Variable exponent version of Hedberg-Wolff inequalities

(Potential Theory and its related Fields)

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Introduction.

Hedberg-Wolff gave the following inequalities in [HW]:

\[ C^{-1} \int_{\mathbb{R}^N} \left[ (G_{\alpha} * \mu)(x) \right]^{p'} dx \leq \int_{\mathbb{R}^N} \mathcal{W}_{\alpha,p}^\mu(x, 1) d\mu(x) \leq C \int_{\mathbb{R}^N} \left[ (G_{\alpha} * \mu)(x) \right]^{p'} dx \]  (1)

for every nonnegative measure \( \mu \) on \( \mathbb{R}^N \) with a positive constant \( C \) independent of \( \mu \), where \( G_{\alpha} \) is the Bessel kernel of order \( \alpha (0 < \alpha < N) \) on \( \mathbb{R}^N \), \( 1 < p < \infty \), \( 1/p + 1/p' = 1 \) and

\[ \mathcal{W}_{\alpha,p}^\mu(x, R) = \int_0^R \left( \frac{\mu(B(x,r))}{r^{N-\alpha p}} \right)^{p'-1} \frac{dr}{r} \quad (R > 0). \]

The function \( \mathcal{W}_{\alpha,p}^\mu(\cdot, R) \) is called the Wolff-potential of \( \mu \) for order \( (\alpha, p) \). Inequalities (1) imply

\[ \mu \in (L^{\alpha,p}(\mathbb{R}^N))^* \iff \int_{\mathbb{R}^N} \mathcal{W}_{\alpha,p}^\mu(x, 1) d\mu(x) < \infty, \]  (2)

where

\[ L^{\alpha,p}(\mathbb{R}^N) = \{ u = G_{\alpha} * f ; f \in L^p(\mathbb{R}^N) \} \]

with the norm \( \|u\|_{\alpha,p} = \|f\|_p \). Since \( L^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N) \) for \( m \in \mathbb{N} \) (A.P. Calderón), (2) shows that

\[ \mu \in (W^{m,p}(\mathbb{R}^N))^* \iff \int_{\mathbb{R}^N} \mathcal{W}_{\alpha,p}^{\mu}(x, 1) d\mu(x) < \infty, \]  (2')

for \( m \in \mathbb{N} \).

In [AH], the proof of (1) is given via the following inequalities

\[ C^{-1} \|M_{\alpha,R} \mu\|_q \leq \|G_{\alpha} * \mu\|_q \leq C\|M_{\alpha,R} \mu\|_q \]  (3)

for \( 0 < q < \infty \) and \( R > 0 \) with a positive constant \( C \) independent of \( \mu \), where

\[ (M_{\alpha,R} \mu)(x) = \sup_{0<r<R} r^{\alpha-N} \mu(B(x,r)). \]

These results have been generalized to the case where \( G_{\alpha} \) is replaced by a general convolution kernel satisfying certain conditions (cf. [JPW], [AE, Part II]).

In the present paper, we consider variable exponents \( p(x) \) on \( \mathbb{R}^N \) and show that inequalities (1) and (3) hold in some restricted forms, and relations (2) and (2') still hold true for \( \mu \) with finite total mass when we replace \( p \) by \( p(x) \) satisfying certain conditions. We discuss these for convolution kernels.

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1. Definitions

As a potential kernel function on $\mathbb{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a nonnegative nonincreasing lower semicontinuous function $k(r)$ on $(0, \infty)$ such that

(k.1) there is $R_0 > 0$ such that $k(r)$ is positive and satisfies the doubling condition on $(0, R_0)$, i.e., $k(r) \leq C_d k(2r)$ for $0 < r < R_0/2$;

(k.2) $\int_0^1 k(r)r^{N-1}dr < \infty$.

By (k.2), $k(x) \in L^1_{loc}(\mathbb{R}^N)$. The $k$-potential of a nonnegative measure $\mu$ on $\mathbb{R}^N$ is defined by

$$(k * \mu)(x) = \int k(x - y)d\mu(y).$$

For $R > 0$, the $(k, R)$-maximal function of $\mu$ is defined by

$$(M_{k,R} \mu)(x) = \sup_{0 < r < R} k(r)\mu(B(x, r)).$$

We consider a variable exponent $p(x)$ on $\mathbb{R}^N$ such that

(P1) $1 < p^- := \inf p(\cdot) \leq p^+ := \sup p(\cdot) < \infty$;

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(1/|x - y|)} \quad \text{for } |x - y| \leq \frac{1}{2}$$

with a constant $C_p \geq 0$, which is referred to as the constant of log-Hölder continuity.

We refer to [KR] for the definition of the $p(\cdot)$-norm $\|f\|_{p(\cdot)}$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$ and the variable exponent Sobolev space $W^{m, p(\cdot)}(\mathbb{R}^N)$ ($m \in \mathbb{N}$).

For $R > 0$, we define the $(k, p(\cdot))$-Wolff potential of $\mu$ by

$$\mathcal{W}^{\mu}_{k, p(\cdot)}(x, R) = \int_0^R k(r)^{p(x)}\mu(B(x, r))^{p(x)-1_{\Gamma}N-1} dr.$$

Example. For $0 < \alpha < N$, the Riesz kernel $I_{\alpha}(x) = 1/|x|^{N-\alpha}$ and the Bessel kernel $G_{\alpha}$ of order $\alpha$ are typical examples of $k(x)$. For these kernels, we can take $R_0$ any positive value.

$$\mathcal{W}^{\mu}_{I_{\alpha}, p(\cdot)}(x, R) := \mathcal{W}^{\mu}_{I_{\alpha}, p(\cdot)'}(x, R) = \int_0^R \left( \frac{\mu(B(x, r))}{r^{N-\alpha p(x)}} \right)^{p(x)'-1}\frac{dr}{r},$$

and

$$\mathcal{W}^{\mu}_{G_{\alpha}, p(\cdot)}(x, R) \sim \mathcal{W}^{\mu}_{I_{\alpha}, p(\cdot)'}(x, R).$$

For a nonnegative measure $\mu$ and $R > 0$, let

$$M(\mu, R) := \sup_{x \in \mathbb{R}^N} \mu(B(x, R)).$$
It is easy to see that if $M(\mu, R) < \infty$ holds for some $R > 0$, then so holds for all $R > 0$.

**Lemma 1.** If either $k * \mu \in L^{p(\cdot)}(\mathbb{R}^N)$ or $M_{k,R} \mu \in L^{p(\cdot)}(\mathbb{R}^N)$ or
\[
\int \mathcal{W}_{k,p(\cdot)}(x,R) \, d\mu(x) < \infty,
\]
then $M(\mu, R) < \infty$ for all $R > 0$.

**Proof.** Suppose that $M(\mu, R) = \infty$ for some $R > 0$. As remarked above, we may assume $0 < R < R_0$ and $M(\mu, R/3) = \infty$. Then, for every $n \in \mathbb{N}$, there exists $\xi_n \in \mathbb{R}^N$ such that $\mu(B(\xi_n, R/3)) \geq n$. If $x \in B(\xi_n, R/3)$, then $\mu(B(x, 2R/3)) \geq n$, so that
\[
(k * \mu)(x) \geq \int_{B(x,2R/3)} k(x-y) \, d\mu(y) \geq k(R)n
\]
and
\[
(M_{k,R} \mu)(x) \geq k(2R/3) \mu(B(x,2R/3)) \geq k(R)n.
\]
Thus
\[
\int [(k * \mu)(x)]^{p(x)} \, dx \geq \int_{B(\xi,R/3)} [(k * \mu)(x)]^{p(x)} \, dx \geq C_1 n^{p^-}
\]
with a constant $C_1 > 0$ independent of $n$. This shows that $k * \mu \notin L^{p(\cdot)}(\mathbb{R}^N)$. Similarly, we see that $M_{k,R} \mu \notin L^{p(\cdot)}(\mathbb{R}^N)$.

Also, if $x \in B(\xi_n, R/3)$, then
\[
\mathcal{W}_{k,p(\cdot)}^{\mu}(x,R) \geq \int_{2R/3}^{R} k(r)^{p(x)} \mu(B(x,r))^{p(x)-1} r^{N-1} dr \geq C_2 n^{p^-}
\]
with a constant $C_2 > 0$ independent of $n$, so that
\[
\int \mathcal{W}_{k,p(\cdot)}^{\mu}(x,R) \, d\mu(x) \geq \int_{B(\xi_n,R/3)} \mathcal{W}_{k,p(\cdot)}^{\mu}(x,R) \, d\mu(x) \geq C_2 n^{p^-}
\]
for all $n \in \mathbb{N}$.

We call $\int \mathcal{W}_{k,p(\cdot)}^{\mu}(x,R) \, d\mu(x)$ the $(k,p(\cdot))$-energy of $\mu$.

2. **Estimate of $(k,p(\cdot))$-energy by $p(\cdot)$-integral of $k$-potential**

**Theorem 1.** Let $M_0 \geq 1$, $0 < R < R_0/2$. Then
\[
\int \mathcal{W}_{k,p(\cdot)}^{\mu}(x,R) \, d\mu(x) \leq C \left( \mu(\mathbb{R}^N) + \int [(k * \mu)(x)]^{p(x)} \, dx \right)
\]
for all nonnegative measure $\mu$ such that $M(\mu, R) \leq M_0$, with a constant $C > 0$ depending only on $N$, $C_d$, $p^+$, $C_p$, $M_0$ and $K_R := \int_{0}^{R} k(r)r^{N-1} \, dr$. 

Proof. We consider a nonlinear potential

\[ V_{k,p(\cdot)}^\mu = k \ast (k \ast \mu)^{p(\cdot)-1}. \]

Since

\[ \int [(k \ast \mu)(x)]^{p(x)} dx = \int V_{k,p(\cdot)}^\mu(x) d\mu(x), \]

it suffices to show

\[ \mathcal{W}_{k,p(\cdot)}^\mu(x, R) \leq C(1 + V_{k,p(\cdot)}^\mu(x)). \quad (2.1) \]

for \( 0 < R < R_0/2 \).

Since \( k(r) \) is nonincreasing and \( K_R < \infty, k(r) \leq NK_Rr^{-N} \) for \( 0 < r < R \). Hence, (P2) implies that

\[ (k(y)\mu(B(x, |y|)))^{p(x)} \leq C(k(y)\mu(B(x, |y|)))^{p(x-y)} \quad (2.2) \]

for \( |y| \leq R \) whenever \( M(\mu, R) \leq M_0 \) and \( k(y)\mu(B(x, |y|))) \geq 1 \), with a constant \( C = C(N, K_R, C_p, M_0) \).

If \( |y| \leq R \), then \( |x - y - \xi| \leq 2|y| \) for \( \xi \in B(x, |y|) \), so that

\[ k(y)\mu(B(x, |y|)) \leq C_d k(2y)\mu(B(x, |y|)) \leq C_d (k \ast \mu)(x - y). \]

Hence, using (2.2) we have

\[
\begin{align*}
\mathcal{W}_{k,p(\cdot)}^\mu(x, R) &= \frac{1}{\sigma_N} \int_{|y|<R} k(y)(k(y)\mu(B(x, |y|)))^{p(x)-1} dy \\
&\leq \frac{1}{\sigma_N} \int_{|y|<R} k(y) dy + C \int_{|y|<R} k(y)(k(y)\mu(B(x, |y|)))^{p(x-y)-1} dy \\
&\leq K_R + C \int_{|y|<R} k(y)[(k \ast \mu)(x - y)]^{p(x-y)-1} dy \\
&\leq K_R + CV_{k,p(\cdot)}^\mu(x),
\end{align*}
\]

with constants \( C = C(N, C_d, K_R, p^+, C_p, M_0) > 0 \), which shows (2.1). (Here, \( \sigma_N \) denotes the surface area of the unit sphere in \( \mathbb{R}^N \).)

Remark. In Theorem 3, it is not known whether the term \( \mu(\mathbb{R}^N) \) is really necessary.

On the other hand, for non-constant exponent \( p(\cdot) \), the following inequality does not hold even if \( M(\mu, R) \leq M_0 \):

\[ \int [(k \ast \mu)(x)]^{p(x)} dx \leq C \int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x). \]

In fact, if \( p(\cdot) \) is continuous and non-constant in \( \mathbb{R}^N \), then we can find nonnegative measures \( \{\mu_j\} \) such that \( M(\mu_j, R) \to 0 \) \( (j \to \infty) \) and

\[
\frac{\int \mathcal{W}_{G_\alpha,p(\cdot)}^{\mu_j}(x, R) d\mu_j(x)}{\int [(G_\alpha \ast \mu_j)(x)]^{p(x)} dx} \to 0 \quad (2.3)
\]
as \( j \to \infty \) for every \( 0 < \alpha < N \) and every \( R > 0 \).

Proof. We can choose two compact sets \( K_1 \) and \( K_2 \) and \( 1 < p_1 < p_2 < \infty \) such that \( |K_1| > 0, \ |K_2| > 0, \)

\[
p(x) \leq p_1 \quad \text{for} \quad x \in K_1 \quad \text{and} \quad p(x) \geq p_2 \quad \text{for} \quad x \in K_2.
\]

Let \( \mu_j = (1/j)\chi_{K_2}dx, \ j = 1, 2, \ldots \). Obviously, \( M(\mu_j, R) \to 0 \).

Since \( \mu_j(B(x, r)) \leq (1/j)C_NR^N \) for any \( x \in \mathbb{R}^N \) and \( r > 0, \)

\[
W_{G_{\alpha,p}(\cdot)}^{\mu_j}(x, R) \leq C(\alpha, N, p^+) \int_0^R (r^{-N})^{p(x)}[(1/j)C_NR^N]^{p(x)-1}r^{N-1}dr
\]

with constants \( C(\ldots) > 0 \). If \( x \in K_2 \), then \( (1/j)C_NR^N \leq (1/j)C_NR^N \), so that

\[
\int W_{G_{\alpha,p}(\cdot)}^{\mu_j}(x, R) d\mu_j(x) \leq C(\alpha, N, p^+, p^-, R)(1/j)^{p_2}|K_2|.
\] (2.4)

On the other hand, since \( G_{\alpha} \) is positive continuous on \( \mathbb{R}^N, \)

\[
A = A(\alpha, K_1, K_2) := \inf\{G_{\alpha}(x-y); x \in K_1, y \in K_2\} > 0.
\]

If \( x \in K_1 \), then

\[
(G_{\alpha}*\mu_j)(x) = (1/j) \int_{K_2} G_{\alpha}(x-y)dy \geq (1/j)A|K_2|.
\]

Thus,

\[
\int (G_{\alpha}*\mu_j)^{p(x)}dx \geq \int_{K_1} [(1/j)A|K_2|]^{p(x)}dx
\]

\[
\geq (1/j)^{p_1} \min(A|K_2|, 1)^{p_1}|K_1|.
\] (2.5)

In view of (2.4) and (2.5), we obtain (2.3), since \( p_2 > p_1 \).

3. Estimate of \( p(\cdot) \)-integral of \((k, R)\)-maximal function by \((k, p(\cdot))\)-energy

**Theorem 2.** Let \( M_0 \geq 1 \) and \( 0 < R < R_0/3 \). Then

\[
\int [(M_{k,R} \mu)(x)]^{p(x)}dx \leq C \left( \mu(\mathbb{R}^N) + \int W_{k,p(\cdot)}^{\mu}(x, 3R) d\mu(x) \right)
\]

for all nonnegative measure \( \mu \) such that \( M(\mu, 3R) \leq M_0 \), with a constant \( C > 0 \) depending only on \( N, C_d, K_R, p^+, C_p \) and \( M_0 \).

**Proof.** Let \( 0 < R < R_0/3 \). For \( 0 < r < R, \)

\[
\int_0^{3R/2} k(t)^{p(x)} \mu(B(x, t))^{p(x)}dt \geq \int_r^{3r/2} k(2r)^{p(x)} \mu(B(x, r))^{p(x)}dt
\]

\[
\geq \log(3/2)C_d^{-p^+} [k(r) \mu(B(x, r))]^{p(x)}.
\]
Hence
\[
[(M_{k,R} \mu)(x)]^{p(x)} \leq 3C_{d}^{p^{+}} \int_{0}^{3R/2} k(t)^{p(x)} \mu(B(x,t))^{p(x)} \frac{dt}{t},
\]
and so
\[
\int [(M_{k,R} \mu)(x)]^{p(x)} dx \leq 3C_{d}^{p^{+}} \int_{0}^{3R/2} \left( \int k(t)^{p(x)} \mu(B(x,t))^{p(x)} dx \right) \frac{dt}{t}.
\]
Now,
\[
\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} dx = \int k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} \left( \int \chi_{B(x,t)}(y) d\mu(y) \right) dx
\]
\[
= \int \left( \int k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} dx \right) d\mu(y).
\]
As in the proof of Theorem 1, we have
\[
[k(t) \mu(B(x,t))]^{p(x)-1} \leq C [k(t) \mu(B(x,t))]^{p(y)-1} \leq C [k(t) \mu(B(y,2t))]^{p(y)-1}
\]
whenever \(|x-y| < t < 3R/2, M(\mu, 3R) \leq M_{0}\) and \(k(t) \mu(B(x,t)) \geq 1\), where constants \(C\) depend only on \(N, C_{d}, K_{3R}, p^{+}, C_{p}\) and \(M_{0}\).

Thus,
\[
\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} dx \leq |B(0,t)| \left( k(t) \mu(\mathbb{R}^{N}) + C \int k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} d\mu(y) \right).
\]
Therefore
\[
\int [(M_{k,R} \mu)(x)]^{p(x)} dx
\]
\[
\leq C \left( \mu(\mathbb{R}^{N}) \int_{0}^{3R/2} k(t)t^{N-1} dt + \int_{0}^{3R/2} t^{N-1} \left( \int k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} d\mu(y) \right) dt \right)
\]
\[
\leq C \left( \mu(\mathbb{R}^{N}) + \int \left( \int_{0}^{3R/2} k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} t^{N-1} dt \right) d\mu(y) \right)
\]
\[
\leq C \left( \mu(\mathbb{R}^{N}) + \int W_{k,p(y)}^{\mu}(y,3R) d\mu(y) \right)
\]
with constants \(C\) depending only on \(N, C_{d}, K_{3R}, p^{+}, C_{p}\) and \(M_{0}\).
4. Estimate of $p(\cdot)$-norm of convolution potential by $p(\cdot)$-norm of $k$-maximal function

The example given in the Remark in section 2 also shows that the following (modular) inequality does not hold whenever $p(\cdot)$ is continuous and non-constant:

$$\int_{\mathbb{R}^N} (G_{\alpha} \ast \mu)^{p(x)} dx \leq C \int_{\mathbb{R}^N} (M_{\alpha,R} \mu)^{p(x)} dx.$$

However, we obtain norm inequality under an additional conditions on $p(x)$:

**Theorem 3.** Suppose $k(r)$ in addition satisfies

(k.3) $\int_1^\infty k(r) r^{N-1} \, dr < \infty$;

(k.4) There is a constant $C_k > 0$ such that

$$\int_0^r k(t) t^{N-1} \, dt \leq C_k r^N k(r) \text{ for } 0 < r < R_0;$$

and suppose $p(x)$ in addition satisfies (P3) (log-Hölder continuity at $\infty$)

$$|p(x) - p(y)| \leq \frac{C_\infty}{\log(c + |x|)} \text{ for } |y| > |x|.$$

Then, for $0 < R < R_0/2$,

$$\|k \ast \mu\|_{p(\cdot)} \leq C \|M_{k,R} \mu\|_{p(\cdot)}$$

with a constant $C > 0$ depending only on $N, C_d, C_k, k(R), K, p^+, p^-, C_{lh}, C_\infty$ and $R$.

Note that the Bessel kernal $G_{\alpha}$ satisfies (k.3) and (k.4).

To prove Theorem 3, given $R > 0$, let

$$k_R(r) = k(r) \chi_{(0,R)}(r) \quad \text{and} \quad \tilde{k}_R(r) = k(r) \chi_{[R,\infty)}(r).$$

We treat $k_R \ast \mu$ and $\tilde{k}_R \ast \mu$ separately. First, we show

**Proposition 1.** Suppose $k(r)$ satisfies (k.1), (k.2) and (k.4), and suppose $p(x)$ satisfies (P1), (P2) and (P3). Then, for $0 < R < R_0/2$,

$$\|k_R \ast \mu\|_{p(\cdot)} \leq C \|M_{k,R} \mu\|_{p(\cdot)}$$

with a constant $C > 0$ depending only on $N, C_d, C_k, k(R), p^+, p^-, C_{lh}, C_\infty$ and $R$.

We prove this proposition applying the following theorem due to D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez [CFMP]:

**C-F-M-P Theorem.** Let $\mathcal{F}$ be a family of ordered pairs $(f, g)$ of nonnegative measurable functions on $\mathbb{R}^N$. Suppose that for some $p_0$, $0 < p_0 < \infty$,

$$\int_{\mathbb{R}^N} f(x)^{p_0} w(x) \, dx \leq C_0 \int_{\mathbb{R}^N} g(x)^{p_0} w(x) \, dx$$

with a constant $C > 0$ depending only on $N, C_d, C_k, k(R), p^+, p^-, C_{lh}, C_\infty$ and $R$. 

for all \((f,g) \in \mathcal{F}\) and for all \(A_1\)-weights \(w\), where \(C_0\) depends only on \(p_0\) and the \(A_1\)-constant of \(w\). Let \(p(\cdot)\) satisfy (P1), (P2) and (P3), and assume further that \(p^- > p_0\). Then
\[
\|f\|_{p(\cdot)} \leq C\|g\|_{p(\cdot)}
\]
for all \((f,g) \in \mathcal{F}\).

**Remark.** In [CFMP], the last phrase in the above theorem is “for all \((f,g) \in \mathcal{F}\) such that \(f \in L^{p(\cdot)}(\mathbb{R}^N)\)”. By examining its proof, we see that \(g \in L^{p(\cdot)}(\mathbb{R}^N)\) (i.e., \(\|g\|_{p(\cdot)} < \infty\)) implies \(f \in L^{p(\cdot)}(\mathbb{R}^N)\), and hence we do not need “such that \(f \in L^{p(\cdot)}(\mathbb{R}^N)\)”.

Thus the proof of Proposition 1 is reduced to the verification of

**Proposition 1'.** Let \(1 < q < \infty\). Under the assumptions on \(k\) in Proposition 1, for \(0 < R < R_0/2\),
\[
\int_{\mathbb{R}^N} (k_R \ast \mu)^q w \, dx \leq C \int_{\mathbb{R}^N} (M_{k,R} \mu)^q w \, dx
\]
for all \(A_1\)-weights \(w\), where \(C\) depends only on \(N\), \(q\), \(C_d\), \(C_k\), \(R\) and the \(A_1\)-constant of \(w\).

In the case \(k(x) = G_\alpha\), this proposition is given in [T]. For general kernels \(k\), we can prove this proposition by combining the arguments given in [T] and [AE, Part II]. Since our setting is different from either of them, we here give details of a proof.

First we recall some properties of \(A_1\)-weights \(w\). \(w\) is, by definition, a nonnegative locally integrable function on \(\mathbb{R}^N\) such that
\[
\int_B w(x) \, dx \leq A_1 |B| \text{ess inf}_B w
\]
for every ball (or cube) \(B\). The constant \(A_1\) is called the \(A_1\)-constant of \(w\). For a measurable set \(E\) in \(\mathbb{R}^N\), we write \(w(E) = \int_E w(x) \, dx\). An \(A_1\)-weight satisfies the \(A_\infty\)-condition:
\[
w(E) \leq C_w \left(\frac{|E|}{|Q|}\right)^{\sigma} w(Q) \quad (4.1)
\]
for every cube \(Q\) and every measurable subset \(E\) of \(Q\), where \(C_w > 0\) and \(\sigma > 0\) are constants depending only on \(N\) and the \(A_1\)-constant of \(w\) (see, e.g., [T; Theorem 1.2.9] or [HKM; Chap.15]).

The following is the key lemma (cf. [T; Lemma 3.1.3] and [AE; Lemma 4.3.2]):

**Lemma 2.** Suppose \(k(r)\) satisfies (k.1), (k.2) and (k.4). Let \(0 < R < R_0/2\) and \(w\) be an \(A_1\)-weight. Set \(a = 4C_d^2\). Then for every \(\eta > 0\) there exists \(\varepsilon \in (0,1]\), depending only on \(N\), the \(A_1\)-constant of \(w\), \(R\), \(C_d\), \(C_k\) and \(\eta\), such that
\[
w(\{x ; (k_R \ast \mu)(x) > a\lambda\})
\leq \eta w(\{x ; (k_R \ast \mu)(x) > \lambda\}) + w(\{x ; (M_{k,R} \mu)(x) > \varepsilon \lambda\})
\]
for all \(\lambda > 0\).

**Proof.** For \(\lambda > 0\), let
\[
E_\lambda = \{x ; (k_R \ast \mu)(x) > \lambda\}.
\]
It is an open set, since $k_R \ast \mu$ is lower-semicontinuous. Let $\{Q_j\}$ be the Whitney decomposition of $E_\lambda$ into closed dyadic cubes; namely, the interiors of $Q_j$ and $Q_j'$ are disjoint for $j \neq j'$, $E_\lambda = \bigcup_j Q_j$ and

$$\text{diam } Q_j \leq \text{dist}(Q_j, E_\lambda^c) \leq 4 \text{ diam } Q_j$$

for each $j$. If diam $Q_j > R/8$, then subdivide it into dyadic cubes with diameter $\leq R/8$ but $> R/16$. We denote this modified decomposition by $\{Q_j\}$ again.

Let $Q \in \{Q_j\}$, $d = \text{diam } Q$ and let $B = B(x_Q, 6d)$, where $x_Q$ be the center of $Q$. Note that $8d \leq R$. Let $\mu_1 = \mu|_B$ and $\mu_2 = \mu - \mu_1$. For every $x \in Q$, $B \subset B(x, 7d)$. Hence,

$$\int_Q (k_R \ast \mu_1)(\xi) \, d\xi = \int_Q \left( \int_B k_R(\xi - y) \, d\mu(y) \right) \, d\xi = \int_B \left( \int_Q k_R(\xi - y) \, d\xi \right) \, d\mu(y)$$

$$\leq \int_{B(x,7d)} \left( \int_{B(0,7d)} k(\xi) \, d\xi \right) \, d\mu(y)$$

$$= C(N) \left( \int_0^{7d} k(t) t^{N-1} \, dt \right) \mu(B(x, 7d))$$

$$\leq C_1(N, C_k) \cdot |Q| \cdot (M_{k,R} \mu)(x) \quad \text{(by (k.4))}$$

Let an $A_1$-weight $w$ and $\eta > 0$ be given. Then, by (4.1), we can find $\epsilon \in (0, 1]$ depending only on $\eta$, $N$, $C_d$, $C_k$ and the $A_1$-constant of $w$ such that if $E \subset Q$ and $|E| \leq C_1(N, C_k)(2\epsilon/a)|Q|$ then $w(E) \leq \eta w(Q)$. If there exists $x \in Q$ such that $(M_{k,R} \mu)(x) \leq \epsilon \lambda$, then the above inequalities imply

$$\left| \{\xi \in Q; (k_R \ast \mu_1)(\xi) > \frac{a}{2} \lambda \} \right|$$

$$\leq \frac{2}{a \lambda} \int_Q (k_R \ast \mu_1)(\xi) \, d\xi \leq C_1(N, C_k)(2\epsilon/a)|Q|,$$

so that

$$w(\{\xi \in Q; (k_R \ast \mu_1)(\xi) > \frac{a}{2} \lambda \}) \leq \eta w(Q).$$

Thus,

$$w(\{x \in Q; (k_R \ast \mu_1)(x) > \frac{a}{2} \lambda, (M_{k,R} \mu)(x) \leq \epsilon \lambda\}) \leq \eta w(Q) \quad (4.2).$$

Next, we show

$$\{x \in Q; (k_R \ast \mu)(x) > a \lambda, (M_{k,R} \mu)(x) \leq \epsilon \lambda\}$$

$$\subset \{x \in Q; (k_R \ast \mu_1)(x) > \frac{a}{2} \lambda, (M_{k,R} \mu)(x) \leq \epsilon \lambda\} \quad (4.3)$$

If $Q$ is one of undivided Whitney cubes, then $\text{dist}(Q, E_\lambda^c) \leq 4d$, so that $B \cap E_\lambda^c \neq \emptyset$. Let $x' \in B \cap E_\lambda^c$. Note that $d \leq R/8$, so that $12d < R_0$. If $x \in Q$ and $y \in B^c$, then
$|x-x'| \leq 7d$ and $|x-y| \geq 5d$, so that $|x-y| \geq (5/12)|x'-y|$. Hence, if $x \in Q$ and $(M_{k,R} \mu)(x) \leq \epsilon \lambda$, then

$$(k_{R} * \mu_{2})(x) = \int_{B(x,R)} k(x-y) \, d\mu_{2}(y) \leq \int_{B(x,R)} k((5/12)|x'-y|) \, d\mu_{2}(y)$$

$$\leq C_{d}^{2} \int_{B(x,R)} k'(x-y) \, d\mu_{2}(y)$$

$$\leq C_{d}^{2} \int_{B(x',R)} k(x'-y) \, d\mu(y) + C_{d}^{2} \int_{B(x,R) \setminus B(x',R)} k(x'-y) \, d\mu(y)$$

$$\leq C_{d}^{2} (k_{R} * \mu)(x) + C_{d}^{2} k(R) \mu(B(x,R))$$

$$\leq C_{d}^{2} (1 + \epsilon) \lambda \leq 2C_{d}^{2} \lambda \leq \frac{\alpha}{2} \lambda.$$ 

Thus we have (4.3) in this case.

Next let $Q$ be one of divided cubes. Recall that $R/16 < d \leq R/8$. If $x \in Q$ and $y \in B^{c}$, then $|y-x| \geq 5d > (5/16)R > R/4$. Hence,

$$(k_{R} * \mu_{2})(x) \leq \int_{\{R/4 < |y-x| < R\}} k(x-y) \, d\mu(y)$$

$$\leq k(R/4) \mu(B(x,R)) \leq C_{d}^{2} k(R) \mu(B(x,R)) \leq \frac{\alpha}{2} (M_{k,R} \mu)(x).$$

Hence, if $(M_{k,R} \mu)(x) \leq \epsilon \lambda$, then

$$(k_{R} * \mu_{2})(x) \leq \frac{\alpha}{2} \epsilon \lambda \leq \frac{\alpha}{2} \lambda,$$

which implies (4.3).

Now, from (4.2) and (4.3) we see that

$$w(\{x \in Q; (k_{R} * \mu)(x) > a \lambda\})$$

$$\leq \eta w(Q) + w(\{x \in Q; (M_{k,R} \mu)(x) > \epsilon \lambda\}).$$

for every $Q \in \{Q_{j}\}$. Summing up over all $Q$, we obtain Lemma 2.

**Proof** of Proposition 1': Let $a$ be as in the above lemma and $E_{\lambda}$ be as in the above proof, i.e., $E_{\lambda} = \{x; (k_{R} * \mu)(x) > \lambda\}$ ($\lambda > 0$). First assume that $\mu$ has compact support. Then, $k_{R} * \mu$ has compact support, too, and hence $\lambda \mapsto w(E_{\lambda})$ is a bounded function on $(0, \infty)$. Applying Lemma 2 with $\eta = a^{-q}/2$, we have, for any $L > 0$,

$$\int_{0}^{aL} w(E_{\lambda}) \lambda^{q-1} \, d\lambda = a^{q} \int_{0}^{L} w(E_{a\lambda}) \lambda^{q-1} \, d\lambda$$

$$\leq \frac{1}{2} \int_{0}^{L} w(E_{\lambda}) \lambda^{q-1} \, d\lambda + a^{q} \int_{0}^{L} w(\{x; (M_{k,R} \mu)(x) > \epsilon \lambda\}) \lambda^{q-1} \, d\lambda$$

with $\epsilon > 0$ depending only on $N$, the $A_{1}$-constant of $w$, $R$, $C_{d}$ and $C_{k}$. Hence,

$$\int_{0}^{aL} w(E_{\lambda}) \lambda^{q-1} \, d\lambda \leq 2a^{q} \epsilon^{-q} \int_{0}^{L} w(\{x; (M_{k,R} \mu)(x) > \lambda\}) \lambda^{q-1} \, d\lambda.$$
Now, letting $L 	o \infty$, we have
\[
\int_{\mathbb{R}^N} (k_R * \mu)^q w \, dx \leq 2a^q \epsilon^{-q} \int_{\mathbb{R}^N} (M_{k,R} \mu)^q w \, dx.
\]

If $\mu$ does not have compact support, let $\mu_m = \chi_{B(0,m)} \mu$ and apply the above result to $\mu_m$, and then let $m \to \infty$. Since $k_R * \mu_m \uparrow k_R * \mu$, the required result follows by the monotone convergence theorem.

To treat $k_R * \mu$, we prepare another lemma. For nonnegative measure $\mu$ on $\mathbb{R}^N$ and $R > 0$, let
\[
\overline{M}_R \mu(x) = \sup_{r \geq R} r^{-N} \mu(B(x, r)).
\]

Lemma 3. If $k(R) > 0$, then
\[
\overline{M}_R \mu \leq C(N, R, k(R)) \mathcal{M}(M_{k,R} \mu),
\]
where, $\mathcal{M}(f)$ denotes the Hardy-Littlewood maximal function of $f$.

**Proof.** Fix $x \in \mathbb{R}^N$ and let $r \geq R > 0$. We can find a finite number of $y_j \in B(x, r)$ such that
\[
B(x, r) \subset \bigcup_j B(y_j, R/2) \quad \text{and} \quad \sum_j \chi_{B(y_j, R/2)} \leq A(N) < \infty.
\]

If $y \in B(y_j, R/2)$, then $B(y_j, R/2) \subset B(y, R)$, so that
\[
\mu(B(y_j, R/2)) \leq \mu(B(y, R)) \leq \frac{(M_{k,R} \mu)(y)}{k(R)}.
\]

Since $B(y_j, R/2) \subset B(x, r + R/2) \subset B(x, 2r)$,
\[
\mu(B(x, r)) \leq \sum_j \mu(B(y_j, R/2)) \leq \frac{1}{k(R)|B(0, R/2)|} \sum_j \int_{B(y_j, R/2)} (M_{k,R} \mu)(y) \, dy \leq \frac{2^N A(N)}{k(R)|B(0, R)|} \int_{B(x, 2r)} (M_{k,R} \mu)(y) \, dy \leq \frac{4^N A(N)}{k(R) R^N} \mathcal{M}(M_{k,R} \mu)(x),
\]
so that
\[
r^{-N} \mu(B(x, r)) \leq C(N, R, k(R)) \mathcal{M}(M_{k,R} \mu)(x)
\]
for $r \geq R$. Thus, we obtain the required estimate.

**Proposition 2.** Suppose $k(r)$ satisfies (k.1), (k.2) and (k.3), and suppose $p(x)$ satisfies (P1), (P2) and (P3). Then, for $0 < R < R_0$,
\[
\|k_R * \mu\|_{p(\cdot)} \leq C\|M_{k,R} \mu\|_{p(\cdot)}
\]
with a constant $C > 0$ depending only on $N, R, k(R), C_{th}, C_{\infty}, p^{+}, p^{-}$ and

$$K := \int_{0}^{\infty} k(r)r^{N-1} dt.$$

Proof.

$$(\tilde{k}_{R} * \mu)(x) = \int_{R^{N} \setminus B(x,R)} k(x-y) d\mu(y)$$

$$= \int_{[R,\infty)} k(r) d[\mu(B(x,\cdot))](r)$$

$$\leq \limsup_{r \to \infty} k(r)\mu(B(x,r)) + \int_{[R,\infty)} \mu(B(x,r)) d(-k)(r).$$

Note that (k.3) implies that $k(r) \leq r^{-N}$ for $r \geq r_0$. Thus, if $r > \max(r_0, R)$, we have

$$k(r)\mu(B(x,r)) \leq r^{N}k(r)(\overline{M}_{R}\mu)(x) \leq (\overline{M}_{R}\mu)(x).$$

Hence,

$$\limsup_{r \to \infty} k(r)\mu(B(x,r)) \leq (\overline{M}_{R}\mu)(x).$$

On the other hand,

$$\int_{[R,\infty)} \mu(B(x,r)) d(-k)(r) \leq \left(\int_{[R,\infty)} r^{N} d(-k)(r)\right) (\overline{M}_{R}\mu)(x)$$

$$\leq \left(R^{N}k(R) + N \int_{R}^{\infty} k(r)r^{N-1} dr\right) (\overline{M}_{R}\mu)(x)$$

$$\leq (R^{N}k(R) + NK)(\overline{M}_{R}\mu)(x).$$

Hence

$$(\tilde{k}_{R} * \mu)(x) \leq C(N, R, k(R), K)(\overline{M}_{R}\mu)(x).$$

Thus, by Lemma 2, we have

$$(\tilde{k}_{R} * \mu)(x) \leq CM(M_{k,R}\mu)(x)$$

with a constant $C = C(N, R, k(R), K) > 0$, which implies

$$\|\tilde{k}_{R} * \mu\|_{p(\cdot)} \leq C\|M(M_{k,R}\mu)\|_{p(\cdot)}$$

with $C = C(N, R, k(R), K, p^{+}) > 0$. Now, under our assumptions on $p(\cdot)$, we know (see [CFN]) that

$$\| \mathcal{M}(f) \|_{p(\cdot)} \leq C\|f\|_{p(\cdot)},$$

and hence we obtain the required estimate.

Combining Propositions 1 and 2, we obtain Theorem 3.

From Theorems 1, 2 and 3, we derive
Corollary 1. Suppose $p(\cdot)$ satisfies (P1), (P2) and (P3), and $k(r)$ satisfies (k.1), (k.2), (k.3) and (k.4). Then, for nonnegative measures $\mu$ in $\mathbb{R}^N$ with $\mu(\mathbb{R}^N) < \infty$, 

$$k \ast \mu \in L^{p(\cdot)}(\mathbb{R}^N) \text{ if and only if } \int W_{k,p(\cdot)}^{\mu}(x,R) \, d\mu(x) < \infty.$$ 

It is known (see [GHN]) that if $p(\cdot)$ satisfies (P1), (P2) and (P3), then 

$$W^{m,p(\cdot)}(\mathbb{R}^N) = \{ u = G_m \ast f ; f \in L^{p(\cdot)}(\mathbb{R}^N) \}$$

for $m \in \mathbb{N}$. Thus we can state

Corollary 2. If $p(\cdot)$ satisfies (P1), (P2) and (P3), then for nonnegative measures $\mu$ on $\mathbb{R}^N$ with $\mu(\mathbb{R}^N) < \infty$, 

$$\mu \in (W^{m,p(\cdot)}(\mathbb{R}^N))^* \text{ if and only if } \int W_{m,p(\cdot)}^{\mu}(x,R) \, d\mu(x) < \infty$$ 

for $m \in \mathbb{N}$ with $0 < m < N$.

References


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