数理解析研究所講究録
第 1669 巻 2009 年 58-67

A characterization of conjugate functions on parabolic Bergman spaces

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1. Introduction

Let H be the upper half space of $\mathbb{R}^{n+1}$ $(n\geq 1)$, that is, $H=\{X=(x,t)\ ;\ x\in \mathbb{R}^{n},\ t>0\}$. For $0<\alpha\leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)}:=\frac{\partial}{\partial t}+(-\Delta_{x})^{\alpha},$$

where $\Delta_{x}:=\frac{\partial^{2}}{\partial x_{1}^{2}}+\cdots+\frac{\partial^{2}}{\partial x_{n}^{2}}$ is the Laplacian on the x-space $\mathbb{R}^{n}$. A real-valued continuous function u on H is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u=0$ in the sense of distributions. (The explicit definition of the $L^{(\alpha)}$-harmonic function is described in section 3.) For $\lambda>-1$ and $1\leq p<\infty$, the α-parabolic Bergman space $\boldsymbol{b}_{\alpha}^{p}(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with

$$\| u \|_{L^{p}(\lambda)}:=\left(\int_{H}|u(x,t)|^{p}t^{\lambda}dV(x,t)\right)^{1/p}<\infty,$$

where V is the Lebesgue volume measure on H and $L^{p}(\lambda):=L^{p}(H,t^{\lambda}dV)$. In particular, we may write $L^{p}=L^{p}(0)$ and $\boldsymbol{b}_{\alpha}^{p}=\boldsymbol{b}_{\alpha}^{p}(0)$, respectively.

Our aim of this paper is to study conjugate systems on α-parabolic Bergman spaces. The α-parabolic Bergman spaces $\boldsymbol{b}_{\alpha}^{p}$ were introduced and studied by Nishio, Shimomura, and Suzuki [7]. It was shown in [7] that $\boldsymbol{b}_{1/2}^{p}$ coincide with the usual harmonic Bergman spaces of Ramey and Yi [11]. Accordingly, usual harmonic Bergman spaces are the classes of L^{p}-solutions of the parabolic equation $L^{(\alpha)}u=0$ with $\alpha=1/2$. In [12], the Cauchy-Riemann equations on a region of the two-dimensional Euclidean space are extended to higher dimensions, and properties of systems of conjugate harmonic functions on Hardy spaces were studied (see also [3]). In the theory of harmonic Bergman spaces, properties of conjugate functions were also studied, and as an application, estimates of tangential derivative norms of harmonic Bergman functions were given (see section 6 in [11]). On the other hand, Yamada [13] studied conjugate functions of parabolic Bergman functions. However, the suitable notion of conjugacy were not extended to α-parabolic Bergman spaces. In this paper, we introduce a suitable extension of conjugacy to α-parabolic Bergman spaces and study their properties. We also give estimates of tangential derivative norms of α-parabolic Bergman functions.

Now, we introduce the extension of conjugacy to α-parabolic Bergman spaces. Let $\partial_{j}=\partial/\partial x_{j}$ $(1\leq j\leq n)$ and $\partial_{t}=\partial/\partial t$. Let $C(\Omega)$ be the set of all real-valued continuous functions on a region Ω, and for a positive integer k, $C^{k}(\Omega)\subset C(\Omega)$ denotes the set of all k times continuously differentiable functions on Ω, and put $C^{\infty}(\Omega) = \cap_k C^k(\Omega)$. Furthermore, for a real number κ, let $\mathcal{D}_t^{\kappa} = (-\partial_t)^{\kappa}$ be the fractional differential operator with respect to t. (The definition of the fractional differential operator and the fundamental properties of fractional calculus for α-parabolic Bergman functions are described in section 2.)

DEFINITION 1. For a function $u \in b_\alpha^p(\lambda)$, we shall say that a vector-valued function $V = (v_1, \ldots, v_n)$ on H is *an α-parabolic conjugate function of u* if $v_j \in C^1(H)$ and V satisfies the equations

$$\text{(C.1)} \qquad \nabla_x u = -\mathcal{D}_t V, \qquad \nabla_x v_j = \partial_j V \ (1 \leq j \leq n),$$

and

$$\text{(C.2)} \qquad \mathcal{D}_t^{\frac{1}{\alpha}-1} u = \nabla_x \cdot V,$$

where $\nabla_x = (\partial_1, \cdots, \partial_n)$ and $\nabla_x \cdot V$ is the divergence of V.

We remark that the fractional derivative $\mathcal{D}_t^{\frac{1}{\alpha}-1} u$ is well defined whenever $u \in b_\alpha^p(\lambda)$ with $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$ (see section 2). Our formulation of the extension of conjugacy is based on the Cauchy-Riemann equations $u_x = v_t$ and $-u_t = v_x$ on a region of the two-dimensional Euclidean space. Evidently, when $\alpha = 1/2$, the equations (C.1) and (C.2) coincide with the generalized Cauchy-Riemann equations for harmonic functions in [12];

$$\text{(1.1)} \qquad \partial_j u = \partial_t v_j, \qquad \partial_k v_j = \partial_j v_k, \qquad 1 \leq j, k \leq n,$$

and

$$\text{(1.2)} \qquad \partial_t u + \sum_{j=1}^{n} \partial_j v_j = 0.$$

Particularly, an $(n+1)$-tuple $(v_1, \ldots, v_n, u)$ which satisfies (1.1) and (1.2) is said to be a system of conjugate harmonic functions on H. We present results of Ramey and Yi [11] concerning with conjugate functions of harmonic Bergman functions.

THEOREM A. (Theorem 6.1 of [11]) *Let $1 \leq p < \infty$ and $u \in b_{1/2}^p$. Then, there exists a unique $1/2$-parabolic conjugate function $V = (v_1, \ldots, v_n)$ of u such that $v_j \in b_{1/2}^p$. Also, there exists a constant $C = C(n,p) > 0$ independent of u such that*

$$C^{-1} \| u \|_{L^p} \leq \| \, |V| \, \|_{L^p} \leq C \| u \|_{L^p},$$

where $|V| := \{v_1^2 + \cdots + v_n^2\}^{1/2}$.

For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The following theorem gives estimates of tangential derivative norms of harmonic Bergman functions.

THEOREM B. (Theorem 6.2 of [11]) *Let $1 \leq p < \infty$ and $u \in b_{1/2}^p$. Then, for each $m \in \mathbb{N}_0$,* *there exists a constant* $C = C(n,p,m) > 0$ *independent of* u *such that*

$$C^{-1} \| u \|_{L^p} \leq \sum_{|\gamma|=m} \| t^m \partial_x^\gamma u \|_{L^p} \leq C \| u \|_{L^p} .$$

We describe the main results of this paper. We remark that the condition $p\left(\frac{1}{2\alpha}-1\right)+\lambda > -1$ in Theorem 1 below holds for all $1 \leq p < \infty$ and $\lambda > -1$ whenever $0 < \alpha \leq 1/2$.

THEOREM 1. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, *and* $u \in \boldsymbol{b}_\alpha^p(\lambda)$. *If* α, p, *and* λ *satisfy the condition* $\eta = p\left(\frac{1}{2\alpha}-1\right)+\lambda > -1$, *then there exists a unique* α*-parabolic conjugate function* $V = (v_1, \ldots, v_n)$ *of* u *such that* $v_j \in \boldsymbol{b}_\alpha^p(\eta)$. *Also, there exists a constant* $C = C(n,p,\alpha,\lambda) > 0$ *independent of* u *such that*

$$C^{-1} \| u \|_{L^p(\lambda)} \leq \| \, |V| \, \|_{L^p(\eta)} \leq C \| u \|_{L^p(\lambda)} . \tag{1.3}$$

We remark that similar statements in Theorem 1 can not hold for the case $\eta = p\left(\frac{1}{2\alpha}-1\right)+\lambda \leq -1$. In fact, we can show that $\boldsymbol{b}_\alpha^p(\lambda) = \{0\}$ when $\lambda \leq 1$. We do not know whether Theorem A is extended to the full range $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. However, we can give estimates of tangential derivative norms of $\boldsymbol{b}_\alpha^p(\lambda)$-functions.

THEOREM 2. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, *and* $u \in \boldsymbol{b}_\alpha^p(\lambda)$. *Then, for each* $m \in \mathbb{N}_0$, *there exists a constant* $C = C(n,p,\alpha,\lambda,m) > 0$ *independent of* u *such that*

$$C^{-1} \| u \|_{L^p(\lambda)} \leq \sum_{|\gamma|=m} \| t^{\frac{m}{2\alpha}} \partial_x^\gamma u \|_{L^p(\lambda)} \leq C \| u \|_{L^p(\lambda)} . \tag{1.4}$$

This paper is constructed as follows. In section 2, we describe properties of fractional calculus on $\boldsymbol{b}_\alpha^p(\lambda)$. In section 3, we define integral operators induced by the fundamental solution of the parabolic operator $L^{(\alpha)}$ and investigate their properties, which are useful for studying α-parabolic conjugate functions. In section 4, we present more properties of α-parabolic conjugate functions.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Fractional calculus on $\boldsymbol{b}_\alpha^p(\lambda)$

In order to extend conjugacy to α-parabolic Bergman spaces, we need fractional calculus on $\boldsymbol{b}_\alpha^p(\lambda)$. First, we describe fractional differential operators for functions on $\mathbb{R}_+ = (0,\infty)$. For a real number $\kappa > 0$, let

$$\mathcal{FC}^{-\kappa} := \{\varphi \in C(\mathbb{R}_+) \, ; \, \exists\varepsilon > 0, \; \exists C > 0 \text{ s.t. } |\varphi(t)| \leq Ct^{-\kappa-\varepsilon}, \; \forall t \in \mathbb{R}_+\}.$$

For a function $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral $\mathcal{D}_t^{-\kappa}\varphi$ of φ by

$$\mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1}\varphi(\tau+t)d\tau = \frac{1}{\Gamma(\kappa)} \int_t^\infty (\tau-t)^{\kappa-1}\varphi(\tau)d\tau, \quad t \in \mathbb{R}_+, \tag{2.1}$$

where Γ is the gamma function. Moreover, let

$$\mathcal{F}C^{\kappa} := \{\varphi \, ; \, d_t^{\lceil\kappa\rceil}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)}\},$$

where $d_t = d/dt$, $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ, and we will write $\mathcal{F}C^0 := C(\mathbb{R}_+)$. We can also define the fractional derivative $\mathcal{D}_t^{\kappa}\varphi$ of $\varphi \in \mathcal{F}C^{\kappa}$ by

$$\mathcal{D}_t^{\kappa}\varphi(t) := \mathcal{D}_t^{-(\lceil\kappa\rceil-\kappa)}\left((-d_t)^{\lceil\kappa\rceil}\varphi\right)(t), \quad t \in \mathbb{R}_+. \tag{2.2}$$

In particular, we will write $\mathcal{D}_t^0\varphi = \varphi$. For a real number κ, we may call both (2.1) and (2.2) *the fractional derivatives of* φ *with order* κ. And, we call $\mathcal{D}_t^{\kappa}$ *the fractional differential operator with order* κ. Some basic properties of the fractional differential operators are the following.

LEMMA 2.1. (Proposition 2.1 of [4]) *For real numbers $\kappa, \nu > 0$, the following statements hold.*

(1) If $\varphi \in \mathcal{F}C^{-\kappa}$, then $\mathcal{D}_t^{-\kappa}\varphi \in C(\mathbb{R}_+)$.

(2) If $\varphi \in \mathcal{F}C^{-\kappa-\nu}$, then $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\kappa-\nu}\varphi$.

(3) If $d_t^k\varphi \in \mathcal{F}C^{-\nu}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$ and $d_t^{\lceil\kappa\rceil}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)-\nu}$, then $\mathcal{D}_t^{\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\nu}\mathcal{D}_t^{\kappa}\varphi = \mathcal{D}_t^{\kappa-\nu}\varphi$.

(4) If $d_t^{k+\lceil\nu\rceil}\varphi \in \mathcal{F}C^{-(\lceil\nu\rceil-\nu)}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$, $d_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)}$ for all integers $0 \le \ell \le \lceil\nu\rceil - 1$, and $d_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, then $\mathcal{D}_t^{\kappa}\mathcal{D}_t^{\nu}\varphi = \mathcal{D}_t^{\kappa+\nu}\varphi$.

Next, we also describe some basic results concerning with the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n}\displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x\cdot\xi)\, d\xi & (t > 0) \\ 0 & (t \le 0), \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and $L^{(\alpha)}$-harmonic on H. We note that $W^{(\alpha)} \ge 0$ on H and $\int_{\mathbb{R}^n} W^{(\alpha)}(x,t)dx = 1$ for all $0 < t < \infty$. Furthermore, $W^{(\alpha)} \in C^{\infty}(H)$. Let $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}_0^n$ be a multi-index and $k \in \mathbb{N}_0$. The following estimate is Lemma 1 of [9]: there exists a constant $C = C(n, \alpha, \gamma, k) > 0$ such that

$$|\partial_x^{\gamma}\partial_t^k W^{(\alpha)}(x,t)| \le C(t + |x|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha}+k\right)} \tag{2.3}$$

for all $(x,t) \in H$. In particular, by (2.3), we note that for each $x \in \mathbb{R}^n$, the function $\varphi(\,\cdot\,) = W^{(\alpha)}(x,\,\cdot\,)$ belongs to $\mathcal{F}C^{\kappa}$ for $\kappa > -\frac{n}{2\alpha}$. The statements in the following lemma are consequences of [4].

LEMMA 2.2. (Theorem 3.1 of [4]) *Let $0 < \alpha \le 1$, $\gamma \in \mathbb{N}_0^n$ be a multi-index, and κ be a real number such that $\kappa > -\frac{n}{2\alpha}$. Then, the following statements hold.*

(1) The derivatives $\partial_x^{\gamma}\mathcal{D}_t^{\kappa}W^{(\alpha)}(x,t)$ and $\mathcal{D}_t^{\kappa}\partial_x^{\gamma}W^{(\alpha)}(x,t)$ are well defined, and

$$\partial_x^{\gamma}\mathcal{D}_t^{\kappa}W^{(\alpha)}(x,t) = \mathcal{D}_t^{\kappa}\partial_x^{\gamma}W^{(\alpha)}(x,t).$$

Furthermore, there exists a constant $C' = C'(n, \alpha, \gamma, \kappa) > 0$ *such that*

$$|\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x,t)| \le C'(t + |x|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha}+\kappa\right)}$$

for all $(x,t) \in H$.

(2) If a real number ν *satisfies the condition* $\nu + \kappa > -\frac{n}{2\alpha}$, *then the derivative* $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}$ *is well defined, and*

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x,t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x,t)$$

for all $(x,t) \in H$.

(3) The derivative $\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x,t)$ *is* $L^{(\alpha)}$*-harmonic on* H.

By the elementary calculation, we also give the following lemma. This lemma plays an important role for the study of conjugate functions on parabolic Bergman spaces.

LEMMA 2.3. *Let* $0 < \alpha \le 1$. *Then,*

$$\left(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x\right) W^{(\alpha)}(x,t) = 0$$

for all $(x,t) \in H$.

We present basic properties of fractional derivatives of $\boldsymbol{b}_\alpha^p(\lambda)$-functions. We begin with describing estimates of ordinary derivatives of $\boldsymbol{b}_\alpha^p(\lambda)$-functions. Let $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, $\gamma \in \mathbb{N}_0^n$ be a multi-index, and $k \in \mathbb{N}_0$. Then, it is known that $\boldsymbol{b}_\alpha^p(\lambda) \subset C^\infty(H)$ (see [13]) and the following estimate is given by Lemma 3.4 of [13]: there exists a constant $C = C(n, \alpha, p, \lambda, \gamma, k) > 0$ such that

$$|\partial_x^\gamma \partial_t^k u(x,t)| \le C t^{-\left(\frac{|\gamma|}{2\alpha}+k\right)-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}} \| u \|_{L^p(\lambda)} \tag{2.4}$$

for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $(x,t) \in H$. The estimate (2.4) implies that the point evaluation is a bounded linear functional on $\boldsymbol{b}_\alpha^p(\lambda)$. Furthermore, the estimate (2.4) also shows that a function $\varphi(\,\cdot\,) = u(x, \,\cdot\,)$ belongs to $\mathcal{FC}^\kappa$ for $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$, so we can define fractional derivatives of $\boldsymbol{b}_\alpha^p(\lambda)$-functions. Some properties of fractional derivatives of $\boldsymbol{b}_\alpha^p(\lambda)$-functions are given in the following.

LEMMA 2.4. (Proposition 4.1 of [4]) *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, $\gamma \in \mathbb{N}_0^n$ *be a multi-index, and* κ *be a real number such that* $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$. *If* $u \in \boldsymbol{b}_\alpha^p(\lambda)$, *then the following statements hold.*

(1) The derivatives $\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)$ *and* $\mathcal{D}_t^\kappa \partial_x^\gamma u(x,t)$ *are well defined, and*

$$\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) = \mathcal{D}_t^\kappa \partial_x^\gamma u(x,t).$$

Furthermore, there exists a constant $C = C(n, \alpha, p, \lambda, \gamma, \kappa) > 0$ *independent of* u *such that*

$$|\partial_x^\gamma \partial_t^\kappa u(x,t)| \le C t^{-\left(\frac{|\gamma|}{2\alpha}+\kappa\right)-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}} \| u \|_{L^p(\lambda)}$$

for all $(x,t) \in H$.

(2) If a real number ν *satisfies the condition* $\nu+\kappa > -\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}$, *then the derivative* $\mathcal{D}_t^{\nu}\partial_x^{\gamma}\mathcal{D}_t^{\kappa}u(x,t)$ *is well defined, and*

$$\mathcal{D}_t^{\nu}\partial_x^{\gamma}\mathcal{D}_t^{\kappa}u(x,t) = \partial_x^{\gamma}\mathcal{D}_t^{\nu+\kappa}u(x,t)$$

for all $(x,t) \in H$.

(3) The derivative $\partial_x^{\gamma}\mathcal{D}_t^{\kappa}u(x,t)$ *is* $L^{(\alpha)}$*-harmonic on* H.

For a real number $\lambda > -1$, let $c_\lambda = 2^{\lambda+1}/\Gamma(\lambda+1)$. The following lemma is also a consequence of [4], and (2.5) is the reproducing formula for $\boldsymbol{b}_\alpha^p(\lambda)$-functions.

LEMMA 2.5. (Theorem 5.2 of [4]) *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *Suppose that* ν *and* κ *are real numbers such that* $\nu > -\frac{\lambda+1}{p}$ *and* $\kappa > \frac{\lambda+1}{p}$. *Then,*

$$u(x,t) = c_{\nu+\kappa-1}\int_H \mathcal{D}_t^{\nu}u(y,s)\mathcal{D}_t^{\kappa}W^{(\alpha)}(x-y,t+s)s^{\nu+\kappa-1}dV(y,s) \tag{2.5}$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ *and* $(x,t) \in H$. *Furthermore, (2.5) also holds whenever* $p=1$ *and* $\kappa = \lambda+1$.

Finally, we present the following lemma. This lemma plays an important role for proving Theorem 2.

LEMMA 2.6. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, *and* $u \in \boldsymbol{b}_\alpha^p(\lambda)$. *Then,*

$$\left(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x\right)u(x,t) = 0$$

for all $(x,t) \in H$.

3. Integral operators induced by the fundamental solution

In this section, we define integral operators induced by the fundamental solution $W^{(\alpha)}$ and investigate their properties. These investigations are useful for studying α-parabolic conjugate functions of $\boldsymbol{b}_\alpha^p(\lambda)$-functions.

First, we recall the definition of $L^{(\alpha)}$-harmonic functions. (For details, see section 2 of [7].) We describe about the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C_c^\infty(H) \subset C(H)$ be the set of all infinitely differentiable functions on H with compact support. Then, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha\psi(x,t) := -C_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y|>\delta}\left(\psi(x+y,t)-\psi(x,t)\right)|y|^{-n-2\alpha}dy \tag{3.1}$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha\pi^{-n/2}\Gamma\left((n+2\alpha)/2\right)/\Gamma(\alpha) > 0$. Let $\widetilde{L}^{(\alpha)} := -\partial t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be

$L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $\int_H |u\widetilde{L}^{(\alpha)}\psi| dV < \infty$ and $\int_H u\widetilde{L}^{(\alpha)}\psi dV = 0$ for all $\psi \in C_c^\infty(H)$. By (3.1) and the compactness of supp(ψ) (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that

$$\mathrm{supp}(\widetilde{L}^{(\alpha)}\psi) \subset S = \mathbb{R}^n \times [t_1, t_2] \text{ and } |\widetilde{L}^{(\alpha)}\psi(x,t)| \leq C(1+|x|)^{-n-2\alpha} \text{ for } (x,t) \in S.$$

Hence, the condition $\int_H |u\widetilde{L}^{(\alpha)}\psi| dV < \infty$ for all $\psi \in C_c^\infty(H)$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dx dt < \infty.$$

Next, we define integral operators induced by the fundamental solution $W^{(\alpha)}$. Let $\gamma \in \mathbb{N}_0^n$ be a multi-index and $\kappa, \rho \in \mathbb{R}$ with $\kappa > -\frac{n}{2\alpha}$. Then, we define the integral operator $P_\alpha^{\gamma,\kappa,\rho}$ by

$$P_\alpha^{\gamma,\kappa,\rho} f(x,t) := \int_H f(y,s)\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x-y, t+s)s^\rho dV(y,s),$$

whenever the integral is well defined. Some properties of $P_\alpha^{\gamma,\kappa,\rho}$ are given in the following theorem.

THEOREM 3.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\sigma \in \mathbb{R}$. Suppose that a multi-index $\gamma \in \mathbb{N}_0^n$ and $\kappa, \rho \in \mathbb{R}$ with $\kappa > -\frac{n}{2\alpha}$ satisfy*

$$\sigma - \rho p < p - 1 < \left(\frac{|\gamma|}{2\alpha} + \kappa\right) p + \sigma - \rho p.$$

Then, for every $f \in L^p(\sigma)$, the following assertions hold.

(1) The function $P_\alpha^{\gamma,\kappa,\rho} f(x,t)$ is well defined for every $(x,t) \in H$ and there exists a constant $C > 0$ independent of f such that

$$\| P_\alpha^{\gamma,\kappa,\rho} f \|_{L^p(\eta)} \leq C \| f \|_{L^p(\sigma)},$$

where $\eta = \left(\frac{|\gamma|}{2\alpha} + \kappa - \rho - 1\right) p + \sigma$. Moreover, $P_\alpha^{\gamma,\kappa,\rho} f$ is $L^{(\alpha)}$-harmonic on H. Consequently, $P_\alpha^{\gamma,\kappa,\rho} f \in \boldsymbol{b}_\alpha^p(\eta)$.

(2) Furthermore, let $\beta \in \mathbb{N}_0^n$ be a multi-index and $\nu \in \mathbb{R}$. If ν satisfies

$$\nu + \kappa > -\frac{n}{2\alpha} \text{ and } p - 1 < \left(\frac{|\gamma|}{2\alpha} + \nu + \kappa\right) p + \sigma - \rho p,$$

then the derivative $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma,\kappa,\rho} f(x,t)$ is well defined for every $(x,t) \in H$ and $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma,\kappa,\rho} f = P_\alpha^{\beta+\gamma,\nu+\kappa,\rho} f$, that is,

$$\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma,\kappa,\rho} f(x,t) = \int_H f(y,s)\partial_x^{\beta+\gamma} \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x-y, t+s)s^\rho dV(y,s).$$

Consequently, put $\eta = \left(\frac{|\beta|+|\gamma|}{2\alpha} + \nu + \kappa - \rho - 1\right)p + \sigma$, *then there exists a constant* $C > 0$ *independent of* f *such that*

$$\| \partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma,\kappa,\rho} f \|_{L^p(\eta)} \leq C \| f \|_{L^p(\sigma)}$$

and $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma,\kappa,\rho} f \in \boldsymbol{b}_\alpha^p(\eta)$.

By the above theorem, we have the following corollary.

COROLLARY 3.2. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, *and* $\lambda > -1$. *Then, the following assertions hold.*

(1) If a real number κ *satisfies* $\kappa > \frac{\lambda+1}{p}$, *then the operator* $R_\alpha^{\kappa-1} = c_{\kappa-1} P_\alpha^{0,\kappa,\kappa-1}$ *is a bounded projection from* $L^p(\lambda)$ *onto* $\boldsymbol{b}_\alpha^p(\lambda)$.

(2) For a real number $\nu > -\frac{\lambda+1}{p}$, *there exists a constant* $C = C(n, p, \alpha, \lambda, \nu) > 0$ *such that*

$$C^{-1} \| u \|_{L^p(\lambda)} \leq \| t^\nu \mathcal{D}_t^\nu u \|_{L^p(\lambda)} \leq \sum_{|\gamma| < \nu + \frac{\lambda+1}{p}} \| t^{\frac{|\gamma|}{2\alpha} + \nu - |\gamma|} \partial_x^\gamma \mathcal{D}_t^{\nu - |\gamma|} u \|_{L^p(\lambda)} \leq C \| u \|_{L^p(\lambda)}$$

for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$, *where* $\gamma \in \mathbb{N}_0^n$ *denotes a multi-index.*

4. More properties of α-parabolic conjugate functions

In this section, we present more properties of α–parabolic conjugate functions. Given a harmonic function u on H, it is well known that a vector-valued function $V = (v_1, \ldots, v_n)$ on H with $v_j \in C^1(H)$ satisfies the equations (1.1) and (1.2) if and only if there exists a function $g \in C^2(H)$ such that

$$g \text{ is harmonic on } H \text{ and } \nabla_{(x,t)} g = (v_1, \ldots, v_n, u), \tag{4.1}$$

where $\nabla_{(x,t)} = (\partial_1, \ldots, \partial_n, \partial_t)$. The following theorem is a analogous result of (4.1) for our case.

THEOREM 4.1. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, *and* $u \in \boldsymbol{b}_\alpha^p(\lambda)$. *Then, a vector-valued function* $V = (v_1, \ldots, v_n)$ *on* H *is an* α*-parabolic conjugate function of* u *if and only if there exists a function* $g \in C^2(H) \cap \mathcal{F}C^{\frac{1}{\alpha}}$ *such that*

$$(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x) g = 0 \text{ on } H \text{ and } \nabla_{(x,t)} g = (v_1, \ldots, v_n, u).$$

Next, we give an inversion theorem, that is, for a vector-valued function $V = (v_1, \ldots, v_n)$ on H we construct a function $u \in \boldsymbol{b}_\alpha^p(\lambda)$ such that V is an α-parabolic conjugate function of u.

THEOREM 4.2. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, *and* $\eta > -1$. *Suppose that a vector-valued function* $V = (v_1, \ldots, v_n)$ *on* H *satisfies* $v_j \in \boldsymbol{b}_\alpha^p(\eta)$ *and* $\nabla_x v_j = \partial_j V$ *for all* $1 \leq j \leq n$. *If* α, p, *and* η *satisfy the condition* $\lambda = p(1 - \frac{1}{2\alpha}) + \eta > -1$, *then there exists a unique function* u

on H such that $u \in b^p_\alpha(\lambda)$ and V is an α-parabolic conjugate function of u. Also, there exists a constant $C = C(n,p,\alpha,\eta) > 0$ independent of V such that

$$C^{-1} \| \, |V| \, \|_{L^p(\eta)} \leq \| \, u \, \|_{L^p(\lambda)} \leq C \, \| \, |V| \, \|_{L^p(\eta)} \, .$$

We also have the following proposition.

PROPOSITION 4.3. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in b^p_\alpha(\lambda)$. Let $1 \leq j \leq n$ be fixed. Suppose that a vector-valued function $V = (v_1, \ldots, v_n)$ on H is an α-parabolic conjugate function of u. Then, $v_j \in \mathcal{F}C^{\frac{1}{\alpha}}$. Furthermore, if $v_k \in C^2(H)$ for all $1 \leq k \leq n$, then $(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x)v_j = 0$ on H.*

Finally, we present a decomposition theorem for α-parabolic conjugate functions. We begin with presenting the following lemma.

LEMMA 4.4. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in b^p_\alpha(\lambda)$. Suppose α, p, and λ satisfy the condition $\eta = p\left(\frac{1}{2\alpha} - 1\right) + \lambda > -1$. Then, for every α-parabolic conjugate function $U = (u_1, \ldots, u_n)$ of u, the function $\mathcal{D}_t^{-1}\mathcal{D}_t u_j$ on H is well defined and belongs to $b^p_\alpha(\eta)$ for all $1 \leq j \leq n$.*

The following theorem is a decomposition theorem for α–parabolic conjugate functions.

THEOREM 4.5. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in b^p_\alpha(\lambda)$. Suppose α, p, and λ satisfy the condition $\eta = p\left(\frac{1}{2\alpha} - 1\right) + \lambda > -1$. Then, every α-parabolic conjugate function $U = (u_1, \ldots, u_n)$ of u can be uniquely expressed in the form*

$$U(x,t) = V(x,t) + F(x), \qquad (x,t) \in H, \tag{4.2}$$

where $V = (v_1, \ldots, v_n)$ is the unique α-parabolic conjugate function of u with $v_j \in b^p_\alpha(\eta)$ in Theorem 1 and $F = (f_1, \ldots, f_n)$ is an n-tuple of harmonic functions on $\mathbb{R}^n$ with $\partial_k f_j = \partial_j f_k$, $1 \leq j,k \leq n$ and $\sum_{j=1}^n \partial_j f_j = 0$ (that is, $F = (f_1, \ldots, f_n)$ is a system of conjugate harmonic functions on $\mathbb{R}^n$, consequently each u_j belongs to $C^\infty(H)$). Conversely, every function U of the form (4.2) is an α-parabolic conjugate function of u.

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