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Spaces of maps from the closed Riemann surface into the 2-sphere

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1 Introduction.

For connected spaces X and Y, let $\mathrm{Map}(X,Y)$ (resp. $\mathrm{Map}^*(X,Y)$) denote the space consisting of all continuous (resp. based continuous) maps $f: X \to Y$ with compact-open topology. Let T_g denote the closed Riemann surface of genus g. Then for each integer $d \in \mathbb{Z} = \pi_0(\mathrm{Map}(T_g, S^2))$ we denote by $\mathrm{Map}_d(T_g, S^2)$ (resp. by $\mathrm{Map}_d^*(T_g, S^2)$) the corresponding path-component of $\mathrm{Map}(T_g, S^2)$ (resp. $\mathrm{Map}^*(T_g, S^2)$) consisting of all maps (resp. of base-point preserving maps) $f: T_g \to S^2$ of degree d. Similarly, we denote by $\mathrm{Hol}_d(T_g, S^2)$ the subspace of $\mathrm{Map}_d(T_g, S^2)$ of all holomorphic maps $f: T_g \to S^2$ of degree d, and by $\mathrm{Hol}_d^*(T_g, S^2)$ the corresponding subspace of $\mathrm{Map}_d^*(T_g, S^2)$ of all base-point preserving holomorphic maps of degree d. Note that $\mathrm{Hol}_d(T_g, S^2) = \emptyset$ if $d < 0$ and that any holomorphic map $f: T_g \to S^2$ of degree zero is a constant map. So in this paper we always assume that $d \geq 1$ and recall the following results.

Theorem 1.1 (L. Larmore and E. Thomas, [7]). (i) *If* $g = 0$, $T_0 = S^2$ *and there are isomorphisms*

$$\pi_1(\mathrm{Map}_d(S^2, S^2)) \cong \mathbb{Z}/2d, \quad \pi_1(\mathrm{Map}_d^*(S^2, S^2)) \cong \mathbb{Z}.$$

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(ii) *If $g \geq 1$, there are isomorphisms*

$$\begin{cases} \pi_1(\mathrm{Map}_d(T_g, S^2)) \cong \langle \alpha, e_j \ (1 \leq j \leq 2g) | [e_k, e_{k+g}] = \alpha^2, \alpha^{2d} = 1 \rangle, \\ \pi_1(\mathrm{Map}_d^*(T_g, S^2)) \cong \langle \alpha, e_j \ (1 \leq j \leq 2g) | [e_k, e_{k+g}] = \alpha^2 \rangle, \end{cases}$$

where $k = 1, 2, \cdots, g$, and $[x, y] = xyx^{-1}y^{-1}$. □

Theorem 1.2 (G. Segal, [9]). *The inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(T_g, S^2) \to \mathrm{Map}_d^*(T_g, S^2) \\ j_d : \mathrm{Hol}_d(T_g, S^2) \to \mathrm{Map}_d(T_g, S^2) \end{cases}$$

are homotopy equivalences up to dimension d if $g = 0$, and they are homology equivalences up to dimension $D(d; g) = d - 2g$ if $g \geq 1$. □

Theorem 1.3 (S. Kallel, [6]). *If $d > 2g$ and $g \geq 1$, the inclusion maps i_d and j_d induce isomorphisms*

$$\begin{cases} i_{d*} : \pi_1(\mathrm{Hol}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d^*(T_g, S^2)), \\ j_{d*} : \pi_1(\mathrm{Hol}_d(T_g, S^2)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d(T_g, S^2)). \end{cases} \quad \square$$

Remark. A map $f : X \to Y$ is called *a homotopy (resp. homology) equivalence up to dimension D* if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and epimorphism for $k = D$.

We expect that the inclusions i_d and j_d will be homotopy equivalences up to dimension $D(d; g)$ for $g \geq 1$. For example, Theorem 1.3 supports that this might be true, and it seems valuable to investigate the homotopy types of the universal coverings of $\mathrm{Hol}_d^*(T_g, S^2)$ and $\mathrm{Hol}_d(T_g, S^2)$.

The main purpose of this note is to announce the recent work of the author given in [13], in which we shall study the homotopy types of universal coverings of the above spaces. Let $\widetilde{X}$ denote the universal covering of a connected space X. Then we can state our results as follows.

Theorem 1.4 ([13]). *If $d \geq 1$, there is a homotopy equivalence*

$$\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}_d^*}(T_g, S^2) \xrightarrow{\simeq} \widetilde{\mathrm{Hol}_d}(T_g, S^2).$$

Corollary 1.5 (([1], [8], [12])). *If $g = 0$ and $d \geq 1$, there is a homotopy equivalence $\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}_d^*}(S^2, S^2) \xrightarrow{\simeq} \widetilde{\mathrm{Hol}_d}(S^2, S^2)$.* □

Let $i : \mathrm{Hol}_d^*(T_g, S^2) \to \mathrm{Hol}_d(T_g, S^2)$ be an inclusion map and let $ev : \mathrm{Hol}_d(T_g, S^2) \to S^2$ denote the evaluation map given by $ev(f) = f(t_0)$, where $t_0 \in T_g$ is the base-point of T_g. Then it is known that there is a evaluation fibration sequence (e.g. [6])

$$\mathrm{Hol}_d^*(T_g, S^2) \xrightarrow[\subset]{i} \mathrm{Hol}_d(T_g, S^2) \xrightarrow{ev} S^2.$$

Corollary 1.6 ([13]). *If $k \geq 2$ and $d \geq 1$, the above sequence induces a split short exact sequence*

$$0 \to \pi_k(\mathrm{Hol}_d^*(T_g, S^2)) \xrightarrow{i_*} \pi_k(\mathrm{Hol}_d(T_g, S^2)) \xrightarrow{ev_*} \pi_k(S^2) \to 0.$$

Theorem 1.7 ([13]). *Let $d \geq 1$ be an integer.*

(i) *There is a homotopy equivalence*

$$\tilde{\Phi} : S^3 \times \widetilde{\mathrm{Map}}^*(T_g, S^3) \xrightarrow{\simeq} \widetilde{\mathrm{Map}}_d(T_g, S^2)$$

and there is a fibration sequence (up to homotopy equivalence)

$$\Omega^2 S^3\langle 3\rangle \to \widetilde{\mathrm{Map}}^*(T_g, S^3) \to (\Omega S^3)^{2g},$$

where $S^3\langle 3\rangle$ denotes the 3-connective covering of S^3.

(ii) *For any $k \geq 2$, there is an isomorphism*

$$\pi_k(\widetilde{\mathrm{Map}}^*(T_g, S^3)) \cong \pi_{k+2}(S^3) \oplus \pi_{k+1}(S^3)^{\oplus 2g}.$$

2 The idea of the proofs.

Let X, Y and Z be connected spaces and let $f : X \to Y$ be a map. Define the map $f^\# : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ by $f^\#(g) = g \circ f$ for $g \in \mathrm{Map}(Y, Z)$. Similarly, we define the map $f^\# : \mathrm{Map}^*(Y, Z) \to \mathrm{Map}^*(X, Z)$ by the restriction. It is well known that there is a cofiber sequence

$$S^1 \xrightarrow{\varphi_g} \vee^{2g} S^1 \xrightarrow{i'} T_g \xrightarrow{q_g} S^2 \xrightarrow{\Sigma\varphi_g} \vee^{2g} S^2 \tag{1}$$

where $\pi_1(\vee^{2g} S^1)$ is the free group on $2g$ generators $\{a_j, b_j : 1 \leq j \leq g\}$ and $\varphi_g = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \in \pi_1(\vee^{2g} S^1)$.

Lemma 2.1. $q^{\#}_{g\,*} : \pi_k(\Omega^2_d S^2) \to \pi_k(\mathrm{Map}^*_d(T_g, S^2))$ *is a monomorphism for any* $k \geq 1$.

Proof. By using an easy diagram chasing we can show the assertion. □

Consider the commutative diagram of evaluation fibration sequences

$$
(2) \qquad
\begin{array}{ccccc}
\mathrm{Map}^*_d(S^2, S^2) & \xrightarrow[\subset]{i_T} & \mathrm{Map}_d(S^2, S^2) & \xrightarrow{ev_S} & S^2 \\
{\scriptstyle q^{\#}_g}\downarrow & & {\scriptstyle q^{\#}_g}\downarrow & & \| \\
\mathrm{Map}^*_d(T_g, S^2) & \xrightarrow[\subset]{i_S} & \mathrm{Map}_d(T_g, S^2) & \xrightarrow{ev_T} & S^2
\end{array}
$$

Lemma 2.2. $ev_{X*} : \pi_2(\mathrm{Map}_d(X, S^2)) \to \pi_2(S^2)$ *is trivial for* $X \in \{S^2, T_g\}$.

Proof. If $X = S^2$, we can easily show the assertion by using Whitehead's Theorem [11] concerning the boundary operator of the homotopy exact sequence induced from the evaluation fibartion. By using this with the diagram chasing of (2), we can also prove the assertion for $X = T_g$. □

Lemma 2.3. *The induced homomorphisms*

$$
\begin{cases}
i_{T*} : \pi_2(\mathrm{Map}^*_d(T_g, S^2)) \to \pi_2(\mathrm{Map}_d(T_g, S^2)) \\
i_* : \pi_2(\mathrm{Hol}^*_d(T_g, S^2)) \to \pi_2(\mathrm{Hol}_d(T_g, S^2))
\end{cases}
$$

are epimorphisms.

Proof. The proof follows from the diagram chasing and Lemma 2.2. □

If we identify $S^2 = \mathbb{C}\mathrm{P}^1$, the group $SU(2)$ acts on S^2 by the right matrix multiplication. By using this right matrix multiplication, define the map

$$
(3) \qquad \Phi : SU(2) \times \mathrm{Map}^*_d(T_g, S^2) \to \mathrm{Map}_d(T_g, S^2)
$$

by $(\Phi(A, f))(t) = f(t) \cdot A$ for $(A, f, t) \in SU(2) \times \mathrm{Map}^*_d(T_g, S^2) \times T_g$.

Since $\Phi(SU(2) \times \mathrm{Hol}^*_d(T_g, S^2)) \subset \mathrm{Hol}_d(T_g, S^2)$, we can define the map

$$
(4) \qquad \Phi_d : SU(2) \times \mathrm{Hol}^*_d(T_g, S^2) \to \mathrm{Hol}_d(T_g, S^2)
$$

by the restriction $\Phi_d = \Phi | SU(2) \times \mathrm{Hol}^*_d(T_g, S^2)$.

Theorem 2.4 ([13]). $\Phi_{d*} : \pi_k(SU(2)\times \mathrm{Hol}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_k(\mathrm{Hol}_d(T_g, S^2))$ *is an isomorphism for any* $k \geq 2$. □

Proof. The detail is omitted and see [13] in detail. □

Now we can prove Theorem 1.4 by using Theorem 2.4.

Proof of Theorem 1.4. Let

$$\begin{cases} \pi : \widetilde{\mathrm{Hol}_d}(T_g, S^2) \to \mathrm{Hol}_d(T_g, S^2) \\ \pi' : \widetilde{\mathrm{Hol}_d^*}(T_g, S^2) \to \mathrm{Hol}_d^*(T_g, S^2) \end{cases}$$

denote projection maps of the universal coverings. If we identify $SU(2) = S^3$, it is easy see that the universal covering of $SU(2) \times \mathrm{Hol}_d^*(T_g, S^2)$ is given by $1 \times \pi' : S^3 \times \widetilde{\mathrm{Hol}_d^*}(T_g, S^2) \to SU(2) \times \mathrm{Hol}_d^*(T_g, S^2)$. Then because π is a projection of the universal covering, there is a lifting $\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}_d^*}(T_g, S^2) \to \widetilde{\mathrm{Hol}_d}(T_g, S^2)$ such that the following diagram is commutative.

$$\begin{array}{ccc} S^3 \times \widetilde{\mathrm{Hol}_d^*}(T_g, S^2) & \xrightarrow{\tilde{\Phi}_d} & \widetilde{\mathrm{Hol}_d}(T_g, S^2) \\ {\scriptstyle 1\times\pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ SU(2) \times \mathrm{Hol}_d^*(T_g, S^2) & \xrightarrow{\Phi_d} & \mathrm{Hol}_d(T_g, S^2) \end{array}$$

Since $\pi_k(\Phi_d)$ is an isomorphism for any $k \geq 2$, an easy diagram chasing shows that $\pi_k(\tilde{\Phi}_d)$ is also an isomorphism for any $k \geq 2$. Because $S^3 \times \widetilde{\mathrm{Hol}_d^*}(T_g, S^2)$ and $\widetilde{\mathrm{Hol}_d}(T_g, S^2)$ are simply connected, $\tilde{\Phi}_d$ is a homotopy equivalence. □

By using the completely similar way as above, we can see that there is a fibration sequence (up to homotopy equivalence)

$$S^1 \longrightarrow SU(2) \times \mathrm{Map}_d^*(T_g, S^2) \xrightarrow{\Phi} \mathrm{Map}_d(T_g, S^2). \tag{5}$$

Theorem 2.5 ([13]). $\Phi_* : \pi_k(SU(2)\times \mathrm{Map}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_k(\mathrm{Map}_d(T_g, S^2))$ *is an isomorphism for any* $k \geq 2$.

Proof. The proof is analogous to that of Theorem 2.4. □

Corollary 2.6 ([13]). *There is a homotopy equivalence*

$$\widetilde{\mathrm{Map}}_d(T_g, S^2) \simeq S^3 \times \widetilde{\mathrm{Map}}_0^*(T_g, S^2). \quad \square$$

Lemma 2.7. *There is a homotopy fibration sequence*

$$\Omega^2 S^3\langle 3\rangle \to \widetilde{\mathrm{Map}}^*(T_g, S^3) \to (\Omega S^3)^{2g}.$$

Proof. This can be proved by using tedious diagram chasing and the detail is omitted. $\square$

Proof of Theorem 1.7. (i) By using Corollary 2.6 and Lemma 2.7, to prove (i) it is sufficient to show that there is a homotopy equivalence

$$\widetilde{\mathrm{Map}}^*(T_g, S^3) \simeq \widetilde{\mathrm{Map}}_0^*(T_g, S^2). \tag{6}$$

However, by using a similar manner as that of the previous Theorem we can prove the assertion (i) and the detail is omitted.

(ii) Assume that $k \geq 2$. Since there is a homotopy equivalence $\Sigma^k T_g \simeq \vee^{2g} S^{1+k} \vee S^{2+k}$ and S^3 is a Lie group, $\mathrm{Map}^*(T_g, S^3)$ is an H-space. Hence, the assertion (ii) easily follows. $\square$

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