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Structure of nonnegative solutions for parabolic equations and perturbation theory for elliptic operators

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This talk is concerned with structure of nonnegative solutions for parabolic equations, perturbation theory for elliptic operators, and their relations.

Let M be a Riemannian manifold of dimension n, and D be a noncompact domain of M. Let L be an elliptic operator on D of the form

$$Lu = -m^{-1} \operatorname{div}(mA\nabla u) + Vu .$$

Here m is a positive measurable function such that $m, m^{-1} \in L^{\infty}_{loc}(D)$, A is a symmetric measurable section on D of $\operatorname{End}(TM)$ such that $^{\forall} K \subset\subset D$ $^{\exists} \lambda > 0$

such that

$$\lambda|\xi|^2 \leq \langle A_x\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2, \quad x\in K,\ (x,\xi)\in TM,$$

and V is a real-valued function with $V \in L^p_{loc}(D, m\,d\nu)$ for some $p > \max(\frac{n}{2}, 1)$, where $d\nu$ is the Riemannian measure.

For simplicity of notations, we assume

$$\lambda_0 \equiv \inf\{ \int_D (\langle A\nabla u, \nabla u\rangle + Vu^2)\, m\,d\nu\ ;$$

$$u\in C_0^\infty(D),\ \int_D u^2\, m\,d\nu = 1\}$$

is positive. (Actually, the condition $\lambda_0 > -\infty$ suffices.)

We consider nonnegative solutions of the parabolic equation

$$(\#) \qquad (\partial_t + L)u = 0 \quad \text{in} \quad Q = D\times(0,T),$$

where $T > 0$.

Our problem is the following.

<Problem> Determine all nonnegative solutions of (#).

We put

$$\underline{P}(Q) = \{ u \geqq 0 ; \ u \text{ is a solution of } (\#) \} .$$

This problem is closely related to the Cauchy problem

$$(\#^\circ) \quad \begin{cases} (\partial_t + L) u = 0 & \text{in } Q \\ u(x,0) = 0 & \text{on } D . \end{cases}$$

We say :

[UP] holds for $(\#^\circ)$ when any solution $u \geqq 0$ of $(\#^\circ)$ must be identically zero.

[NUP] holds for $(\#^\circ)$ when there exists a solution $u \gneqq 0$ of $(\#^\circ)$.

<Example> (Widder '44) $D = M = \mathbb{R}^n$, $L = -\Delta$
$\Rightarrow$ [UP] holds for $(\#^\circ)$

When [UP] holds, our problem has an extremly simple answer.

<Fact> (Ancona–Taylor '92)

[UP] holds for (#^{0})

$\Rightarrow$ $\forall u \in \underline{P}(Q)$ $\exists_1 \mu$: Borel measure on D such that

$$u(x,t) = \int_D p(x,y,t)\,d\mu(y), \quad (x,t) \in Q .$$

Here p is a minimal fundamental solution for $\partial_t + L$ with respect to the measure $m\,d\nu$.

Thus we need to consider what happens in the case [NUP]. For this purpose let us introduce the following condition [SSP] (i.e. the constant function 1 is a semismall perturbation of L).

[SSP] $\forall \varepsilon > 0$ $\exists K \subset\subset D$ such that

$$\int_{D \setminus K} G(x^0, z) \cdot 1 \cdot G(z, y)\, m(z)\, d\nu(z)$$

$$\leqq \varepsilon\, G(x^0, y), \qquad y \in D \setminus K,$$

where x^0 is a fixed reference point in D and G is the positive Green function of L on D.

Recall that we have assumed $\lambda_0 > 0$. Thus the Green function exists. Furthermore we can define the selfadjoint operator L_D on $L^2(D; m dv)$ associated with the quadratic form generated by L.

Let

$\partial_M D$: Martin boundary of D for L

$\partial_m D$: minimal Martin boundary of D for L

$D_L^* = D \cup \partial_M D$: Martin compactification of D for L

$K(x, \xi)$: Martin kernel with pole $\xi \in \partial_M D$

Then we see : [SSP] $\Rightarrow$ The following $1°$, $2°$, $3°$ hold.

$1°$ The spectrum of L_D consists of discrete eigenvalues with finite multiplicity. Let $\lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ be the eigenvalues of L_D repeated according to the multiplicity, and ϕ_j be an eigenfunction for λ_j such that $\phi_0 > 0$ and $\{\phi_j\}_{j=0}^{\infty}$ is a complete orthonormal system of $L^2(D; m dv)$.

$2°$ There exists a continuous extension $[\phi_j/\phi_0]$ of ϕ_j/ϕ_0 up to $\partial_M D$.

3° [NUP] holds for (#°).

Furthermore, we have the following

<Theorem 1> [SSP] holds

$\Longrightarrow$ ${}^{\forall}\xi \in \partial_M D$

$$^{\exists}\lim_{D \ni y \to \xi} \frac{p(x,y,t)}{\phi_0(y)} \equiv q(x,\xi,t), \quad x \in D,\ t \in \mathbb{R}$$

q is a continuous function on $D \times \partial_M D \times \mathbb{R}$ such that

$$q > 0 \quad \text{on} \quad D \times \partial_M D \times (0,\infty)$$
$$q = 0 \quad \text{on} \quad D \times \partial_M D \times (-\infty, 0]$$
$$(\partial_t + L)\, q(\cdot,\xi,\cdot) = 0 \quad \text{in} \quad D \times \mathbb{R}$$

We are now ready to give an answer to our problem.

<Theorem 2> [SSP] holds

$\Longrightarrow$ ${}^{\forall}u \in \mathcal{P}(Q)$ $\ {}^{\exists_1}(\mu, \lambda)$ such that

μ is a Borel measure on D

λ is a Borel measure on $\partial_M D \times [0, T)$ which is supported by $\partial_m D \times [0, T)$

$$u(x,t) = \int_D p(x,y,t)\, d\mu(y)$$

$$+ \int_{\partial_M D \times [0,T)} q(x, \xi, t-s)\, d\lambda(\xi, s),$$

$$(x,t) \in Q$$

I should mention that Theorems 1 and 2 are results of [Mendez - Murata '98, preprint].

The proof of Theorem 2 is based upon the abstract parabolic Martin representation theorem and Choquet's theorem. Its key step is to identify the parabolic Martin boundary.

Here let's see simple examples.

< Example 1 > $D \subset \mathbb{R}^2$ with $|D| < \infty$

$L = -\Delta$

$\Rightarrow$ [SSP] holds and so Theorem 2 holds.

For the higher dimensional case, we need some regularity of a boundary.

<Example 2> $D \subset \mathbb{R}^n$: bounded John domain

$$L = -\Delta$$

$\Rightarrow$ [SSP] holds and so Theorem 2 holds.

<Example 3> $L = -\Delta + 1$ on $\mathbb{R}^n$, $\beta \in \mathbb{R}$,

$$D = \{ (x_1, x') \in \mathbb{R}^n ; \ x_1 > 1, \ |x'| < x_1^\beta \}$$

Then

$$[SSP] \text{ holds} \iff \beta < -1$$

Now let us introduce a condition which is weaker than [SSP].

Put $E_L(D) = \{ h > 0 ; \ Lh = 0 \text{ on } D \}$.
Fix $h \in E_L(D)$.

[NhB] (i.e. 1 is non-h-big)

$$\exists v \in E_{L+1}(D) \ \text{ s.t. } \ 0 < v \leqq h \text{ on } D$$

By virtue of a nice characterization of [NhB] by Grigor'yan – Hansen '98, we can show

$$[SSP] \Longrightarrow [NhB] \quad \forall h \in E_L(D)$$

Furthermore, we have

<Theorem> Fix $h \in E_L(D)$. Then

$[NhB] \iff \exists\, u$: solution of $(\#^\circ)$ such that
$0 < u(x,t) < h(x)$ on $D \times (0,\infty)$

Obviously, this implies [NUP].

We say [hB] (i.e. 1 is h-big) holds when
$(L+1)v = 0,\ 0 \leqq v \leqq h$ on $D \Rightarrow v \equiv 0$

Then a direct consequence of the above theorem is the following

<Corollary> $[UP] \Longrightarrow [hB] \quad {}^\forall h \in E_L(D)$

This observation is useful in showing [hB] since we have a general and sharp sufficient condition for [UP] by Ishige-Murata '01.

As for [SSP], we have powerful theorems by Ancona '97.

Summing up, we have :

$$[SSP] \Rightarrow [NhB] \Rightarrow [NUP]$$
$$[UP] \Rightarrow [hB]$$

$[NUP] \Longleftrightarrow$ 1 is a "small" perturbation
$[UP] \Longleftrightarrow$ 1 is a "big" perturbation

Now let us see some more examples.

< Example 4 > (generalized Poincaré disc)

$$D = \{ x \in \mathbb{R}^2 \,;\, |x| < 1 \}, \qquad \gamma \in \mathbb{R}$$

$$L = -(1-|x|^2)^2 \left(\log \frac{2}{1-|x|^2} \right)^{\gamma} \Delta_{\mathbb{R}^2}$$

Then

(i) $[SSP] \Longleftrightarrow \gamma > 1$
(ii) $[UP] \Longleftrightarrow \gamma \leqq 1$

< Example 5 > $L = -\sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j) + V(x)$

: uniformly elliptic operator on $D = \mathbb{R}^n$

$0 < {}^{\exists}c < 1$, ${}^{\exists}\rho$: positive continuous increasing function on $[0, \infty)$

such that

$$c\,[\rho(|x|)]^2 \leqq V(x) \leqq [\rho(|x|)]^2, \quad x \in \mathbb{R}^n$$

$$c\,\rho\left(r + \frac{c}{\rho(r)}\right) \leqq \rho(r), \quad r \geqq 0$$

Then

(i) [SSP] $\Longleftrightarrow$ $\displaystyle\int_1^\infty \frac{dr}{\rho(r)} < \infty$

(ii) [UP] $\Longleftrightarrow$ $\displaystyle\int_1^\infty \frac{dr}{\rho(r)} = \infty$