Spaces of maps from the closed Riemann surface into the 2-sphere

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1 Introduction.

For connected spaces $X$ and $Y$, let $\text{Map}(X, Y)$ (resp. $\text{Map}^*(X, Y)$) denote the space consisting of all continuous (resp. based continuous) maps $f : X \to Y$ with compact-open topology. Let $T_g$ denote the closed Riemann surface of genus $g$. Then for each integer $d \in \mathbb{Z} = \pi_0(\text{Map}(T_g, S^2))$ we denote by $\text{Map}_d(T_g, S^2)$ (resp. by $\text{Map}_d^*(T_g, S^2)$) the corresponding path-component of $\text{Map}(T_g, S^2)$ (resp. $\text{Map}^*(T_g, S^2)$) consisting of all maps (resp. of base-point preserving maps) $f : T_g \to S^2$ of degree $d$. Similarly, we denote by $\text{Hol}_d(T_g, S^2)$ the subspace of $\text{Map}_d(T_g, S^2)$ of all holomorphic maps $f : T_g \to S^2$ of degree $d$, and by $\text{Hol}_d^*(T_g, S^2)$ the corresponding subspace of $\text{Map}_d^*(T_g, S^2)$ of all base-point preserving holomorphic maps of degree $d$. Note that $\text{Hol}_d(T_g, S^2) = \emptyset$ if $d < 0$ and that any holomorphic map $f : T_g \to S^2$ of degree zero is a constant map. So in this paper we always assume that $d \geq 1$ and recall the following results.

Theorem 1.1 (L. Larmore and E. Thomas, [7]). (i) If $g = 0$, $T_0 = S^2$ and there are isomorphisms

$$
\pi_1(\text{Map}_d(S^2, S^2)) \cong \mathbb{Z}/2d, \quad \pi_1(\text{Map}_d^*(S^2, S^2)) \cong \mathbb{Z}.
$$

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(ii) If $g \geq 1$, there are isomorphisms
\[
\begin{align*}
\pi_1(\text{Map}_d(T_g, S^2)) &\cong \langle \alpha, e_j \ (1 \leq j \leq 2g) \ | \ [e_k, e_{k+g}] = \alpha^2, \alpha^{2d} = 1 \rangle, \\
\pi_1(\text{Map}_d^*(T_g, S^2)) &\cong \langle \alpha, e_j \ (1 \leq j \leq 2g) \ | \ [e_k, e_{k+g}] = \alpha^2 \rangle,
\end{align*}
\]
where $k = 1, 2, \ldots, g$, and $[x, y] = x y x^{-1} y^{-1}$. □

**Theorem 1.2** (G. Segal, [9]). The inclusion maps
\[
\begin{align*}
i_d & : \text{Hol}_d^*(T_g, S^2) \to \text{Map}_d^*(T_g, S^2) \\
.j_d & : \text{Hol}_d(T_g, S^2) \to \text{Map}_d(T_g, S^2)
\end{align*}
\]
are homotopy equivalences up to dimension $d$ if $g = 0$, and they are homology equivalences up to dimension $D(d; g) = d - 2g$ if $g \geq 1$. □

**Theorem 1.3** (S. Kallel, [6]). If $d > 2g$ and $g \geq 1$, the inclusion maps $i_d$ and $j_d$ induce isomorphisms
\[
\begin{align*}
i_{d*} : \pi_1(\text{Hol}_d^*(T_g, S^2)) &\cong \pi_1(\text{Map}_d^*(T_g, S^2)), \\
.j_{d*} : \pi_1(\text{Hol}_d(T_g, S^2)) &\cong \pi_1(\text{Map}_d(T_g, S^2)).
\end{align*}
\]

**Remark.** A map $f : X \to Y$ is called a homotopy (resp. homology) equivalence up to dimension $D$ if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and epimorphism for $k = D$.

We expect that the inclusions $i_d$ and $j_d$ will be homotopy equivalences up to dimension $D(d; g)$ for $g \geq 1$. For example, Theorem 1.3 supports that this might be true, and it seems valuable to investigate the homotopy types of the universal coverings of $\text{Hol}_d^*(T_g, S^2)$ and $\text{Hol}_d(T_g, S^2)$.

The main purpose of this note is to announce the recent work of the author given in [13], in which we shall study the homotopy types of universal coverings of the above spaces. Let $\tilde{X}$ denote the universal covering of a connected space $X$. Then we can state our results as follows.

**Theorem 1.4** ([13]). If $d \geq 1$, there is a homotopy equivalence
\[
\tilde{\Phi}_d : S^3 \times \text{Hol}_d^*(T_g, S^2) \xrightarrow{\simeq} \text{Hol}_d(T_g, S^2).
\]

**Corollary 1.5** ([1], [8], [12]). If $g = 0$ and $d \geq 1$, there is a homotopy equivalence $\tilde{\Phi}_d : S^3 \times \text{Hol}_d(S^2, S^2) \xrightarrow{\simeq} \text{Hol}_d(S^2, S^2)$. □
Let \( i : \text{Hol}_d^*(T_g, S^2) \to \text{Hol}_d(T_g, S^2) \) be an inclusion map and let \( ev : \text{Hol}_d(T_g, S^2) \to S^2 \) denote the evaluation map given by \( ev(f) = f(t_0) \), where \( t_0 \in T_g \) is the base-point of \( T_g \). Then it is known that there is an evaluation fibration sequence (e.g. [6])

\[
\text{Hol}_d^*(T_g, S^2) \overset{i}{\longrightarrow} \text{Hol}_d(T_g, S^2) \overset{ev}{\longrightarrow} S^2.
\]

**Corollary 1.6 ([13]).** If \( k \geq 2 \) and \( d \geq 1 \), the above sequence induces a split short exact sequence

\[
0 \to \pi_k(\text{Hol}_d^*(T_g, S^2)) \overset{i_*}{\longrightarrow} \pi_k(\text{Hol}_d(T_g, S^2)) \overset{ev_*}{\longrightarrow} \pi_k(S^2) \to 0.
\]

**Theorem 1.7 ([13]).** Let \( d \geq 1 \) be an integer.

(i) There is a homotopy equivalence

\[
\tilde{\Phi} : S^3 \times \widetilde{\text{Map}}^*(T_g, S^3) \overset{\sim}{\longrightarrow} \text{Map}_d(T_g, S^2)
\]

and there is a fibration sequence (up to homotopy equivalence)

\[
\Omega^2 S^3\langle 3 \rangle \to \widetilde{\text{Map}}^*(T_g, S^3) \to (\Omega S^3)^{2g},
\]

where \( S^3\langle 3 \rangle \) denotes the 3-connective covering of \( S^3 \).

(ii) For any \( k \geq 2 \), there is an isomorphism

\[
\pi_k(\widetilde{\text{Map}}^*(T_g, S^3)) \cong \pi_{k+2}(S^3) \oplus \pi_{k+1}(S^3)^{\oplus 2g}.
\]

## 2 The idea of the proofs.

Let \( X, Y \) and \( Z \) be connected spaces and let \( f : X \to Y \) be a map. Define the map \( f^\# : \text{Map}(Y, Z) \to \text{Map}(X, Z) \) by \( f^\#(g) = g \circ f \) for \( g \in \text{Map}(Y, Z) \). Similarly, we define the map \( f^\# : \text{Map}^*(Y, Z) \to \text{Map}^*(X, Z) \) by the restriction. It is well known that there is a cofiber sequence

\[
S^1 \overset{\varphi_g}{\longrightarrow} \vee^{2g} S^1 \overset{t'}{\longrightarrow} T_g \overset{q_g}{\longrightarrow} S^2 \overset{\Sigma \varphi_g}{\longrightarrow} \vee^{2g} S^2
\]

where \( \pi_1(\vee^{2g} S^1) \) is the free group on \( 2g \) generators \( \{a_j, b_j : 1 \leq j \leq g\} \) and \( \varphi_g = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \in \pi_1(\vee^{2g} S^1) \).
Lemma 2.1. $q_{g}^\# : \pi_{k}(\Omega_{d}^{2}S^{2}) \to \pi_{k}(Map_{d}^{*}(T_{g}, S^{2}))$ is a monomorphism for any $k \geq 1$.

Proof. By using an easy diagram chasing we can show the assertion. □

Consider the commutative diagram of evaluation fibration sequences

\[
\begin{array}{ccc}
\text{Map}_{d}^{*}(S^{2}, S^{2}) & \xrightarrow{i_{r}} & \text{Map}_{d}(S^{2}, S^{2}) & \xrightarrow{ev_{S}} S^{2} \\
\text{Map}_{d}(T_{g}, S^{2}) & \xrightarrow{i_{s}} & \text{Map}_{d}(T_{g}, S^{2}) & \xrightarrow{ev_{T}} S^{2}
\end{array}
\]

Lemma 2.2. $ev_{X*} : \pi_{2}(Map_{d}(X, S^{2})) \to \pi_{2}(S^{2})$ is trivial for $X \in \{S^{2}, T_{g}\}$.

Proof. If $X = S^{2}$, we can easily show the assertion by using Whitehead’s Theorem [11] concerning the boundary operator of the homotopy exact sequence induced from the evaluation fibration. By using this with the diagram chasing of (2), we can also prove the assertion for $X = T_{g}$. □

Lemma 2.3. The induced homomorphisms

\[
\begin{cases}
i_{T*} : \pi_{2}(Map_{d}^{*}(T_{g}, S^{2})) \to \pi_{2}(Map_{d}(T_{g}, S^{2})) \\
i_{*} : \pi_{2}(\text{Hol}_{d}^{*}(T_{g}, S^{2})) \to \pi_{2}(\text{Hol}_{d}(T_{g}, S^{2}))
\end{cases}
\]

are epimorphisms.

Proof. The proof follows from the diagram chasing and Lemma 2.2. □

If we identify $S^{2} = \mathbb{C}P^{1}$, the group $SU(2)$ acts on $S^{2}$ by the right matrix multiplication. By using this right matrix multiplication, define the map

\[
\Phi : SU(2) \times Map_{d}^{*}(T_{g}, S^{2}) \to Map_{d}(T_{g}, S^{2})
\]

by $(\Phi(A, f))(t) = f(t) \cdot A$ for $(A, f, t) \in SU(2) \times Map_{d}^{*}(T_{g}, S^{2}) \times T_{g}$.

Since $\Phi(SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2})) \subset \text{Hol}_{d}(T_{g}, S^{2})$, we can define the map

\[
\Phi_{d} : SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2}) \to \text{Hol}_{d}(T_{g}, S^{2})
\]

by the restriction $\Phi_{d} = \Phi|SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2})$. 
**Theorem 2.4** ([13]). \( \Phi_{d*} : \pi_{k}(SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2})) \to \pi_{k}(\text{Hol}_{d}(T_{g}, S^{2})) \) is an isomorphism for any \( k \geq 2 \).

**Proof.** The detail is omitted and see [13] in detail.

Now we can prove Theorem 1.4 by using Theorem 2.4.

**Proof of Theorem 1.4.** Let

\[
\begin{align*}
\pi & : \overline{\text{Hol}}_{d}(T_{g}, S^{2}) \to \text{Hol}_{d}(T_{g}, S^{2}) \\
\pi' & : \overline{\text{Hol}}_{d}^{*}(T_{g}, S^{2}) \to \text{Hol}_{d}^{*}(T_{g}, S^{2})
\end{align*}
\]

denote projection maps of the universal coverings. If we identify \( SU(2) = S^{3} \), it is easy to see that the universal covering of \( SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2}) \) is given by \( 1 \times \pi' : S^{3} \times \overline{\text{Hol}}_{d}^{*}(T_{g}, S^{2}) \to SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2}) \). Then because \( \pi \) is a projection of the universal covering, there is a lifting \( \tilde{\Phi}_{d} : S^{3} \times \overline{\text{Hol}}_{d}^{*}(T_{g}, S^{2}) \to \overline{\text{Hol}}_{d}(T_{g}, S^{2}) \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
S^{3} \times \overline{\text{Hol}}_{d}^{*}(T_{g}, S^{2}) & \xrightarrow{\Phi_{d}} & \overline{\text{Hol}}_{d}(T_{g}, S^{2}) \\
1 \times \pi' \downarrow & & \downarrow \pi \\
SU(2) \times \text{Hol}_{d}^{*}(T_{g}, S^{2}) & \xrightarrow{\Phi_{d}} & \text{Hol}_{d}(T_{g}, S^{2})
\end{array}
\]

Since \( \pi_{k}(\Phi_{d}) \) is an isomorphism for any \( k \geq 2 \), an easy diagram chasing shows that \( \pi_{k}(\tilde{\Phi}_{d}) \) is also an isomorphism for any \( k \geq 2 \). Because \( S^{3} \times \overline{\text{Hol}}_{d}^{*}(T_{g}, S^{2}) \) and \( \overline{\text{Hol}}_{d}(T_{g}, S^{2}) \) are simply connected, \( \tilde{\Phi}_{d} \) is a homotopy equivalence.

By using the completely similar way as above, we can see that there is a fibration sequence (up to homotopy equivalence)

\[
S^{1} \to SU(2) \times \text{Map}_{d}^{*}(T_{g}, S^{2}) \xrightarrow{\Phi} \text{Map}_{d}(T_{g}, S^{2}).
\]

**Theorem 2.5** ([13]). \( \Phi_{*} : \pi_{k}(SU(2) \times \text{Map}_{d}^{*}(T_{g}, S^{2})) \to \pi_{k}(\text{Map}_{d}(T_{g}, S^{2})) \) is an isomorphism for any \( k \geq 2 \).

**Proof.** The proof is analogous to that of Theorem 2.4.
Corollary 2.6 ([13]). There is a homotopy equivalence
$$\widetilde{\text{Map}}_{d}(T_{g}, S^{2}) \simeq S^{3} \times \widetilde{\text{Map}}_{0}(T_{g}, S^{2}). \square$$

Lemma 2.7. There is a homotopy fibration sequence
$$\Omega^{2}S^{3}\langle 3 \rangle \to \widetilde{\text{Map}}^{*}(T_{g}, S^{3}) \to (\Omega S^{3})^{2g}.$$  

Proof. This can be proved by using tedious diagram chasing and the detail is omitted. \square

Proof of Theorem 1.7. (i) By using Corollary 2.6 and Lemma 2.7, to prove (i) it is sufficient to show that there is a homotopy equivalence

(6) $$\widetilde{\text{Map}}^{*}(T_{g}, S^{3}) \simeq \widetilde{\text{Map}}_{0}(T_{g}, S^{2}).$$

However, by using a similar manner as that of the previous Theorem we can prove the assertion (i) and the detail is omitted.

(ii) Assume that $k \geq 2$. Since there is a homotopy equivalence $\Sigma^{k}T_{g} \simeq \vee^{2g}S^{1+k} \vee S^{2+k}$ and $S^{3}$ is a Lie group, $\text{Map}^{*}(T_{g}, S^{3})$ is an H-space. Hence, the assertion (ii) easily follows. \square

参考文献


