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Lattices of Non-Compact Lie Groups

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1 Introduction

Consider solvable Lie groups of type $H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1}$ ($n \geq m$). Here $\psi$ is a homomorphism from $\mathbb{R}^m$ to $GL(n+1, \mathbb{R})$ and the group structure of $H$ is given by

$$(s, x)(t, y) = (s + t, x + \psi(t)(y)), \quad (s, t \in \mathbb{R}^m, \ x, y \in \mathbb{R}^{n+1}).$$

We call Lie groups of this type 1-step solvable Lie groups. In this paper we study about the automorphisms groups of lattices (cocompact discrete subgroups) of 1-step solvable Lie groups $H$.

The unimodularization of $n$ products $\text{Aff}^+(\mathbb{R})^n$ of the affine group $\text{Aff}^+(\mathbb{R})$ is a 1-step solvable Lie group which takes the form of $\mathbb{R}^n \ltimes \psi \mathbb{R}^{n+1}$. In this case, the homomorphism $\psi$ is injective and splits as a direct sum of non-equivariant real 1-dimensional representations. Conversely, if the homomorphism $\psi$ of $H = \mathbb{R}^n \ltimes \psi \mathbb{R}^{n+1}$ has all of these properties, then $H$ is isomorphic to $\text{Aff}^+(\mathbb{R})^n$. Let $\Gamma$ be a lattice of $H = \mathbb{R}^n \ltimes \psi \mathbb{R}^{n+1} \cong \text{Aff}^+(\mathbb{R})^n$. In [2], we defined an algebraic number field $k(\Gamma)$ of degree $n + 1$ which is associated with a lattice $\Gamma$, and showed that the automorphism group $\text{Aut}(\Gamma')$ of a lattice $\Gamma'$ commensurable with $\Gamma$ is essentially identified with a subgroup of the automorphism group $\text{Aut}(k(\Gamma)/\mathbb{Q})$. More precisely, there is a surjection from the set $\{\text{Aut}(\Gamma') \mid \Gamma' < H, \ \Gamma' \in \text{Com}(\Gamma)\}$ to the set $\{F \mid F < \text{Aut}(k(\Gamma)/\mathbb{Q}) \}$ (Theorem 1.2 in [2]). Here $\text{Com}(\Gamma)$ denotes the set of lattices $\Gamma'$ which are commensurable with $\Gamma$ (see §4). But, when $n > m$, we have quite different results from those in the case of $n = m$.

In the first half of this paper, we review basic facts about lattices of 1-step solvable Lie groups $H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1}$, and in §4 we state an interesting Theorem 1.2 in [2]. In the last two sections, we study the case of $m < n$, especially the case of $n = m + 1$.

From now on, let $H$ denote 1-step solvable Lie groups of type $\mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1}$. Moreover we assume that $\psi$ is injective and splits as a direct sum of non-equivariant real 1-dimensional representations.
2 Structure matrix of $H$

From the assumption that $\psi$ splits as a direct sum of non-equivariant real 1-dimensional representations, for a basis $\{e_1, e_2, \cdots, e_m\}$ of $\mathbb{R}^m$, $A_j := \psi(e_j)$ ($1 \leq j \leq m$) are simultaneously conjugate to diagonal matrices $\text{diag}(e^\lambda_{1j}, e^\lambda_{2j}, \cdots, e^\lambda_{nj})$. Put

$$
\Lambda_{\psi} := \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n+1,1} & \lambda_{n+1,2} & \cdots & \lambda_{n+1,m}
\end{pmatrix} = \begin{pmatrix}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_{n+1}
\end{pmatrix},
$$

and call $\Lambda_{\psi}$ the \textit{structure matrix} of $H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1}$. Clearly $H$ is determined by the structure matrix.

We note here some fundamental facts on $\Lambda_{\psi}$.

1. Changing bases of $\mathbb{R}^m$ and $\mathbb{R}^{n+1}$, the new structure matrix $\Lambda'_{\psi}$ is written as $\Lambda'_{\psi} = T \Lambda_{\psi} P$, where $T$ is a row exchanging matrix and $P$ is an $m$-square non-singular matrix, that is $P \in \text{GL}(m, \mathbb{R})$. If $\Lambda'_{\psi} = T \Lambda_{\psi} P$ holds, then we say $\Lambda_{\psi}$ and $\Lambda'_{\psi}$ to be \textit{equivalent} and identify $\Lambda_{\psi}$ with $\Lambda'_{\psi}$.

2. Let $\Delta : Garrow \mathbb{R}_+$ be the modular function of a Lie group $G$ defined by $\Delta(g) = |\det Ad_g|$. For $H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1}$, the modular function $\Delta : H \rightarrow \mathbb{R}_+$ is given by $\Delta(t, x) = \exp(\sum_{i=1}^{n+1} \Lambda_i \cdot t)$.

3. If there exists a cocompact discrete subgroup (i.e. a lattice) $\Gamma$ of $H$, then $\Delta(t, x) = 1$ for $\forall (t, x) \in H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1}$. This shows $\sum_{i=1}^{n+1} \Lambda_i \cdot t = 0$ ($\forall t \in \mathbb{R}^m$), and thus $\sum_{i=1}^{n+1} \Lambda_i = 0$.

In this paper, we study about lattices of $H$. Thus, from now on, we assume that the structure matrix $\Lambda_{\psi}$ satisfies $\sum_{i=1}^{n+1} \Lambda_i = 0$.

3 Lattices and algebraic number fields

In this section, we define the algebraic number field $k(\Gamma)$ associated with a lattice $\Gamma$ of $H$. Let $H_0 := [H, H]$ and $H_1 := H/H_0$. Then $H_0 \cong \mathbb{R}^{n+1}$, $H_1 \cong \mathbb{R}^m$ and $H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1} = H_1 \ltimes H_0$ holds. The following is a known result.

\textbf{Lemma 3.1} (\cite[Lemma 2.3]{3}) \textit{Let $\Gamma < H$ be a lattice. Put $\Gamma_0 := \Gamma \cap H_0 = \Gamma \cap \mathbb{R}^{n+1}$ and $\Gamma_1 := \Gamma / \Gamma_0$. Then $\Gamma_0$ and $\Gamma_1$ are lattices of $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$, respectively.}
From Lemma 3.1, we can see $\Gamma_0 \cong \mathbb{Z}^{n+1}$ and $\Gamma_1 \cong \mathbb{Z}^m$. Moreover we have the exact sequences

$$
1 \to \mathbb{R}^{n+1} \to H \to \mathbb{R}^m \to 1 \\
1 \to \Gamma_0 \to \Gamma \to \Gamma_1 \to 1
$$

In general, $\Gamma$ is not a semi-direct product group. But the restriction $\psi|\Gamma_1$ becomes a homomorphism from $\Gamma_1$ to $\text{Aut}(\Gamma_0)$, and hence, $\psi(t) \in SL(n+1, \mathbb{Z})$ ($t \in \Gamma_1$). Thus we may assume that, in the structure matrix

$$
\Lambda_{\psi} = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n+1,1} & \lambda_{n+1,2} & \cdots & \lambda_{n+1,m}
\end{pmatrix},
$$

the numbers $e^{\lambda_{1j}}, e^{\lambda_{2j}}, \ldots, e^{\lambda_{n+1,j}}$ are eigenvalues of an integer matrix $A_j = \psi(t_j) \in SL(n+1, \mathbb{Z})$, that is, those numbers are algebraic integers. Here $\{t_1, t_2, \ldots, t_m\}$ is a $\mathbb{Z}$-basis of $\Gamma_1 \cong \mathbb{Z}^m$.

We suppose the following conditions on $\Lambda_{\psi}$.

**Assumption A on $\psi$ (i.e. on $\Lambda_{\psi}$)**

1. $\psi$ is injective.
2. There exists $t_0 \in \Gamma_1$ such that each eigenvalue $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ of $\psi(t_0) = A$ is an algebraic integer of degree $n+1$. Here $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ are each other conjugate elements.

**Remark 3.1.**

1. When $n = m$, the assumption 2 is automatically derived from the assumption 1.
2. For each $t \in \Gamma_1$, the matrix $\psi(t)$ can be described as $g(A)$ ($g[X] \in \mathbb{Q}[X]$) because $\psi(t)$ and $\psi(t_0) = A$ are commutative ([2, Corollary 3.2]).

Under Assumption A, we can assign a totally real algebraic number field $k(\Gamma) = \mathbb{Q}(\alpha)$ of degree $n+1$ to a lattice $\Gamma < H$, where $\alpha = \alpha_1$ in the above assumption 2. Call $k(\Gamma)$ the *algebraic number field associated with* $\Gamma$. We note that, from Remark 3.1-(2), $k(\Gamma)$ does not depends on the choice of $t_0$.

Lattices $\Gamma$ and $\Gamma'$ are called to be *commensurable* and denoted by $\Gamma \sim \Gamma'$ if $|\Gamma : \Gamma \cap \Gamma'| < \infty$ and $|\Gamma' : \Gamma \cap \Gamma'| < \infty$. From Remark 3.1-(2), it follows
that \( k(\Gamma) = k(\Gamma') \) if \( \Gamma \sim \Gamma' \). Furthermore we say that \( \Gamma \) and \( \Gamma' \) are weakly commensurable if there exists \( \varphi \in \text{Aut}(H) \) such that \( \varphi(\Gamma) \sim \Gamma' \). When \( n = m \), \( k(\Gamma') = \varphi_{*}(k(\Gamma)) \) holds ([2, Lemma 3.3]). From those facts, we obtain the following theorem.

**Theorem 3.2** Suppose \( n = m \). Then the map

\[
\left\{ \begin{array}{ll}
\text{the set of all weakly} \\
\text{commensurable classes} \\
of \text{lattices of } H
\end{array} \right\} \rightarrow \left\{ \begin{array}{ll}
\text{the set of all isomorphism} \\
\text{classes of totally real algebraic} \\
\text{number fields of degree } n + 1
\end{array} \right\}
\]

induced from the map \( \Gamma \rightarrow k(\Gamma) \) is bijective.

4 \quad \textbf{Aut}(\Gamma)

Let \( \Gamma \) be a lattice of \( H \), and take \( \varphi \in \text{Aut}(\Gamma) \). Then the following hold.

1. \( \varphi \) naturally induces automorphisms \( \varphi_1 : \Gamma_1 \rightarrow \Gamma_1 \) and \( \varphi_0 : \Gamma_0 \rightarrow \Gamma_0 \).
2. \( \psi(\varphi_1(t)) = \varphi_0\psi(t)\varphi_0^{-1} \) \((\forall t \in \Gamma_1 = \mathbb{Z}^m)\).

The equality 2 follows from that \( \varphi \) is a homomorphism. We call this equality in 2 the compatibility condition for \((\varphi_1, \varphi_0)\).

**Remark 4.1.** It is known that \( \varphi \in \text{Aut}(\Gamma) \) is uniquely extended to \( \tilde{\varphi} \in \text{Aut}(H) \) (e.g. [1][2]). Clearly the compatibility condition holds for \((\tilde{\varphi}_1, \tilde{\varphi}_0)\).

Using 1 and 2 above, we can define a homomorphism

\[ A_{\Gamma} : \text{Aut}(\Gamma) \rightarrow \text{Aut}(k(\Gamma)/\mathbb{Q}) \]

by

\[ A_{\Gamma}(\varphi)(\psi(t_0)) = \psi(\varphi_1(t_0)) = \varphi_0\psi(t_0)\varphi_0^{-1}. \]

From the definition, the map \( A_{\Gamma}(\varphi) \) induces a permutation of the set \( \{\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \).

**Theorem 4.1** Suppose \( m = n \). Let \( \Gamma \) be a lattice of \( H \). Then, for each subgroup \( F < \text{Aut}(k(\Gamma)/\mathbb{Q}) \), there exists a lattice \( \Gamma' < H \) such that

1. \( \Gamma' \) is commensurable with \( \Gamma \),
2. \( A_{\Gamma'}(\text{Aut}(\Gamma')) = F \).
Outline of the proof. Let $k$ be a totally real algebraic number field of degree $n+1$ and let $\{f^{(1)}, f^{(2)}, \cdots, f^{(n+1)}\}$ be the set of all imbeddings of $k$ into $\mathbb{R}$. Let $\mathcal{O}(k)$ be the subring of algebraic integers in $k$. The ring $\mathcal{O}(k)$ is isomorphic to $\mathbb{Z}^{n+1}$ as additive groups. Denote by $\mathcal{E}(k)$ the unit group of $\mathcal{O}(k)$ and put

$$\mathcal{E}^{+}(k) := \{ \varepsilon \in \mathcal{E}(k) \mid f^{(i)}(\varepsilon) > 0 \ (1 \leq i \leq n+1) \}.$$  

Define an injective map $\ell_k : \mathcal{E}^{+}(k) \to \mathbb{R}^{n+1}$ by

$$\ell_k(\varepsilon) = (\log (f^{(1)}(\varepsilon)), \log (f^{(2)}(\varepsilon)), \cdots, \log (f^{(n+1)}(\varepsilon)))$$

The Dirichlet's unit theorem asserts that $\ell_k(\mathcal{E}^{+}(k))$ is a lattice of $V = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \}$. Put

$$\Gamma_k = \ell_k(\mathcal{E}^{+}(k)) \ltimes \psi_k \mathcal{O}(k), \quad H_k = (\ell_k(\mathcal{E}^{+}(k)) \otimes \mathbb{R}) \ltimes \overline{\psi}_k (\mathcal{O}(k) \otimes \mathbb{R}).$$

The homomorphism $\psi_k : \ell_k(\mathcal{E}^{+}(k)) \to \text{Aut}(\mathcal{O}(k))$ is given by $\psi_k \circ \ell_k = \iota_k$, where $\iota_k$ is the tautological map defined by $\iota_k(\varepsilon)(\gamma) = \varepsilon \gamma$ ($\gamma \in k$). The homomorphism $\psi_k$ is the natural extension of $\psi_k$.

Now take groups $H$ and $\Gamma$ in the theorem. Then we can construct an isomorphism $\Psi_{\Gamma}$ from $H$ to $H_k$ such that $\varphi_k(\Psi_{\Gamma}(\Gamma)) \cong \Gamma_k$ for $\varphi_k \in \text{Aut}(H_k)$ ([2, Lemma 3.6]). From this, to prove the theorem, we may assume $\Gamma = \Gamma_k \subset H_k = H$.

Suppose $H = H_k, \Gamma = \Gamma_k$. From the definition, $A_{\Gamma_k}(\text{Aut}(\Gamma_k)) = \text{Aut}(k/\mathbb{Q})$ holds. Take a subgroup $\mathcal{E}_1 < \mathcal{E}^{+}(k)$ with $|\mathcal{E}^{+}(k) : \mathcal{E}_1| < \infty$, and put $\Gamma' := \ell_k(\mathcal{E}_1) \ltimes \psi_k \mathcal{O}(k)$. Clearly $\Gamma_k$ and $\Gamma'$ are commensurable. Moreover it is seen that

$$\text{Ad}(\iota_k^{-1})A_{\Gamma'}(\text{Aut}(\Gamma')) = \{ \sigma \in \text{Aut}(k/\mathbb{Q}) \mid \sigma(\mathcal{E}_1) = \mathcal{E}_1 \}.$$  

Thus, for a given $F < \text{Aut}(k/\mathbb{Q})$, we only have to construct $\mathcal{E}_1$ such that

$$F = \{ \sigma \in \text{Aut}(k/\mathbb{Q}) \mid \sigma(\mathcal{E}_1) = \mathcal{E}_1 \}.$$  

Such an $\mathcal{E}_1$ can be constructed by using Artin's theorem on relative fundamental units (e.g., [2, Theorem 4.1]).

When $n = m$, we showed that if $\varphi \in \text{Ker}A_{\Gamma}$, then $\varphi^2 = \text{Ad}(h_0)$ for some $h_0 \in H$ ([2, Corollary 3.11]).

5 $\text{Aut}(\Gamma)$ when $n > m$

In the rest of the paper, we treat the case where $n > m$, that is, $H = \mathbb{R}^m \ltimes \psi \mathbb{R}^{n+1} (n > m)$. We add one more assumption on the structure matrix $\Lambda_\psi$.  

\[\square\]
Assumption B on $\Lambda_{\psi}$

Every $m$ row vectors $\Lambda_{i_1}, \Lambda_{i_2}, \ldots, \Lambda_{i_m}$ of $\Lambda_{\psi}$ are linearly independent over $\mathbb{R}$.

Remark 5.1. When $n = m$, Assumption B is automatically derived from the injectivity of $\psi$.

Let $\Gamma$ be a lattice of $H$, and take $\varphi \in \text{Aut}(\Gamma)$. Then, from the compatibility condition $\psi(\varphi_1(t)) = \varphi_0 \psi(t) \varphi_0^{-1}$, the map $A_\Gamma(\varphi) \in \text{Aut}(k(\Gamma)/\mathbb{Q})$ induces a permutation $\sigma \in S_{n+1}$. Moreover $\varphi \in \text{Aut}(\Gamma)$ acts on the structure matrix $\Lambda_{\psi}$ as follows.

$$T_\sigma \Lambda_{\psi} = T_\sigma \left( \begin{array}{c} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{array} \right) = \left( \begin{array}{c} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{array} \right) P_\sigma \quad (5.1)$$

where $T_\sigma$ is the row exchanging matrix corresponding to $\sigma$ and $P_\sigma \in GL(m, \mathbb{Z})$. Let

$$\left( \begin{array}{c} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{array} \right) = \left( \begin{array}{c} I_m \\ c_{11}, c_{12}, \ldots, c_{1m} \\ c_{p1}, c_{p2}, \ldots, c_{pm} \end{array} \right) \left( \begin{array}{c} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{array} \right)$$

where $p = n + 1 - m$.

Putting $P'_\sigma = P_\sigma^{-1}$, the above relation (5.1) is re-written as

$$T_\sigma \left( \begin{array}{c} I_m \\ c_{11}, c_{12}, \ldots, c_{1m} \\ \cdots \\ c_{p1}, c_{p2}, \ldots, c_{pm} \end{array} \right) = \left( \begin{array}{c} I_m \\ c_{11}, c_{12}, \ldots, c_{1m} \\ \cdots \\ c_{p1}, c_{p2}, \ldots, c_{pm} \end{array} \right) P'_\sigma. \quad (5.2)$$

From the condition $\sum_{i=1}^{n+1} \Lambda_i = 0$, we have

$$1 + \sum_{i=1}^{p} c_{ij} = 0 \quad (1 \leq j \leq m). \quad (5.3)$$

Remark 5.2. When $n = m$, for every permutation $\sigma \in S_{n+1}$, the conditions (5.1)(5.2) are satisfied.

Divide each permutation $\sigma \in S_{n+1}$ into a product of distinct cyclic permutations, $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$. We say cyclic permutations $\sigma_i = (i_1, i_2, \cdots, i_k)$ and
\[ \sigma_j = (j_1, j_2, \ldots, j_k) \] to be distinct if \( i_s \neq j_t \ (i \neq j) \ (1 \leq s \leq \ell, 1 \leq t \leq k) \), and define the length of \( \sigma_i = (i_1, i_2, \ldots, i_\ell) \) to be \( \ell \). For example, \((123)\) and \((4567)\) are distinct cyclic permutations, and \((123)\) and \((3456)\) are not distinct.

The following propositions are given by simple calculations (see [5] for the details). In the propositions and corollaries below, we suppose Assumption B.

**Proposition 5.1** ([5]) Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in S_{n+1} \) \((\sigma \neq \text{trivial})\) be the product of distinct cyclic permutations. Suppose that, for the \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \), there exists \( P_\sigma' \) satisfying the conditions (5.2)(5.3). Then all \( \sigma_i \) have the same length, or the length of \( \sigma_1 \) is 1 and the other \( \sigma_i \)'s have the same length.

When each non-trivial cyclic permutation \( \sigma_i \) of \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \) has the same length, we also say the length to be the size of \( \sigma \). For example, the size of \( \sigma = (123)(456)(789) \) is 3.

**Proposition 5.2** ([5]) If \((n+1) - m = s \) is even, then the size of \( \sigma \) satisfying the conditions (5.2)(5.3) is not larger than \( s \).

**Corollary 5.3** ([5]) Suppose that \( n + 1 \) is even and \((n+1) - m = 2 \). Then a permutation \( \sigma \) satisfying the conditions (5.2)(5.3) is one of

\[
(1) \text{ trivial,} \quad (2) \ \sigma = (12)(34) \cdots (m+1, m+2),
\]

by renumbering row vectors \( \Lambda_1, \Lambda_2, \ldots, \Lambda_{n+1} \) of \( \Lambda_\psi \) if necessary. Moreover, in the case of (2), the structure matrix \( \Lambda_\psi \) is equivalent to the form

\[
\left( \begin{array}{cccc}
I_m \\
1_1 a_1 b_1 a_2 b_2 \cdots a_k b_k \\
b_1 a_1 b_2 a_2 \cdots b_k a_k \\
\end{array} \right) \quad \left( \begin{array}{c}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_m
\end{array} \right).
\]

We say \( \Lambda_\psi \) to be type (S) if it is equivalent to the above form (S). Corollary 5.3 implies that \( \text{Aut}(\Gamma) \) in the case \( n > m \) is quite different from that in the case \( n = m \).

**Corollary 5.4** ([5]) Let \( \Gamma \) be a lattice of \( H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1} \). Suppose that \( n + 1 \) is even and \((n+1) - m = 2 \). Then

\[ |A_\Gamma(\text{Aut}(\Gamma))| \leq 2. \]
6 Case of $H = \mathbb{R}^2 \ltimes \psi \mathbb{R}^4$

In this section we treat $H = \mathbb{R}^2 \ltimes \psi \mathbb{R}^4$, and give two examples of $\Gamma$ such that $|A_{\Gamma}(\text{Aut}(\Gamma))| = 2$ and $|A_{\Gamma}(\text{Aut}(\Gamma))| = 1$. See [4] for another examples.

Let $A_j (1 \leq j \leq 3)$ be the $4 \times 4$ integer matrices given by

$$A_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 7 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 5 & 1 & 0 & -1 \\ -12 & -2 & 1 & 7 \\ 7 & 1 & 2 & -12 \\ -1 & 0 & 1 & 5 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} -9 & -104 & -575 & -2742 \\ 69 & 719 & 3921 & 18619 \\ -153 & -1283 & -6756 & -31725 \\ 104 & 575 & 2742 & 12438 \end{pmatrix}.$$

Then we can see the following.

1. $\det A_j = 1$ ($j = 1, 2, 3$), that is, $A_j \in SL(4, \mathbb{Z})$.

2. Let $f_2(x) = -x^3 + 7x^2 - 12x + 5$, $f_3(x) = 104x^3 - 153x^2 + 69x - 9$. Then $A_2 = f_2(A_1)$, $A_3 = f_3(A_1)$.

3. Let $g_j(x)$ be the characteristic polynomial of $A_j$. Each $g_j(x)$ is given as

$$g_1(x) = x^4 - 7x^3 + 13x^2 - 7x + 1,$$
$$g_2(x) = (x^2 - 3x + 1)^2,$$
$$g_3(x) = x^4 - 6392x^2 + 1515658x^2 - 11717x + 1,$$

and thus all of the eigenvalues of the matrices $A_j (1 \leq j \leq 3)$ are positive real numbers.

4. The eigenvalues of $A_1$ are

$$\alpha_1 = \frac{7 - \sqrt{5}}{4} - \frac{1}{2} \sqrt{\frac{19 - 7\sqrt{5}}{2}}$$
$$\alpha_2 = \frac{7 - \sqrt{5}}{4} + \frac{1}{2} \sqrt{\frac{19 - 7\sqrt{5}}{2}}$$
$$\alpha_3 = \frac{7 - \sqrt{5}}{4} - \frac{1}{2} \sqrt{\frac{19 + 7\sqrt{5}}{2}}$$
$$\alpha_4 = \frac{7 + \sqrt{5}}{4} + \frac{1}{2} \sqrt{\frac{19 + 7\sqrt{5}}{2}}$$

Clearly the eigenvalues of $A_2$ and $A_3$ are $f_2(\alpha_i)$ and $f_3(\alpha_i)$ ($1 \leq i \leq 4$). The numerical values of $\alpha_i$ are

$$\alpha_1 \doteqdot 0.544113$$
$$\alpha_2 \doteqdot 1.8378528$$
$$\alpha_3 \doteqdot 0.2277777$$
$$\alpha_4 \doteqdot 4.390257$$
5. Let $\Lambda$ be the $4 \times 2$ matrix whose $(ij)$ entry is $\log(f_j(\alpha_i))$ ($1 \leq i \leq 4, 1 \leq j \leq 2$). (We put $f_1(x) := x$). Then the numerical value of $\Lambda$ is the following:

$$
\Lambda \doteqdot \begin{pmatrix}
-0.608598 & -0.962424 \\
0.608598 & -0.962424 \\
-1.47939 & 0.962424 \\
1.47939 & 0.969242
\end{pmatrix}
$$

6. Let $\Lambda_u$ be the “upper half” of $\Lambda$. That is, let $\Lambda_u$ be the $2 \times 2$ matrix whose $(ij)$ entry is $\log(f_j(\alpha_i))$ ($1 \leq i \leq 2, 1 \leq j \leq 2$). Then we have

$$
\Lambda(\Lambda_u)^{-1} \doteqdot \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0.71541 & -1.71541 \\
-1.71541 & 0.71541
\end{pmatrix}.
$$

7. Let $\Lambda'$ be the $4 \times 2$ matrix whose $(i1)$ entry is $\log \alpha_i$ and $(i2)$ entry is $\log f_3(\alpha_i)$ ($1 \leq i \leq 4$). Then the numerical value of $\Lambda'$ is the following:

$$
\Lambda' \doteqdot \begin{pmatrix}
-0.608598 & -9.35757 \\
0.608598 & 5.50787 \\
-1.47939 & -4.87376 \\
1.47939 & 8.72346
\end{pmatrix}
$$

**Lemma 6.1** (1) $\Lambda$ in 5 is of type (S), (2) $\Lambda'$ in 7 is not of type (S).

**Proof** It is seen that

$$
\Lambda = \begin{pmatrix}
\lambda_1 & \mu_1 \\
-\lambda_1 & \mu_1 \\
\lambda_2 & \mu_2 \\
-\lambda_2 & \mu_2
\end{pmatrix}, \quad (\mu_1 + \mu_2 = 0).
$$

Thus

$$
\Lambda(\Lambda_u)^{-1} = \begin{pmatrix}
\lambda_1 & \mu_1 \\
-\lambda_1 & \mu_1 \\
\lambda_2 & \mu_2 \\
-\lambda_2 & \mu_2
\end{pmatrix} \frac{1}{2\lambda_1 \mu_1} \begin{pmatrix}
\mu_1 & -\mu_1 \\
\lambda_1 & -\lambda_1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\frac{\lambda_1 \mu_2 + \lambda_2 \mu_1}{2\lambda_1 \mu_1} & \frac{\lambda_1 \mu_2 - \lambda_2 \mu_1}{2\lambda_1 \mu_1} \\
\frac{\lambda_1 \mu_2 - \lambda_2 \mu_1}{2\lambda_1 \mu_1} & \frac{\lambda_1 \mu_2 + \lambda_2 \mu_1}{2\lambda_1 \mu_1}
\end{pmatrix}.
$$

We omit the proof of (2).
Proposition 6.2 Let $H = \mathbb{R}^2 \ltimes_{\psi} \mathbb{R}^4$ be a 1-step solvable Lie group such that the structure matrix $\Lambda_\psi$ is $\Lambda$ (resp. $\Lambda'$) in Lemma 6.1. Let $\Gamma$ be the lattice of $H$ given by $\mathbb{Z}^2 \ltimes_{\psi} \mathbb{Z}^4$. Then $|A_\Gamma(\text{Aut}(\Gamma))| = 2$ (resp. $|A_\Gamma(\text{Aut}(\Gamma))| = 1$).

Proof Let $\Lambda_\psi = \Lambda$, and let $\sigma = (12)(34)$. Then the homomorphism $\varphi \in \text{Aut}(\Gamma)$ given by $\varphi_0 = T_\sigma$, $\varphi_1 = P_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ clearly satisfies the relation $T_\sigma \Lambda = \Lambda P_\sigma$, and thus $A_\Gamma(\varphi) = \sigma$. Let $\Lambda_\psi = \Lambda'$. Then Corollary 5.3 and Lemma 6.1 show $|A_\Gamma(\text{Aut}(\Gamma))| = 1$. 

References

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