \lambda_i C_{\varphi_i} - C_\psi$ is compact if and only if $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} \equiv C_\psi \pmod{\mathcal{K}}$.

In a recent paper, MacCluer, Zhao and the author [12] studied the topological structure of the set $C(H^\infty)$ of composition operators on the Banach space $H^\infty$ of bounded analytic functions on $\mathbb{D}$. In [7], Hosokawa, Izuchi and Zheng showed that $C_\varphi$ is not isolated in $C(H^\infty)$ if and only if $\varphi$ is not an extreme point of the closed unit ball of $H^\infty$, and that $C_\varphi$ and $C_\psi$ are in the same connected component in $C(H^\infty)$ if and only if $C_\varphi$ and $C_\psi$ are in the same connected component in $C(H^\infty)/\mathcal{K}$. In [6], Hosokawa and Izuchi studied the estimate of the essential norm which is the norm in $\mathcal{B}(H^\infty)/\mathcal{K}$. 
After these works, $H^\infty$ has attracted much attention in the study of this area. In particular, Toews [16] extended the results of [12] and [8] to the setting of several variables. Gorkin, Mortini and Suárez [5] gave upper and lower bounds for the essential norm of difference of two composition operators on $H^\infty$, where the setting is on the unit ball of $\mathbb{C}^n (n \geq 1)$. Now, furthermore, linear relations of composition operators have been studied in some cases. In [4], Gorkin and Mortini studied norms and essential norms of linear combinations of endomorphisms on uniform algebras. Kriete and Moorhouse [11] considered linear relations of composition operators on the Hilbert Hardy space. Hosokawa, Nieminen and the author [9] have done in the Bloch space case.

In this article, we investigate properties of linear combinations of composition operators on $H^\infty$. In the next section we will review on the results of compact differences on $H^\infty$ to study the linear relations of composition operators. In Section 3 we will characterize the compactness of linear combinations of composition operators on $H^\infty$. These results are due to a part of the joint-work [10] with K.J. Izuchi.

2 Reviews on results of compact differences

Let $H^\infty = H^\infty(D)$ be the space of all bounded analytic functions on the open unit disk $D$. Then $H^\infty$ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)|: z \in D\}.$$  

Denote by ball $H^\infty$ the closed unit ball of $H^\infty$. For $\varphi \in S(D)$, we define the composition operator $C_\varphi$ on $H^\infty$ by

$$C_\varphi f = f \circ \varphi \quad \text{for } f \in H^\infty.$$  

It is clear that $C_\varphi$ is linear and bounded on $H^\infty$. and that $C_\varphi$ is compact on $H^\infty$ if and only if $\|\varphi\|_\infty < 1$ ([13]).

Our results involve the pseudo-hyperbolic metric. For $z$ and $w \in D$, the pseudo-hyperbolic distance between $z$ and $w$ is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$
MacCluer, Zhao and the author [12] showed the following.

**Theorem 2.1.** Let \( \varphi \) and \( \psi \in S(\mathbb{D}) \) with \( \varphi \neq \psi \). Suppose that \( \|\varphi\|_{\infty} = \|\psi\|_{\infty} = 1 \). Then \( C_{\varphi} - C_{\psi} \) is compact on \( H^\infty \) if and only if
\[
\limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z)) = 0.
\]

Here we can show that the conjecture posed in Section 1 is not true for the case of \( H^\infty \).

**Example 2.2.** Let
\[
\varphi(z) = sz + 1 - s, \quad 0 < s < 1
\]
and
\[
\psi(z) = \varphi(z) + t(z - 1)^b,
\]
where \( |t| \) is so small that \( \psi \) maps \( \mathbb{D} \) into \( \mathbb{D} \).

Then

(i) If \( 0 < b \leq 2 \), then \( C_{\varphi} - C_{\psi} \) is not compact on \( H^\infty \).

(ii) If \( 2 < b \), then \( C_{\varphi} - C_{\psi} \) is compact on \( H^\infty \). But \( C_{\varphi} \) and \( C_{\psi} \) are not in the same component of \( \mathcal{C}(H^\infty) \).

### 3 Linear combinations of composition operators

We here characterize the compactness of linear combinations of composition operators on \( H^\infty \). This work is a part of the joint-work [10] with K.J. Izuchi.

We shall need the following proposition whose proof is an easy modification of that of Proposition 3.11 in [2].

**Proposition 3.1.** Let \( \varphi_1, \varphi_2, \cdots, \varphi_N \) be distinct functions in \( S(\mathbb{D}) \), and \( \lambda_i \in \mathbb{C} \) with \( \lambda_i \neq 0 \) for every \( i \). Then \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} \) is compact on \( H^\infty \) if and only if whenever \( \{f_n\}_n \) is a bounded sequence in \( H^\infty \) such that \( \{f_n\}_n \) converges to 0 uniformly on any compact subset of \( \mathbb{D} \), then \( \|\sum_{i=1}^{N} \lambda_i C_{\varphi_i} f_n\|_{\infty} \) tends to 0 as \( n \to \infty \).

Let \( \varphi_1, \varphi_2, \cdots, \varphi_N \) be distinct functions in \( S(\mathbb{D}) \) and \( N \geq 2 \). Let \( Z = Z(\varphi_1, \varphi_2, \cdots, \varphi_N) \) be the family of sequences \( \{z_n\}_n \) in \( \mathbb{D} \) satisfying the following three conditions;
(a) \(|\varphi_i(z_n)| \to 1\) as \(n \to \infty\) for some \(i\),

(b) \(\{\varphi_i(z_n)\}_{n}\) is a convergent sequence for every \(i\),

(c)
\[
\frac{\varphi_j(z_n) - \varphi_i(z_n)}{1 - \varphi_j(z_n)\varphi_i(z_n)}
\]

is a convergent sequence for every \(i, j\).

Condition (c) implies that
\[(c') \{\rho(\varphi_i(z_n), \varphi_j(z_n))\}_{n}\] is a convergent sequence for every \(i, j\).

Note that if \(|\varphi_i(z_n)| \to 1\) as \(n \to \infty\) for some \(i\), then it is easy to see that there exists a subsequence \(\{z_{n_j}\}_{j}\) of \(\{z_n\}_{n}\) satisfying \(\{z_{n_j}\}_{j} \in \mathcal{Z}\).

For \(\{z_n\}_{n} \in \mathcal{Z}\), we write
\[
I(\{z_n\}) = \{i : 1 \leq i \leq N, |\varphi_i(z_n)| \to 1\}.
\]

By condition (a), \(I(\{z_n\}) \neq \emptyset\). By (b), there exists \(\delta\) with \(0 < \delta < 1\) such that \(|\varphi_j(z_k)| < \delta\) for every \(j \notin I(\{z_n\})\) and \(k\). For each \(t \in I(\{z_n\})\), we write
\[
(3.1) \quad I_0(\{z_n\}, t) = \{j \in I(\{z_n\}) : \rho(\varphi_j(z_n), \varphi_t(z_n)) \to 0\}.
\]

For \(s, t \in I(\{z_n\})\), we have either \(I_0(\{z_n\}, s) = I_0(\{z_n\}, t)\) or \(I_0(\{z_n\}, s) \cap I_0(\{z_n\}, t) = \emptyset\). Hence there is a subset \(\{t_1, t_2, \cdots, t_{\ell}\} \subset I(\{z_n\})\) such that
\[
I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p)
\]
and \(I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset\) for \(p \neq q\).

When we consider the compactness of linear combinations \(\sum_{i=1}^{N} \lambda_i C_{\varphi_i}\), some \(C_{\varphi_i}\) could be compact, that is, \(\|\varphi_i\|_{\infty} < 1\). We may exclude such trivial ones from our linear combinations.

Gorkin and Mortini [4, Theorem 11] characterized necessary conditions for linear combinations of composition operators to be compact on some uniform algebras. We here obtain necessary and sufficient conditions on the compactness.
Theorem 3.2. Let $\varphi_1, \varphi_2, \cdots, \varphi_N \ (N \geq 2)$ be distinct functions in $S(\mathbb{D})$ with $\|\varphi_i\|_{\infty} = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every $i$. Then the following conditions are equivalent.

(i) $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$.

(ii) $\sum \{\lambda_i : i \in I_0(\{z_n\}, t)\} = 0$ for every $\{z_n\} \in \mathbb{Z} = \mathbb{Z}(\varphi_1, \varphi_2, \cdots, \varphi_N)$ and $t \in I(\{z_n\})$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$. Let $\{z_n\} \in \mathbb{Z}$ and $t \in I(\{z_n\})$. For each positive integer $k$, we write

$$f_k(z) = \frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k)\overline{z}} \prod_{j \notin I_0(\{z_n\}, t)} \frac{\varphi_j(z_k) - z}{1 - \overline{\varphi_j(z_k)} z}.$$  

Then $f_k \in H^\infty$, $\|f_k\|_{\infty} \leq 2$, and $\{f_k\}_k$ converges to 0 uniformly on every compact subset of $\mathbb{D}$. We have

$$\left\| \sum_{i=1}^{N} \lambda_i C_{\varphi_i} f_k \right\|_{\infty} \geq \left| \sum_{i=1}^{N} \lambda_i f_k(\varphi_i(z_k)) \right| = \left| \sum_{i \in I_0(\{z_n\}, t)} \lambda_i \frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k)\varphi_i(z_k)} \prod_{j \notin I_0(\{z_n\}, t)} \frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \overline{\varphi_j(z_k)}\varphi_i(z_k)} \right|.$$  

Here

$$\frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k)\varphi_i(z_k)} = 1 + \frac{\varphi_t(z_k) - \varphi_t(z_k)}{1 - \varphi_t(z_k)\varphi_i(z_k)}.$$  

For $i \in I_0(\{z_n\}, t)$, by (3.1) $\rho(\varphi_i(z_k), \varphi_t(z_k)) \to 0$ as $k \to \infty$. Hence

$$\frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k)\varphi_i(z_k)} \to 1$$  

as $k \to \infty$.  

On the other hand,
\[
\frac{\phi_j(z_k) - \phi_l(z_k)}{1 - \phi_j(z_k)\phi_l(z_k)} - \frac{\phi_j(z_k) - \phi_l(z_k)}{1 - \phi_j(z_k)\phi_l(z_k)} = \frac{\phi_k(z_k) - \phi_l(z_k)}{1 - \phi_k(z_k)\phi_l(z_k)} \left( 1 + \frac{\phi_i(z_k) - \phi_j(z_k)}{1 - \phi_i(z_k)\phi_j(z_k)} \right)
\times \left( 1 + \frac{\phi_l(z_k) - \phi_j(z_k)}{1 - \phi_l(z_k)\phi_j(z_k)} \right).
\]

Since \(\rho(\phi_i(z_k), \phi_l(z_k)) \to 0\), by (c) we have
\[
\lim_{k \to \infty} \frac{\phi_j(z_k) - \phi_l(z_k)}{1 - \phi_j(z_k)\phi_l(z_k)} = \lim_{k \to \infty} \frac{\phi_j(z_k) - \phi_l(z_k)}{1 - \phi_j(z_k)\phi_l(z_k)}.
\]

Since \(j \not\in I_0(\{z_n\}, t)\), by (3.1) and (c)
\[
\lim_{k \to \infty} \frac{\phi_j(z_k) - \phi_l(z_k)}{1 - \phi_j(z_k)\phi_l(z_k)} = \beta_{j,t} \neq 0
\]
for some \(\beta_{j,t} \in \mathbb{C}\).

By condition (i) and Proposition 3.1,
\[
\left\| \sum_{i=1}^{N} \lambda_i C_{\phi_i} f_k \right\|_\infty \to 0
\]
as \(k \to \infty\). Therefore we get
\[
\left( \sum_{i \in I_0(\{z_n\}, t)} \lambda_i \right) \prod_{j \not\in I_0(\{z_n\}, t)} \beta_{j,t} = 0.
\]

Consequently, we have
\[
\sum_{i \in I_0(\{z_n\}, t)} \lambda_i = 0.
\]

(ii) \(\Rightarrow\) (i). Suppose that \(\sum_{i=1}^{N} \lambda_i C_{\phi_i}\) is not compact on \(H^\infty\). Then there exists a sequence \(\{f_n\}_n\) in ball \(H^\infty\) such that \(f_n \to 0\) uniformly on every compact subset of \(\mathbb{D}\) and
\[
\left\| \sum_{i=1}^{N} \lambda_i f_n \circ \phi_i \right\|_\infty \not\to 0
\]
as $n \to \infty$. For some $\varepsilon > 0$, considering a subsequence of $\{f_n\}_n$, we may assume that
\[ \left\| \sum_{i=1}^{N} \lambda_i f_n \circ \varphi_i \right\|_{\infty} > \varepsilon > 0 \]
for every $n$. Take $z_n \in \mathbb{D}$ with $|z_n| \to 1$ and
\[ \left| \sum_{i=1}^{N} \lambda_i f_n(\varphi_i(z_n)) \right| > \varepsilon. \]
Considering subsequence of $\{z_n\}_n$, we may assume that $\varphi_i(z_n) \to \alpha_i$ as $n \to \infty$ for every $i$. Since $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$, $|\alpha_i| = 1$ for some $i$. Moreover we may assume that $\{z_n\}_n \in \mathcal{Z}$. Also we have
\begin{equation}
(3.2) \quad \liminf_{k \to \infty} \left| \sum_{i \in I(\{z_n\})} \lambda_i f_k(\varphi_i(z_k)) \right| \geq \varepsilon.
\end{equation}
Recall that there exists a subset $\{t_1, t_2, \cdots, t_\ell\} \subset I(\{z_n\})$ such that
\[ I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p) \]
and $I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. Let $i \in I_0(\{z_n\}, t_p)$. Then
\[ \rho(\varphi_i(z_k), \varphi_{t_p}(z_k)) \to 0 \text{ as } k \to \infty. \]
By Schwarz’s lemma, see [3, p. 2],
\begin{equation}
(3.3) \quad \rho(f_k(\varphi_i(z_k)), f_k(\varphi_{t_p}(z_k))) \leq \rho(\varphi_i(z_k), \varphi_{t_p}(z_k)) \to 0
\end{equation}
as $k \to \infty$. Since $\{f_k(\varphi_i(z_k))\}_k$ is bounded, considering a subsequence of $\{z_k\}_k$, we may assume that $f_k(\varphi_i(z_k)) \to \beta_i$ as $k \to \infty$ for every $i$. By (3.3), $\beta_i = \beta_{t_p}$ for every $i \in I_0(\{z_n\}, t_p)$. Therefore
\[
\lim_{k \to \infty} \sum_{i \in I(\{z_n\})} \lambda_i f_k(\varphi_i(z_k)) = \lim_{k \to \infty} \sum_{p=1}^{\ell} \sum_{i \in I_0(\{z_n\}, t_p)} \lambda_i f_k(\varphi_i(z_k))
\]
\[ = \sum_{p=1}^{\ell} \sum_{i \in I_0(\{z_n\}, t_p)} \lambda_i \beta_{t_p} \]
\[ = \sum_{p=1}^{\ell} \beta_{t_p} \sum_{i \in I_0(\{z_n\}, t_p)} \lambda_i \]
\[ = 0 \quad \text{by condition (ii).} \]
This contradicts condition (3.2).
The following corollaries follow from Theorem 3.2.

**Corollary 3.3.** Let $\varphi_1, \varphi_2, \cdots, \varphi_N \ (N \geq 2)$ be distinct functions in $S(\mathbb{D})$ with $\|\varphi_i\|_{\infty} = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every $i$. If $\sum_{i \in J} \lambda_i \neq 0$ for every subset $J$ of $\{1, 2, \cdots, N\}$, then $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is not compact on $H^\infty$.

This says that the sum $\sum_{i=1}^{N} C_{\varphi_i}$ is never compact on $H^\infty$ for every $\varphi_i \in S(\mathbb{D})$ with $\|\varphi_i\|_{\infty} = 1$, $i = 1, \cdots, N$.

**Corollary 3.4.** Let $\varphi_1, \varphi_2, \cdots, \varphi_N \ (N \geq 2)$ be distinct functions in $S(\mathbb{D})$ with $\|\varphi_i\|_{\infty} = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every $i$. Suppose that $\sum_{i=1}^{N} \lambda_i = 0$ and $\sum_{i \in J} \lambda_i \neq 0$ for every non-empty proper subset $J$ of $\{1, 2, \cdots, N\}$. Then $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$ if and only if $C_{\varphi_i} - C_{\varphi_j}$ is compact on $H^\infty$ for every $i, j$ with $i \neq j$.

**Proof.** Suppose that $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$. Then by Theorem 3.2 (ii), for every $\{z_n\} \in \mathcal{Z}$, $I\{\{z_n\}\} = \{1, 2, \cdots, N\}$ and $I_0(\{z_n\}, t) = \{1, 2, \cdots, N\}$ for every $t \in I(\{z_n\})$. Hence

$$\lim_{|\varphi_i(z)| \to 1} \rho(\varphi_i(z), \varphi_j(z)) = 0.$$  

By [12], $C_{\varphi_i} - C_{\varphi_j}$ is compact for every $i, j$.

Suppose that $C_{\varphi_i} - C_{\varphi_j}$ is compact for every $i, j$. Since

$$\sum_{i=1}^{N} \lambda_i C_{\varphi_i} = \left(\sum_{i=1}^{N} \lambda_i\right) C_{\varphi_1} + \sum_{i=2}^{N} \lambda_i (C_{\varphi_i} - C_{\varphi_1}) = \sum_{i=2}^{N} \lambda_i (C_{\varphi_i} - C_{\varphi_1}),$$

we have that $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact. \qed

We recall that the Bloch space $B$ consists of all analytic functions $f$ on $\mathbb{D}$ such that $\|f\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty$. It is well known that $B$ is a Banach space under the norm $\|f\| = |f(0)| + \|f\|_B$. Then, under the assumption of Corollary 3.4, we obtain the following by Theorem 3 in [12].

**Corollary 3.5.** Let $\varphi_1, \varphi_2, \cdots, \varphi_N \ (N \geq 2)$ be distinct functions in $S(\mathbb{D})$ with $\|\varphi_i\|_{\infty} = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every $i$. Suppose that $\sum_{i=1}^{N} \lambda_i = 0$ and $\sum_{i \in J} \lambda_i \neq 0$ for every non-empty proper subset $J$ of $\{1, 2, \cdots, N\}$. Then the following conditions are equivalent.

(i) $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} : H^\infty \to H^\infty$ is compact.
\( \sum_{i=1}^{N} \lambda_{i} C_{\varphi_{i}} : \mathcal{B} \to H^{\infty} \) is compact.

It would be another problem to characterize the boundedness and compactness of \( \sum_{i=1}^{N} \lambda_{i} C_{\varphi_{i}} \), acting from \( \mathcal{B} \) to \( H^{\infty} \) in general. The boundedness and compactness of the differences of two composition operators acting from \( \mathcal{B} \) to \( H^{\infty} \) is concerning to the component problem of the set \( \mathcal{C}(H^{\infty}) \) of composition operators on \( H^{\infty} \) ([12]).

**Example 3.6.** We show the existence of \( \varphi_{1}, \varphi_{2}, \varphi_{3} \in \mathcal{S}(\mathbb{D}) \) with \( \|\varphi_{i}\|_{\infty} = 1 \) such that \( C_{\varphi_{1}} - C_{\varphi_{2}} - C_{\varphi_{3}} \) is compact.

Let \( \sigma(z) = (1+z)/(1-z) \) and

\[
\varphi_{1}(z) = \frac{\sqrt{\sigma(z)} - 1}{\sqrt{\sigma(z)} + 1}
\]

be a lens map ([14]). Also let

\[
\varphi_{2}(z) = 1 - \sqrt{1-z}.
\]

Denote by \( \partial \mathbb{D} \) the boundary of \( \mathbb{D} \). Then \( \varphi_{1}, \varphi_{2} \in \mathcal{S}(\mathbb{D}), \varphi_{1}(\pm 1) = \pm 1, |\varphi_{1}(e^{i\theta})| < 1 \) for \( e^{i\theta} \in \partial \mathbb{D} \) with \( e^{i\theta} \neq \pm 1, \varphi_{2}(1) = 1, \) and \( |\varphi_{2}(e^{i\theta})| < 1 \) for \( e^{i\theta} \in \partial \mathbb{D} \) with \( e^{i\theta} \neq 1 \). As Example (i) in [7, p. 513],

\[
\rho(\varphi_{1}(z), \varphi_{2}(z)) = \left| \frac{\sqrt{\sigma(z)}(1 - \varphi_{2}(z)) - (1 + \varphi_{2}(z))}{\sqrt{\sigma(z)}(1 - \varphi_{2}(z)) + (1 + \varphi_{2}(z))} \right|.
\]

Since

\[
Re \frac{\sqrt{1-z}}{\sqrt{1-\overline{z}}} > 0 \quad \text{for} \quad z \in \mathbb{D},
\]

we have

\[
\lim_{z \to 1} \rho(\varphi_{1}(z), \varphi_{2}(z)) = 0.
\]

Let

\[
\varphi_{3}(z) = -1 + \sqrt{1+z}.
\]

Then \( \varphi_{3} \in \mathcal{S}(\mathbb{D}), \varphi_{3}(-1) = -1, \) and \( |\varphi_{3}(e^{i\theta})| < 1 \) for \( e^{i\theta} \in \partial \mathbb{D} \) with \( e^{i\theta} \neq -1 \). Similarly we have

\[
\lim_{z \to -1} \rho(\varphi_{1}(z), \varphi_{3}(z)) = 0.
\]
Hence by Theorem 3.2, $C_{\varphi_1}' - C_{\varphi_2}' - C_{\varphi_3}'$ is compact.

References


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