LINEAR RELATIONS OF COMPOSITION OPERATORS

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Abstract. We will characterize the compactness of linear combinations of composition operators on the Banach algebra of bounded analytic functions on the open unit disk.

1 Introduction

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Denote by $S(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Then, for $\varphi \in S(\mathbb{D})$, the composition operator $C_{\varphi}$ is defined by

$$C_{\varphi}f(z) = (f \circ \varphi)(z)$$

for $z \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$. During the past few decades, many authors have investigated operator theoretic properties of composition operator $C_{\varphi}$ on various analytic function spaces using function theoretic properties of symbol $\varphi$. For an overview of the study of composition operators, we refer to the books [2], [14] and [17].

Presently some of the long standing open questions in this field are related to the topological structure of the set of composition operators. For a Banach space $\mathcal{X}$ in $\mathcal{H}(\mathbb{D})$, we write $C(\mathcal{X})$ for the set of composition operators on $\mathcal{X}$ with the operator norm topology. Berkson [1] focused

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attention on the topological structure with his isolation results on composition operators on the Hardy spaces. In the case of the Hilbert Hardy space, Shapiro and Sundberg [15] gave further progress, obtained results on compact differences and isolation and suggested questions in the case of the Hilbert Hardy space.

The problems are the following in the general case:

1. Characterize the components of $C(X)$.
2. Which composition operators are isolated in $C(X)$?
3. Which composition differences are compact on $X$?

   One conjecture was proposed: for $\varphi$ and $\psi \in S(D)$, $C_\varphi - C_\psi$ is compact on $X$ if and only if $C_\varphi$ and $C_\psi$ are in the same component in $C(X)$. The topological structure of $C(X)$ has been studied on various analytic function spaces $X$. These problems seem quite hard.

   In view of the other, for $\varphi$ and $\psi \in S(D)$, it holds that $C_\varphi C_\psi = C_{\psi \circ \varphi}$, that is, the product of two composition operators becomes a composition operator. But the sum $C_\varphi + C_\psi$ is not necessarily a composition operator. The set of composition operators has no obvious additive or linear structure. Note that Toeplitz-Hankel operators have additive and linear structure but their products are not clear.

   Let $\mathcal{B}(X)$ be the set of bounded linear operators on $X$ and $\mathcal{K}$ the set of all compact operators on $X$. Then $\mathcal{B}(X)/\mathcal{K}$ is called the Calkin algebra. The compactness of $C_\varphi - C_\psi$ is that $C_\varphi \equiv C_\psi \pmod{\mathcal{K}}$. Topological structure problem (compact difference problem) implies linear relations problem. That is, $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} - C_\psi$ is compact if and only if $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} \equiv C_\psi \pmod{\mathcal{K}}$.

   In a recent paper, MacCluer, Zhao and the author [12] studied the topological structure of the set $C(H^\infty)$ of composition operators on the Banach space $H^\infty$ of bounded analytic functions on $D$. In [7], Hosokawa, Izuchi and Zheng showed that $C_\phi$ is not isolated in $C(H^\infty)$ if and only if $\phi$ is not an extreme point of the closed unit ball of $H^\infty$, and that $C_\phi$ and $C_\psi$ are in the same connected component in $C(H^\infty)$ if and only if $C_\phi$ and $C_\psi$ are in the same connected component in $C(H^\infty)/\mathcal{K}$. In [6], Hosokawa and Izuchi studied the estimate of the essential norm which is the norm in $\mathcal{B}(H^\infty)/\mathcal{K}$.
After these works, $H^\infty$ has attracted much attention in the study of this area. In particular, Toews [16] extended the results of [12] and [8] to the setting of several variables. Gorkin, Mortini and Suárez [5] gave upper and lower bounds for the essential norm of difference of two composition operators on $H^\infty$, where the setting is on the unit ball of $\mathbb{C}^n (n \geq 1)$. Now, furthermore, linear relations of composition operators have been studied in some cases. In [4], Gorkin and Mortini studied norms and essential norms of linear combinations of endomorphisms on uniform algebras. Kriete and Moorhouse [11] considered linear relations of composition operators on the Hilbert Hardy space. Hosokawa, Nieminen and the author [9] have done in the Bloch space case.

In this article, we investigate properties of linear combinations of composition operators on $H^\infty$. In the next section we will review on the results of compact differences on $H^\infty$ to study the linear relations of composition operators. In Section 3 we will characterize the compactness of linear combinations of composition operators on $H^\infty$. These results are due to a part of the joint-work [10] with K.J. Izuchi.

2 Reviews on results of compact differences

Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on the open unit disk $\mathbb{D}$. Then $H^\infty$ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$ 

Denote by ball $H^\infty$ the closed unit ball of $H^\infty$. For $\varphi \in S(\mathbb{D})$, we define the composition operator $C_\varphi$ on $H^\infty$ by

$$C_\varphi f = f \circ \varphi \quad \text{for } f \in H^\infty.$$ 

It is clear that $C_\varphi$ is linear and bounded on $H^\infty$. and that $C_\varphi$ is compact on $H^\infty$ if and only if $\|\varphi\|_\infty < 1$ ([13]).

Our results involve the pseudo-hyperbolic metric. For $z$ and $w \in \mathbb{D}$, the pseudo-hyperbolic distance between $z$ and $w$ is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$
MacCluer, Zhao and the author [12] showed the following.

**Theorem 2.1.** Let $\varphi$ and $\psi \in S(D)$ with $\varphi \neq \psi$. Suppose that $\|\varphi\|_{\infty} = \|\psi\|_{\infty} = 1$. Then $C_\varphi - C_\psi$ is compact on $H^\infty$ if and only if

$$\limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z)) = 0.$$

Here we can show that the conjecture posed in Section 1 is not true for the case of $H^\infty$.

**Example 2.2.** Let

$$\varphi(z) = sz + 1 - s, \quad 0 < s < 1$$

and

$$\psi(z) = \varphi(z) + t(z - 1)^b,$$

where $|t|$ is so small that $\psi$ maps $D$ into $D$.

Then

(i) If $0 < b \leq 2$, then $C_\varphi - C_\psi$ is not compact on $H^\infty$.

(ii) If $2 < b$, then $C_\varphi - C_\psi$ is compact on $H^\infty$. But $C_\varphi$ and $C_\psi$ are not in the same component of $C(H^\infty)$.

### 3 Linear combinations of composition operators

We here characterize the compactness of linear combinations of composition operators on $H^\infty$. This work is a part of the joint-work [10] with K.J. Izuchi.

We shall need the following proposition whose proof is an easy modification of that of Proposition 3.11 in [2].

**Proposition 3.1.** Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ be distinct functions in $S(D)$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every $i$. Then $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$ if and only if whenever $\{f_n\}_n$ is a bounded sequence in $H^\infty$ such that $\{f_n\}_n$ converges to 0 uniformly on any compact subset of $D$, then $\|\sum_{i=1}^{N} \lambda_i C_{\varphi_i} f_n\|_{\infty}$ tends to 0 as $n \to \infty$.

Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ be distinct functions in $S(D)$ and $N \geq 2$. Let $Z = Z(\varphi_1, \varphi_2, \cdots, \varphi_N)$ be the family of sequences $\{z_n\}_n$ in $D$ satisfying the following three conditions;
(a) \(|\varphi_i(z_n)| \to 1\) as \(n \to \infty\) for some \(i\),

(b) \(\{\varphi_i(z_n)\}_n\) is a convergent sequence for every \(i\),

(c) \[
\left\{ \frac{\varphi_j(z_n) - \varphi_i(z_n)}{1 - \varphi_j(z_n)\varphi_i(z_n)} \right\}_n
\]

is a convergent sequence for every \(i, j\).

Condition (c) implies that

(c') \(\{\rho(\varphi_i(z_n), \varphi_j(z_n))\}_n\) is a convergent sequence for every \(i, j\).

Note that if \(|\varphi_i(z_n)| \to 1\) as \(n \to \infty\) for some \(i\), then it is easy to see that there exists a subsequence \(\{z_{n_j}\}_j\) of \(\{z_n\}_n\) satisfying \(z_{n_j} \in \mathcal{Z}\).

For \(\{z_n\}_n \in \mathcal{Z}\), we write

\[I(\{z_n\}) = \{i : 1 \leq i \leq N, |\varphi_i(z_n)| \to 1\}.
\]

By condition (a), \(I(\{z_n\}) \neq \emptyset\). By (b), there exists \(\delta\) with \(0 < \delta < 1\) such that \(|\varphi_j(z_k)| < \delta < 1\) for every \(j \notin I(\{z_n\})\) and \(k\). For each \(t \in I(\{z_n\})\), we write

\[(3.1) \quad I_0(\{z_n\}, t) = \{j \in I(\{z_n\}) : \rho(\varphi_j(z_n), \varphi_t(z_n)) \to 0\}.
\]

For \(s, t \in I(\{z_n\})\), we have either \(I_0(\{z_n\}, s) = I_0(\{z_n\}, t)\) or \(I_0(\{z_n\}, s) \cap I_0(\{z_n\}, t) = \emptyset\). Hence there is a subset \(\{t_1, t_2, \cdots, t_\ell\} \subset I(\{z_n\})\) such that

\[I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p)
\]

and \(I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset\) for \(p \neq q\).

When we consider the compactness of linear combinations \(\sum_{i=1}^{N} \lambda_i C_{\varphi_i}\), some \(C_{\varphi_i}\) could be compact, that is, \(\|\varphi_i\|_{\infty} < 1\). We may exclude such trivial ones from our linear combinations.

Gorkin and Mortini [4, Theorem 11] characterized necessary conditions for linear combinations of composition operators to be compact on some uniform algebras. We here obtain necessary and sufficient conditions on the compactness.
Theorem 3.2. Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ ($N \geq 2$) be distinct functions in $S(D)$ with $\|\varphi_i\|_{\infty} = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every $i$. Then the following conditions are equivalent.

(i) $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$.

(ii) $\sum \{\lambda_i : i \in I_0(\{z_n\}, t)\} = 0$ for every $\{z_n\} \in Z = Z(\varphi_1, \varphi_2, \cdots, \varphi_N)$ and $t \in I(\{z_n\})$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ is compact on $H^\infty$. Let $\{z_n\} \in Z$ and $t \in I(\{z_n\})$. For each positive integer $k$, we write

$$f_k(z) = \frac{1 - |\varphi_t(z_k)|^2}{1 - \overline{\varphi_t(z_k)} z} \prod_{j \notin I_0(\{z_n\}, t)} \frac{\varphi_j(z_k) - z}{1 - \overline{\varphi_j(z_k)} z}.$$

Then $f_k \in H^\infty$, $\|f_k\|_{\infty} \leq 2$, and $\{f_k\}_k$ converges to 0 uniformly on every compact subset of $D$. We have

$$\|\sum_{i=1}^{N} \lambda_i C_{\varphi_i} f_k\|_{\infty}$$

$$\geq |\sum_{i=1}^{N} \lambda_i f_k(\varphi_i(z_k))|$$

$$= |\sum_{i \in I_0(\{z_n\}, t)} \lambda_i \frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k) \varphi_i(z_k)} \prod_{j \notin I_0(\{z_n\}, t)} \frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \varphi_j(z_k) \varphi_i(z_k)}|.$$}

Here

$$\frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k) \varphi_i(z_k)} = 1 + \frac{\varphi_t(z_k) - \varphi_i(z_k)}{1 - \varphi_t(z_k) \varphi_i(z_k)}.$$}

For $i \in I_0(\{z_n\}, t)$, by (3.1) $\rho(\varphi_i(z_k), \varphi_t(z_k)) \to 0$ as $k \to \infty$. Hence

$$\frac{1 - |\varphi_t(z_k)|^2}{1 - \varphi_t(z_k) \varphi_i(z_k)} \to 1$$

as $k \to \infty$. 


On the other hand,

\[
\frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \varphi_j(z_k)\varphi_i(z_k)} - \frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \varphi_j(z_k)\varphi_i(z_k)} = \frac{\varphi_i(z_k) - \varphi_i(z_k)}{1 - \varphi_i(z_k)\varphi_i(z_k)} \left(1 + \overline{\varphi_{j}(z_k)} \frac{\varphi_i(z_k) - \varphi_j(z_k)}{1 - \varphi_i(z_k)\varphi_i(z_k)} \right) \times \left(1 + \overline{\varphi_i(z_k)} \frac{\varphi_i(z_k) - \varphi_j(z_k)}{1 - \varphi_i(z_k)\varphi_i(z_k)} \right).
\]

Since \( \rho(\varphi_i(z_k), \varphi_t(z_k)) \to 0 \), by (c) we have

\[
\lim_{k \to \infty} \frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \varphi_j(z_k)\varphi_i(z_k)} = \lim_{k \to \infty} \frac{\varphi_j(z_k) - \varphi_t(z_k)}{1 - \varphi_j(z_k)\varphi_t(z_k)}.
\]

Since \( j \not\in I_0(\{z_n\}, t) \), by (3.1) and (c)

\[
\lim_{k \to \infty} \frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \varphi_j(z_k)\varphi_i(z_k)} = \beta_{j,t} \neq 0
\]

for some \( \beta_{j,t} \in \mathbb{C} \).

By condition (i) and Proposition 3.1,

\[
\left\| \sum_{i=1}^{N} \lambda_i C_{\varphi_i} f_k \right\|_\infty \to 0
\]

as \( k \to \infty \). Therefore we get

\[
\left( \sum_{i \in I_0(\{z_n\}, t)} \lambda_i \right) \prod_{j \not\in I_0(\{z_n\}, t)} \beta_{j,t} = 0.
\]

Consequently, we have

\[
\sum_{i \in I_0(\{z_n\}, t)} \lambda_i = 0.
\]

(ii) \( \Rightarrow \) (i). Suppose that \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} \) is not compact on \( H^\infty \). Then there exists a sequence \( \{f_n\}_n \) in ball \( H^\infty \) such that \( f_n \to 0 \) uniformly on every compact subset of \( \mathbb{D} \) and

\[
\| \sum_{i=1}^{N} \lambda_i f_n \circ \varphi_i \|_\infty \not\to 0
\]
as $n \to \infty$. For some $\varepsilon > 0$, considering a subsequence of $\{f_n\}_n$, we may assume that
\[
\left\| \sum_{i=1}^{N} \lambda_i f_n \circ \varphi_i \right\|_\infty > \varepsilon > 0
\]
for every $n$. Take $z_n \in \mathbb{D}$ with $|z_n| \to 1$ and
\[
\left| \sum_{i=1}^{N} \lambda_i f_n(\varphi_i(z_n)) \right| > \varepsilon.
\]
Considering subsequence of $\{z_n\}_n$, we may assume that $\varphi_i(z_n) \to \alpha_i$ as $n \to \infty$ for every $i$. Since $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$, $|\alpha_i| = 1$ for some $i$. Moreover we may assume that $\{z_n\}_n \in Z$. Also we have
\[
(3.2) \quad \liminf_{k \to \infty} \left| \sum_{i \in I(\{z_n\})} \lambda_i f_k(\varphi_i(z_k)) \right| \geq \varepsilon.
\]
Recall that there exists a subset $\{t_1, t_2, \cdots, t_\ell\} \subset I(\{z_n\})$ such that
\[
I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p)
\]
and $I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. Let $i \in I_0(\{z_n\}, t_p)$. Then
\[
\rho(\varphi_i(z_k), \varphi_{t_p}(z_k)) \to 0 \quad \text{as} \quad k \to \infty.
\]
By Schwarz's lemma, see [3, p. 2],
\[
(3.3) \quad \rho(f_k(\varphi_i(z_k)), f_k(\varphi_{t_p}(z_k))) \leq \rho(\varphi_i(z_k), \varphi_{t_p}(z_k)) \to 0
\]
as $k \to \infty$. Since $\{f_k(\varphi_i(z_k))\}_k$ is bounded, considering a subsequence of $\{z_k\}_k$, we may assume that $f_k(\varphi_i(z_k)) \to \beta_i$ as $k \to \infty$ for every $i$. By (3.3), $\beta_i = \beta_{t_p}$ for every $i \in I_0(\{z_n\}, t_p)$. Therefore
\[
\lim_{k \to \infty} \sum_{i \in I(\{z_n\})} \lambda_i f_k(\varphi_i(z_k)) = \lim_{k \to \infty} \sum_{p=1}^{\ell} \sum_{i \in I_0(\{z_n\}, t_p)} \lambda_i f_k(\varphi_i(z_k))
\]
\[
= \sum_{p=1}^{\ell} \sum_{i \in I_0(\{z_n\}, t_p)} \lambda_i \beta_{t_p}
\]
\[
= \sum_{p=1}^{\ell} \beta_{t_p} \sum_{i \in I_0(\{z_n\}, t_p)} \lambda_i
\]
\[
= 0 \quad \text{by condition (ii)}.
\]
This contradicts condition (3.2). \(\square\)
The following corollaries follow from Theorem 3.2.

**Corollary 3.3.** Let \( \varphi_1, \varphi_2, \cdots, \varphi_N \) \((N \geq 2)\) be distinct functions in \( S(\mathbb{D}) \) with \( \| \varphi_i \|_{\infty} = 1 \), and \( \lambda_i \in \mathbb{C} \) with \( \lambda_i \neq 0 \) for every \( i \). If \( \sum_{i \in J} \lambda_i \neq 0 \) for every subset \( J \) of \( \{1, 2, \cdots, N\} \), then \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} \) is not compact on \( H^\infty \).

This says that the sum \( \sum_{i=1}^{N} C_{\varphi_i} \) is never compact on \( H^\infty \) for every \( \varphi_i \in S(\mathbb{D}) \) with \( \| \varphi_i \|_{\infty} = 1 \), \( i = 1, \cdots, N \).

**Corollary 3.4.** Let \( \varphi_1, \varphi_2, \cdots, \varphi_N \) \((N \geq 2)\) be distinct functions in \( S(\mathbb{D}) \) with \( \| \varphi_i \|_{\infty} = 1 \), and \( \lambda_i \in \mathbb{C} \) with \( \lambda_i \neq 0 \) for every \( i \). Suppose that \( \sum_{i=1}^{N} \lambda_i = 0 \) and \( \sum_{i \in J} \lambda_i \neq 0 \) for every non-empty proper subset \( J \) of \( \{1, 2, \cdots, N\} \). Then \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} \) is compact on \( H^\infty \) if and only if \( C_{\varphi_i} - C_{\varphi_j} \) is compact on \( H^\infty \) for every \( i, j \) with \( i \neq j \).

**Proof.** Suppose that \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} \) is compact on \( H^\infty \). Then by Theorem 3.2 (ii), for every \( \{z_n\}_{n} \in \mathcal{Z} \), \( I(\{z_n\}) = \{1, 2, \cdots, N\} \) and \( I_{0}(\{z_n\}, t) = \{1, 2, \cdots, N\} \) for every \( t \in I(\{z_n\}) \). Hence

\[
\lim_{|\varphi_i(z)| \to 1} \rho(\varphi_i(z), \varphi_j(z)) = 0.
\]

By [12], \( C_{\varphi_i} - C_{\varphi_j} \) is compact for every \( i, j \).

Suppose that \( C_{\varphi_i} - C_{\varphi_j} \) is compact for every \( i, j \). Since

\[
\sum_{i=1}^{N} \lambda_i C_{\varphi_i} = \left( \sum_{i=1}^{N} \lambda_i \right) C_{\varphi_1} + \sum_{i=2}^{N} \lambda_i (C_{\varphi_i} - C_{\varphi_1}) = \sum_{i=1}^{N} \lambda_i (C_{\varphi_i} - C_{\varphi_j}),
\]

we have that \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} \) is compact.

We recall that the Bloch space \( \mathcal{B} \) consists of all analytic functions \( f \) on \( \mathbb{D} \) such that \( \| f \|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \). It is well known that \( \mathcal{B} \) is a Banach space under the norm \( \| f \| = |f(0)| + \| f \|_{\mathcal{B}} \). Then, under the assumption of Corollary 3.4, we obtain the following by Theorem 3 in [12].

**Corollary 3.5.** Let \( \varphi_1, \varphi_2, \cdots, \varphi_N \) \((N \geq 2)\) be distinct functions in \( S(\mathbb{D}) \) with \( \| \varphi_i \|_{\infty} = 1 \), and \( \lambda_i \in \mathbb{C} \) with \( \lambda_i \neq 0 \) for every \( i \). Suppose that \( \sum_{i=1}^{N} \lambda_i = 0 \) and \( \sum_{i \in J} \lambda_i \neq 0 \) for every non-empty proper subset \( J \) of \( \{1, 2, \cdots, N\} \). Then the following conditions are equivalent.

(i) \( \sum_{i=1}^{N} \lambda_i C_{\varphi_i} : H^\infty \to H^\infty \) is compact.
(ii) $\sum_{i=1}^{N} \lambda_{i} C_{\varphi_{i}} : B \rightarrow H^{\infty}$ is compact.

It would be another problem to characterize the boundedness and compactness of $\sum_{i=1}^{N} \lambda_{i} C_{\varphi_{i}}$ acting from $B$ to $H^{\infty}$ in general. The boundedness and compactness of the differences of two composition operators acting from $B$ to $H^{\infty}$ is concerning to the component problem of the set $C(H^{\infty})$ of composition operators on $H^{\infty}$ ([12]).

**Example 3.6.** We show the existence of $\varphi_{1}, \varphi_{2}, \varphi_{3} \in S(\mathbb{D})$ with $\|\varphi_{i}\|_{\infty} = 1$ such that $C_{\varphi_{1}} - C_{\varphi_{2}} - C_{\varphi_{3}}$ is compact.

Let $\sigma(z) = (1 + z)/(1 - z)$ and

$$\varphi_{1}(z) = \frac{\sqrt{\sigma(z)} - 1}{\sqrt{\sigma(z)} + 1}$$

be a lens map ([14]). Also let

$$\varphi_{2}(z) = 1 - \sqrt{1 - z}.$$  

Denote by $\partial \mathbb{D}$ the boundary of $\mathbb{D}$. Then $\varphi_{1}, \varphi_{2} \in S(\mathbb{D})$, $\varphi_{1}(\pm 1) = \pm 1$, $|\varphi_{1}(e^{i\theta})| < 1$ for $e^{i\theta} \in \partial \mathbb{D}$ with $e^{i\theta} \neq \pm 1$, $\varphi_{2}(1) = 1$, and $|\varphi_{2}(e^{i\theta})| < 1$ for $e^{i\theta} \in \partial \mathbb{D}$ with $e^{i\theta} \neq 1$. As Example (i) in [7, p. 513],

$$\rho(\varphi_{1}(z), \varphi_{2}(z)) = \frac{|\sqrt{\sigma(z)}(1 - \varphi_{2}(z)) - (1 + \varphi_{2}(z))|}{|\sigma(z)(1 - \varphi_{2}(z)) + (1 + \varphi_{2}(z))|}.$$  

Since

$$Re \frac{\sqrt{1 - z}}{\sqrt{1 + z}} > 0 \quad \text{for} \quad z \in \mathbb{D},$$

we have

$$\lim_{z \rightarrow 1} \rho(\varphi_{1}(z), \varphi_{2}(z)) = 0.$$  

Let

$$\varphi_{3}(z) = -1 + \sqrt{1 + z}.$$  

Then $\varphi_{3} \in S(\mathbb{D})$, $\varphi_{3}(-1) = -1$, and $|\varphi_{3}(e^{i\theta})| < 1$ for $e^{i\theta} \in \partial \mathbb{D}$ with $e^{i\theta} \neq -1$. Similarly we have

$$\lim_{z \rightarrow -1} \rho(\varphi_{1}(z), \varphi_{3}(z)) = 0.$$
Hence by Theorem 3.2, \( C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3} \) is compact.

References


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