数理解析研究所講究録
第 1670 巻 2009 年 69-71

COUNTING ON REAL BOTT MANIFOLDS

SUYOUNG CHOI

ABSTRACT. We survey the recent researches about the number of real Bott manifolds. Moreover, we find a lower-bound of the number of topological types of real Bott manifolds.

1. THE NUMBER OF SMALL COVERS

A *small cover*, defined by Davis and Januszkiewicz in [DJ91], is an n-dimensional closed smooth manifold M with a smooth standard real torus $\mathbb{Z}_2^n$ action such that the action is locally isomorphic to a standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ and the orbit space $M/\mathbb{Z}_2^n$ is a simple (combinatorial) polytope. In this paper, we are interested in the small covers over cubes. Here, a *cube* is a simple polytope which is combinatorially equivalent to the Cartesian product of finitely many intervals. We denote n dimensional cube by I^n.

A small cover over cubes, called a *real Bott manifold*, is obtained as iterated $\mathbb{R}P^1$ bundles starting with a point, where each stage is the projectivization of a Whitney sum of two real line bundles. The topological classification of real Bott manifolds is already known. Surprisingly, two real Bott manifolds are (affinely) diffeomorphic if and only if their cohomology rings with $\mathbb{Z}_2$ coefficients are isomorphic as graded rings, which was shown by Kamishima and Masuda in [KM08].

On the other hand, one of interesting properties of real Bott manifolds is that there is one-to-one correspondence between the set of real Bott manifolds (up to Davis-Januszkiewicz equivalence) and the set of acyclic digraphs with labelled finite nodes. Using this relation, the author has computed the number of real Bott manifolds up to Davis-Januszkiewicz equivalence and up to equivariant homeomorphism types, see [Cho08] for details. Moreover, he also computes the number of orientable small covers in [Cho09].

However, we still don't know the number of diffeomorphism types of real Bott manifolds although we know their classification by cohomology rings with $\mathbb{Z}_2$ coefficients. Let D_n be the number of diffeomorphism type of real Bott manifolds of dimension n. The author found a brief upper-bound of D_n in [Cho08, Corollary 4.2]. His upper-bound is the number of acyclic digraphs with n unlabeled nodes.

On the other hand, it is well-known that every real Bott manifold can be represented by a lower triangular matrix over $\mathbb{Z}_2$ all of whose diagonals are 0. Masuda has described the diffeomorphism class of real Bott manifolds in terms of matrices operations in [Mas08]. Using these operatoins, he found

Date: November 1, 2009.

The author was supported by the Japanese Society for the Promotion of Sciences (JSPS grant no. P09023).

a lower-bound of D_n by

$$D_n \geq 2^{(n-2)(n-3)/2}.$$

Recently, the author and his colleague, Oum, are researching on real Bott manifolds from the viewpoint of Graph theory and are preparing the paper [CO] about new invariants of the real Bott manifolds. Using these invariants, we can find a slightly improved lower-bound of D_n.

Theorem 1.1. *Let D_n be the number of real Bott manifolds up to diffeomorphisms. Then,*

$$2^{\binom{n-2}{2}} < \sum_{\ell=2}^{n} 2^{\binom{\ell-2}{2}} \leq D_n.$$

Moreover, here, we list up the number of real Bott manifolds up to $n = 8$. Indeed, Nazra [Naz08] has computed up to $n = 5$. Further, we compute $n = 6, 7$ and 8 in [CO] by computer.

n	0	1	2	3	4	5	6	7	8
D_n	1	1	2	5	12	54	472	8512	328416

Let E_n be the number of orientable real Bott manifolds up to diffeomorphism. Here is the list of E_n's up to $n = 8$.

n	0	1	2	3	4	5	6	7	8
E_n	1	1	1	2	3	8	29	222	3607

2. Proof of Theorem 1.1

Let $\mathcal{A}_n$ be the set of upper triangular matrices all of whose diagonal entries are 0. Here we define 3 operations Φ_S, Φ^k and Φ_C^I on $\mathcal{A}_n$;

(1) $\Phi_S(A) = SAS^{-1}$, where S is a permutation matrix
(2) Denote j-th row of the matrix A by A_j and its (i,j)-th element by A_j^i. Then, for $k \in \{1, \ldots, n\}$,
$$\Phi^k(A)_j = A_j + A_j^k A_k, \text{ for } j = 1, \ldots, n.$$
(3) Let I be a subset of $\{1, \ldots, n\}$ such that $A_i = A_j$ for $i, j \in I$. Let C be an invertible matrix of size $|I|$ over $\mathbb{Z}_2$. Then,
$$\Phi_C^I(A)_j^i := \begin{cases} \sum_{k \in I} C_k^i A_j^k, & \text{if } i \in I; \\ A_j^i, & \text{if } i \notin I. \end{cases}$$
Roughly speaking, Φ_C^I is the left matrix multiplication of C to the submatrix of A which consists of i-th rows for $i \in I$.

We say that two elements in $\mathcal{A}_n$ are *Bott equivalent* if one is transformed to the other through a sequence of the three operations.

Theorem 2.1. [Mas08, Theorem 1.1] *A and B are Bott equivalent if and only if real Bott manifolds $M(A)$ and $M(B)$ are (affinely) diffeomorphic, where $M(A)$ (resp, $M(B)$) is a real Bott manifold corresponding to $A \in \mathcal{A}_n$ (resp, B).*

By [CO], the rank of $A \in \mathcal{A}_n$ is an invariant under the 3 operations. Now, we consider $\ell \times \ell$ matrices of rank $\ell - 1$ all of whose diagonals are 0. Then, for each $A \in \mathcal{A}_\ell$, A_{i+1}^i must be 1 for $i = 1, \ldots, \ell - 1$. Note that, Φ_S and

Φ^I_C cannot be available. By the sequence of Φ^k's, we may change A^i_{i+2} into 0. In each equivalence class, there is one and only one element A satisfying $A^i_{i+2} = 0$ for all $i = 1, \dots, \ell - 2$. Therefore, there are exactly $2^{(\ell-2)(\ell-3)/2}$ equivalence classes.

Now, we consider $A \in \mathcal{A}_n$ such that the rank of submatrix consist of the first ℓ columns is $\ell - 1$ and the last $n - \ell$ columns are 0 vectors. By above arguments, there are $2^{(\ell-2)(\ell-3)/2}$. Since the rank is invariant under the 3 operations, there are at least $\sum_{\ell=2}^{n} 2^{\binom{\ell-2}{2}}$ equivalence classes in $\mathcal{A}_n$, which proves the theorem.

References

[Cho08] Suyoung Choi, *The number of small covers over cubes*, Algebr. Geom. Topol. **8** (2008), no. 4, 2391–2399. MR MR2465745 (2009i:57072)

[Cho09] ______, *The number of orientable small covers over cubes*, arXiv:0812.3861 (2009).

[CO] Suyoung Choi and Sang-il Oum, *Real Bott manifolds and acyclic digraphs.*

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451. MR MR1104531 (92i:52012)

[KM08] Yoshinobu Kamishima and Mikiya Masuda, *Cohomological rigidity of real Bott manifolds*, arXiv:0807.4263 (2008).

[Mas08] Mikiya Masuda, *Classiffcation of real Bott manifolds*, arXiv:0809.2178 (2008).

[Naz08] A. Nazra, *Real Bott tower*, Tokyo Metropolitan University, Master Thesis (2008).

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

E-mail address: choi@sci.osaka-cu.ac.jp