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Author(s)
SUZUKI, TAKASHI; TAKAHASHI, RYO

Citation
数理解析研究所講究録 (2009), 1671: 96-112

Issue Date
2009-12

URL
http://hdl.handle.net/2433/141164

Type
Departmental Bulletin Paper

Textversion
publisher
Kyoto University
On a semilinear elliptic equation with subcritical exponent in higher dimensional space

TAKASHI SUZUKI (鈴木貴)
RYO TAKAHASHI (高橋亮)

Division of Mathematical Science, Department of Systems Innovation,
Graduate School of Engineering Science, Osaka University

Abstract

We study some properties of the solution to a semilinear elliptic equation with subcritical exponent in higher dimensions. Classification of the bounded energy solution in whole space, an inequality of sup + inf type, a theorem of Brezis-Merle type, and the quantized blowup mechanism are presented.

1 Introduction

In this paper, we study the semilinear elliptic equation

\[
\begin{cases}
-\Delta u = v_+^{\gamma} & \text{in } \Omega \\
\int_\Omega v_+^{\frac{n+2}{n-2}} dx < +\infty,
\end{cases}
\]

where \( \gamma \in \left(1, \frac{n+2}{n-2}\right) \), \( n \geq 3 \), and \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \) or \( \Omega = \mathbb{R}^n \). In the case \( \gamma = \frac{n}{n-2} \), classification of the solution to (1.1) with \( \Omega = \mathbb{R}^n \), inequalities of sup + inf and Trudinger-Moser type, and blowup analysis of the solution are done in [21]. As stated there, equation (1.1) is close to Liouville’s equation in two dimensions,

\[
\begin{cases}
-\Delta v = e^v & \text{in } \Omega \subset \mathbb{R}^2 \\
\int_\Omega e^v dx < +\infty.
\end{cases}
\]

In fact, equations (1.1) and (1.2) have the following common properties:

(A) Scaling invariance concerning the equation and the energy

(B) Classification of the bounded energy solution in whole space

(C) Existence of a sup + inf type inequality

(D) Alternatives concerning convergence of the solutions

(E) Quantized blowup mechanism
In what follows, we look over these properties.

(A) For a solution \( v = v(x) \) to (1.2), the transformation \( v_{\mu}(x) = v(\mu x) + 2 \log \mu, \mu > 0 \), satisfies
\[
\begin{cases}
-\Delta v_{\mu} = e^{v_{\mu}} & \text{in } \Omega_{\mu} \\
\int_{\Omega_{\mu}} e^{v_{\mu}} dx = \int_{\Omega} e^{v} dx,
\end{cases}
\]
where \( \Omega_{\mu} = \{ y \in \mathbb{R}^{2} | \mu y \in \Omega \} \). Similarly, for a solution \( v = v(x) \) to (1.1), the transformation \( v_{\mu}(x) = \mu^{q} v(\mu x), \mu > 0, q = \frac{2}{\gamma - 1} \), satisfies
\[
\begin{cases}
-\Delta v_{\mu} = (v_{\mu})_{+}^{\gamma} & \text{in } \Omega_{\mu} \\
\int_{\Omega_{\mu}} (v_{\mu})_{+}^{\gamma} dx = \int_{\Omega} t_{+}^{\gamma} dx,
\end{cases}
\]
where \( \Omega_{\mu} = \{ y \in \mathbb{R}^{n} | \mu y \in \Omega \}, n \geq 3 \). These scale invariances are important extremely in the proof of the properties (B)-(E), and, in particular, allow us to the blowup analysis and the hierarchical argument.

(B) Any nontrivial classical solution to (1.2) in whole space (i.e., \( \Omega = \mathbb{R}^{2} \)) has the form
\[
v(x) = \log \left( \frac{8\mu^{2}}{(1 + \mu^{2}|x-x_{0}|^{2})} \right)
\]
for some \( x_{0} \in \mathbb{R}^{2} \). This fact is shown by Chen and Li [4]. Similar fact for (1.1) with \( \gamma = \frac{n}{n-2} \) is done by Wang and Ye [21]. A crucial difference between (1.3) and (1.4) below is whether a support of the positive part of the solution is compact or not. This makes several arguments for (1.1) simpler. We now state the first result.

**Theorem 1** Assume that \( \gamma \in \left( 1, \frac{n+2}{n-2} \right) \) and \( n \geq 3 \). Then, any non-constant classical solution \( v = v(x) \) to (1.1) with \( \Omega = \mathbb{R}^{n} \) is radially symmetric, and the nonnegative part \( v_{+} \) has a compact support. More precisely, there exist \( x_{0} \in \mathbb{R}^{n} \) and \( \mu > 0 \) such that
\[
v(x) = \begin{cases}
\mu^{q} \phi(\mu|x-x_{0}|) & (\mu|x-x_{0}| \leq r_{\gamma}^{*}) \\
\lambda_{\gamma}^{*} \frac{1}{\omega_{n-1}(n-2)} [1 + \frac{1}{(\mu^{2} - 1)(n-2)}] & (\mu|x-x_{0}| > r_{\gamma}^{*})
\end{cases}
\]
with \( \omega_{n-1} \) standing for the area of the boundary of the unit ball in \( \mathbb{R}^{n} \), where \( r_{\gamma}^{*} \) is the first zero point of the unique solution \( \phi = \phi(r) \) to
\[
\begin{cases}
\phi''(r) + \frac{n-1}{r} \phi'(r) + \phi_{+}^{\gamma}(r) = 0, & r > 0 \\
\phi(0) = 1, & \phi'(0) = 0,
\end{cases}
\]
and
\[
\lambda_{\gamma}^{*} = \omega_{n-1} \int_{0}^{r_{\gamma}^{*}} \phi \frac{u^{n-1}}{2} r^{n-1} dr.
\]
The general entire solution to
\[
-\Delta v = v^{\gamma} \quad \text{in } \mathbb{R}^{n}, n \geq 3
\]
is concerned with the critical Sobolev exponent, i.e., $p_s = \frac{n-2}{n+2}$. Gidas and Spruck showed [8] that there is no positive solution to (1.7) in subcritical case $1 \leq p < p_s$. On the other hand, it was shown by Caffarelli, Gidas, and Spruck [3] that (1.7) has the positive solutions in critical case $p = p_s$. Furthermore, the solution to $v = v(x)$ to (1.7) with $p = p_s$ has the form

$$v(x) = \frac{n(n - 2)\mu^2}{\mu^2 + |x - x_0|^2}$$

for some $x_0 \in \mathbb{R}^n$ and $\mu > 0$ if $v(x) = O(|x|^{2-n})$ as $|x| \to +\infty$. In super critical case $p > p_s$, radial symmetry of the positive solution to (1.7) no longer hold generally, see [11, 22] for details.

(C) The sup + inf type inequality for (1.2) was shown by Shafrir [16], see also [2, 6]. Several sup × inf type inequalities for equations concerning the critical Sobolev exponent are found in [5, 12, 14]. The inequality of sup + inf type for (1.1) with $\gamma = \frac{n}{n-2}$ was established in [21]. We extend it to the case $\gamma \in \left(1, \frac{n+2}{n-2}\right)$.

**Theorem 2** Assume that $\gamma \in \left(1, \frac{n+2}{n-2}\right)$ and $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, for any compact set $K \subset \Omega$ and any number $T > 0$, there exist $C_1 = C_1(n, \gamma) > 0$ and $C_2 = C_2(n, \gamma, K, T) > 0$ such that

$$\sup_{K} v + C_1 \inf_{\Omega} v \leq C_2$$

for any solution $v = v(x)$ to (1.1) with the property

$$\int_{\Omega} v_{+}^{\frac{n(\gamma-1)}{2}} \, dx \leq T.$$

(D) Convergence of the solutions to (1.2) was studied by Brezis and Merle [1], and then the stronger result was obtained by Li and Shafrir [13]. We note that the sup + inf type inequality is a crucial component of the proof of the latter result, see [13]. The corresponding results for (1.1) with $\gamma = \frac{n}{n-2}$ are shown in [21]. They are extend as follows.

**Theorem 3** Assume that $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ and $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$ and $\{v_k\}$ be a sequence of the classical solutions satisfying

$$\begin{cases}
-\Delta v_k = (v_k)^{+} & \text{in } \Omega \\
\int_{\Omega} (v_k)^{+} \, dx \leq T
\end{cases}$$

for some $T > 0$. Then there exists a subsequence, still denoted by the same symbol $\{v_k\}$, such that the following alternatives occur:

(i) $\{v_k\}$ is locally uniformly bounded.

(ii) $v_k \to -\infty$ locally uniformly in $\Omega$. 


There exists a finite set $S = \{x_i\}_{i=1}^m$ such that $v_k \to -\infty$ locally uniformly in $\Omega \setminus S$ and that

$$(v_k)_{+}^{\frac{n(\gamma-1)}{2}} dx \to \sum_{i=1}^m \alpha_*(x_i) \delta_{x_i}(dx)$$

in $\mathcal{M}(\Omega)$ with $\alpha_*(x_i) = l, \lambda_*$ for some $l, \in \mathbb{N}$ and for all $i = 1, \cdots, m$, where $\delta_{x_i}$ and $\mathcal{M}(\Omega)$ denote the Dirac measure and the space of measure, respectively, and $\lambda_*$ is as in (1.6).

(E) Nagasaki and Suzuki [15] studied the quantized blowup mechanism for

$$\begin{cases}
-\Delta v = \sigma e^u & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega
\end{cases}$$

in (1.11) by combining the results by [1, 13, 7]. Then the quantized blowup mechanism also arises for (1.11), see [19] for details. Here, we consider

$$\begin{cases}
-\Delta v = v_+^\gamma & \text{in } \Omega \\
v = (\text{unknown}) \text{ constant} & \text{on } \partial \Omega \\
\int_{\Omega} v^{n(\gamma-1)/2} dx = \lambda
\end{cases}$$

(1.12)

The corresponding result for $\gamma = \frac{n}{n-2}$ is shown in [19]. This property holds even in the case $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$.

**Theorem 4** Assume that $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ and $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$, and $(\lambda_k, v_k)$ be a solution sequence to (1.12) satisfying $\lambda_k \to \lambda_0$. Then, passing to a subsequence, we have the following properties:

(i) $v_k$ is uniformly bounded in $\Omega$.

(ii) $\sup_{\Omega} v_k \to -\infty$.

(iii) $\lambda_0 = \lambda_* l$ for some $l \in \mathbb{N}$, and there exist $x_j^* \in \Omega$ and $x_k^{(j)}$ for all $1 \leq j \leq l$ such that the following (a)-(e) hold:

(a) $S = \{x_j^*\}^l_{j=1} = \{x_0 \in \Omega \mid \text{there are } x_k \in \Omega \text{ such that } v_k(x_k) \to +\infty\}$.

(b) $\frac{1}{2} \nabla R(x_j^*) + \sum_{j \neq j} \nabla x_j G(x_j^*, x_j^*) = 0$ for all $1 \leq j \leq l$.

(c) $x = x_k^{(j)}$ is a local maximum point of $v_k = v_k(x)$.

(d) $v_k(x_k^{(j)}) \to +\infty$ and $v_k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus S$ for all $1 \leq j \leq l$.

(e) $\int_{\Omega} v_{+}^{n(\gamma-1)/2} dx \to \sum_{j=1}^l \lambda_0 \delta_{x_j^*}(dx)$ in $\mathcal{M}(\Omega)$. 
Here, $G = G(x, x')$ denotes the Green function of $-\Delta$ on $\Omega$ with the Dirichlet boundary condition and 

$$R(x) = |G(x, x') - \Gamma(x - x')|_{x' = x}$$

for

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)|x^{n-2}|}.$$  

with $\omega_{n-1}$ standing for the area of the boundary of the unit ball in $\mathbb{R}^n$.

This paper is composed of four sections. Theorems 1 and 2 are proven in Section 2 and 3, respectively. Sketch of the proof of Theorem 3 is described in Section 4. In the following, $C_i$ ($i = 1, 2, \cdots$) denote positive constants whose subscripts are renewed in each section.

## 2 Proof of Theorem 1

In this section, we shall assume that $n \geq 3$ and $\gamma \in \left(1, \frac{n+2}{n-2}\right)$.

In order to show Theorem 1, we shall provide several lemmas. The following lemma is shown similarly to [21].

**Lemma 1** For any $R > 0$ and $A > 0$, there exists a number $C_1 = C_1(\gamma, R, A) > 0$ such that

$$\inf_{\overline{B}_{R/4}} v \leq -C_1$$  

(2.1)

for all solutions $v \in C^2(B_R) \cap C(\overline{B}_R)$ to

$$\begin{cases}
-\Delta v = v^\gamma & \text{in } B_R \\
v(x_0) = 1 & \text{for some } x_0 \in B_{R/2} \\
v \leq A & \text{in } B_R.
\end{cases}$$  

(2.2)

Next, we show a uniform estimate which is crucial to obtain the boundedness from above of the solution to (1.1) with $\Omega = \mathbb{R}^n$.

**Lemma 2** There are $C_0 = C_0(\nu, \gamma) > 0$ and $\delta_0 = \delta_0 > 0$ such that

$$\max_{\overline{B}_{\nu/4}} v \leq C_0$$  

(2.3)

for all solutions $v \in C^2(B_1)$ to

$$\begin{cases}
-\Delta v = v^\gamma & \text{in } B_1 \\
\int_{B_1} v^{\frac{\nu(n-1)}{n(n-2)}} < \delta_0
\end{cases}$$  

(2.4)

**Proof.** If the assertion is false, then there exists a sequence $\{v_k\} \subset C^2(B_1)$ such that

$$\begin{cases}
-\Delta v_k = (v_k)^\gamma & \text{in } B_1 \\
\int_{B_1} (v_k)^{\frac{\nu(n-1)}{n(n-2)}} \, dx \leq \frac{1}{k} \\
\max_{\overline{B}_{\nu/4}} v_k \geq k
\end{cases}$$  

(2.5)
For each $k$, we can take $h_k \in C^2(B_1)$ and $y_k \in B_{1/2}$ such that
\begin{equation}
 h_k(y) = \left(\frac{1}{2} - r\right)^q v_k(y), \quad h_k(y_k) = \max_{B_{1/2}} h_k(y), \tag{2.6}
\end{equation}
where $q = \frac{2}{\gamma - 1}$ and $r = |y|$. It follows from (2.5)-(2.6) that
\begin{align*}
 h_k(y_k) &= \left(\frac{1}{2} - r_k\right)^q v_k(y_k) \geq \max_{B_{1/4}} \left(\frac{1}{2} - r\right)^q v_k(y) \\
 &\geq \left(\frac{1}{4}\right)^q \max_{B_{1/4}} v_k(y) \geq \left(\frac{1}{4}\right)^q, \tag{2.7}
\end{align*}
for all $k$, where $r_k = y_k$.

Here, we consider the following function for each $k$: \begin{equation}
 w_k(y) = \mu_k^q v_k(y_k + \mu_k y) \tag{2.8}
\end{equation}
with
\begin{equation}
 \sigma_k = \frac{1}{2} - r_k, \quad d_k^q = h_k(y_k) = \sigma_k^q v_k(y_k), \quad \mu_k = \sigma_k / d_k. \tag{2.9}
\end{equation}
We have
\begin{equation}
 \frac{1}{2} - |y| \geq \frac{1}{2} - (|y_k| + |y - y_k|) = \left(\frac{1}{2} - r_k\right) - |y - y_k| \geq \sigma_k - \frac{\sigma_k}{2} = \frac{\sigma_k}{2}
\end{equation}
for all $y \in B_{\sigma_k/2}(y_k)$, and hence
\begin{equation}
 d_k^q = h_k(y_k) \geq \left(\frac{1}{2} - |y|\right)^q v_k(y) \geq \left(\frac{\sigma_k}{2}\right)^q v_k(y) \tag{2.10}
\end{equation}
for all $y \in B_{\sigma_k/2}(y_k)$.

Noting that the function $w_k = w_k(y)$ defined by (2.8) has the scale invariance, we find
\begin{equation}
 \begin{cases}
 -\Delta w_k = (w_k)^q & \text{in } B_{d_k/2} \\
 \int_{B_{d_k/2}} (w_k)^{\frac{q+1}{q}} \, dx = \int_{B_{\sigma_k/2}(y_k)} (v_k)^{\frac{q+1}{q}} \, dx \leq \frac{1}{k} \\
 w_k(0) = \mu_k^q v_k(y_k) = 1 \\
 w_k \leq 2^q
\end{cases} \quad \text{in } B_{d_k/2} \tag{2.11}
\end{equation}
by using (2.5), (2.9) and (2.10). It is also clear that $d_k \to +\infty$ by (2.7). Thus Lemma 1 and the elliptic regularity guarantee that there exist a subsequence, still denoted by $\{w_k\}$, and $\tilde{w} \in C^2(\mathbb{R}^n)$ such that
\begin{equation}
 w_k \to \tilde{w} \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n), \tag{2.12}
\end{equation}
\begin{equation}
 \begin{cases}
 -\Delta \tilde{w} = 0 & \text{in } \mathbb{R}^n \\
 \tilde{w}(0) = 1 \\
 \tilde{w} \leq 2^q
\end{cases} \quad \text{in } \mathbb{R}^n. \tag{2.13}
\end{equation}
Since $\tilde{w} = \tilde{w}(x)$ is harmonic and bounded from above in $\mathbb{R}^n$ because of (2.13), it holds that
\begin{equation}
 \tilde{w} \equiv 1 \quad \text{in } \mathbb{R}^n
\end{equation}
by Liouville's theorem, see [10], and hence (2.12) shows that $w_k \to 1$ in $C^2_{\text{loc}}(\mathbb{R}^n)$. This contradicts to the second of (2.11). \hfill \blacksquare
**Proposition 1** Any classical solution to (1.1) with $\Omega = \mathbb{R}^n$ is bounded from above.

**Proof.** Let $v = v(x)$ be a classical solution to (1.1) with $\Omega = \mathbb{R}^n$. Then there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} |v_{+}^\gamma|^\frac{n-1}{2} < \delta_0$$

because of the constraint of (1.1), where $\delta_0$ is as in Lemma 2. Therefore it follows that

$$\sup_{\mathbb{R}^n \setminus B_{R+1}} v \leq C_0$$

from Lemma 2, where $C_0$ is a positive constant appeared there. Hence the assertion holds.

By virtue of Proposition 1, operating (1.1) with $(-\Delta)^{-1}$ is justified.

**Lemma 3** There exist positive numbers $c_\gamma$ and $c_\gamma'$ such that any nontrivial classical solution $v = v(x)$ to (1.1) with $\Omega = \mathbb{R}^n$ has the relation

$$v(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |x-y|^{2-n} v_{+}^\gamma(y)dy - c_\gamma$$

Moreover, we have the asymptotic profile

$$v(x) = -c_\gamma + c_\gamma' |x|^{2-n} + o(|x|^{2-n}), \quad |x| \gg 1,$$

and especially the nonnegative part $v_+ = v_+(x)$ has a compact support.

**Proof.** We introduce the function $w = w(x)$ defined by

$$0 \leq w(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |x-y|^{2-n} v_{+}^\gamma(y)dy.$$  

We shall show that (2.16) is well-defined, and that

$$\lim_{|x| \to +\infty} w(x) = 0,$$

It follows that

$$v_+ \in L^s(\mathbb{R}^n) \quad \text{for any } s \in \left[\frac{n(\gamma-1)}{2}, \infty\right],$$

from the constraint of (1.1) and Proposition 1. We fix $R > 0$ and represent $w$ as

$$0 \leq w(x) = \frac{1}{(n-2)\omega_{n-1}} (w_1(x) + w_2(x)),$$

$$w_1(x) = \int_{|y-x| \geq R} |x-y|^{2-n} v_+^\gamma(y)dy, \quad w_2(x) = \int_{|y-x| < R} |x-y|^{2-n} v_+^\gamma(y)dy.$$
Since $\gamma(n-1) \in \left[ \frac{n(\gamma-1)}{2}, \infty \right)$ for $n \geq 3$, we have

$$0 \leq w_2(x) \leq \left( \int_{|z| < R} |z|^{1-n} \right) \left( \int_{|z| < R} v_+^{\gamma(n-1)}(x-z) \right) \frac{R^{n-2}}{R^{n-1}}$$

$$\leq C_2(n, R)\|v_+\|^\gamma_{L^\gamma(B(x,R))} \to 0 \quad \text{as } |x| \to +\infty \quad (2.19)$$

by (2.18). The term $w_1$ is estimated by

$$0 \leq w_1(x) \leq \begin{cases} \frac{R^{2-n}}{R^{n-1}} \int_{|z| \geq R} |z|^{-n} \left( 1 + \frac{|z|^{n-2}}{R^{n-2}} \right) dz & \text{if } \gamma \in \left( 1, \frac{n}{n-2} \right] \\
\frac{2n}{(n-2)n^{n-1}} & \text{if } \gamma \in \left( \frac{n}{n-2}, \frac{n+2}{n-2} \right) \\
\frac{1}{(n-2)\omega_{n-1}^{\gamma-1}} \int_{R^{n}} v_+^\gamma dx & \text{if } \gamma \in \left( \frac{n}{n-2}, \frac{n+2}{n-2} \right) \end{cases} \quad (2.20)$$

Combining (2.18)-(2.20), and noting that $\gamma \in \left[ \frac{n(\gamma-1)}{2}, \infty \right)$ for $\gamma \in \left( 1, \frac{n}{n-2} \right]$, we see that (2.16) is well-defined, and that

$$0 \leq \lim_{|x| \to +\infty} w(x) \leq \begin{cases} C_4(n, \gamma)R^{2-n} & \text{if } \gamma \in \left( 1, \frac{n}{n-2} \right] \\
C_5(n, \gamma)R^{\frac{1}{n-1}} & \text{if } \gamma \in \left( \frac{n}{n-2}, \frac{n+2}{n-2} \right) \end{cases}$$

which implies (2.17) since $R > 0$ is arbitrary.

We have now

$$-\Delta(v-w) = 0 \quad \text{in } \mathbb{R}^n, \quad \sup_{\mathbb{R}^n} (v-w) < +\infty$$

by (2.16) and Proposition 1. Then, Liouville’s theorem, see [10], guarantees that there exists $c_\gamma \in \mathbb{R}^n$ such that $v - w = c_\gamma$. We claim that $c_\gamma < 0$. If this is not the case then

$$-\Delta v = v^\gamma, \quad v \geq 0 \quad \text{in } \mathbb{R}^n,$$

which is impossible because of $1 < \gamma < \frac{n+2}{n-2}$ and the result from [8]. Thus we obtain (2.14) for $c_\gamma = -c_1 > 0$.

It holds by (2.14) and the dominated convergence theorem that

$$|x|^{n-2}(v(x) - c_\gamma) = w(x)$$

$$= \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|x|^{n-2}}{|x-y|^{n-2}} v_+^\gamma(y) dy$$

$$\to \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} v_+^\gamma dx$$

as $|x| \to +\infty$, which implies (2.15) for $c_\gamma = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} v_+^\gamma dx$. 

\textbf{Proof of Theorem 1}: First, we shall show the radial symmetricity of the solution $v = v(x)$ to (1.1) with $\Omega = \mathbb{R}^n$. To show this, we have only to show
that \( w = w(x) \) defined by (2.16) also satisfies the same property. We introduce the function
\[
f(t) = (t - c_\gamma)_+. \tag{2.21}
\]
where \( c_\gamma > 0 \) is a positive constant in (2.14). Then, it holds that
\[
\begin{cases}
-\Delta w = f(w) & \text{in } \mathbb{R}^n \\
w > 0 \\
\lim_{|x| \to +\infty} w(x) = 0
\end{cases}
\tag{2.22}
\]
by virtue of Lemma 3. Noting (2.21) and the asymptotic profile (2.15), we can apply the result from [9] and conclude that the solution \( w = w(x) \) to (2.22) has the desired property. Namely, there exist a point \( x_0 \in \mathbb{R}^n \) and a function \( V = V(r) \) defined on \([0, +\infty)\) such that
\[
\begin{aligned}
v(x) &= V(r), \\
v(x_0) &= V(0) = \sup_{x \in \mathbb{R}^n} v(x), \\
V'(r) &= 0 \quad (\text{for } r > 0),
\end{aligned} \tag{2.23}
\]
where \( r = |x - x_0| \).

We can readily deduce the remainder of the assertions of Theorem 1 from (2.23) and some direct computations. The proof is complete. 

3 Proof of Theorem 2

In this section, we shall assume that \( n \geq 3 \) and \( \gamma \in \left(1, \frac{n+2}{n-2}\right) \), again.

We begin with an a priori bound of the solution to (2.4).

**Lemma 4** For any \( \delta \in (0, \lambda_{\gamma}^*) \), we have a constant \( C_\delta = C_\delta(n, \gamma, \delta) > 0 \) such that
\[
\max_{B_{1/\delta}} v \leq C_\delta \tag{3.1}
\]
for any solution \( v = v(x) \) to (2.4) with \( \delta_0 = \delta \).

**Proof.** Fix \( \delta \in (0, \lambda_{\gamma}^*) \) and suppose that the assertion is false. Then we can discuss as in the proof of Lemma 2 and find that there exists \( w \in C^2(\mathbb{R}^n) \) such that
\[
\begin{cases}
-\Delta w = w^\gamma & \text{in } \mathbb{R}^n \\
\int_{\mathbb{R}^n} w^\gamma \frac{dx}{|x|^{n-1}} \leq \delta < \lambda_{\gamma}^* \\
w(0) = 1 \\
w \leq 2q, \quad q = \frac{\gamma}{\gamma - 1} \text{ in } \mathbb{R}^n,
\end{cases}
\]
which is a contradiction by Theorem 1.

One can see that Theorem 2 is a direct consequence of the following lemma.

**Lemma 5** Let \( T \) be a positive constant. Then we have \( C_1 = C_1(n, \gamma) > 0 \) and \( C_2 = C_2(n, \gamma, T) > 0 \) such that
\[
v(0) + \inf_{B_1} v \leq C_2 \tag{3.2}
\]
for any solution $v = v(x) \in C^2(B_1)$ to
\begin{equation}
\begin{cases}
-\Delta v = v_+^\gamma & \text{in } B_1 \\
\int_{B_1} v_+\frac{\gamma(\gamma-1)}{2}\,dx \leq T.
\end{cases}
\end{equation}

**Proof.** Suppose that the assertion does not hold. Then for any $\hat{C} > 0$, there exists a sequence $\{v_k\} \subset C^2(B_1)$ such that
\begin{equation}
\begin{cases}
-\Delta v_k = (v_k)_+^\gamma & \text{in } B_1 \\
\int_{B_1} (v_k)_+\frac{\gamma(\gamma-1)}{2}\,dx \leq T \\
v_k(0) + \hat{C}\inf_{B_1} v_k \geq k.
\end{cases}
\end{equation}

It is obvious that
\begin{equation}
v_k(0) \geq \frac{k}{1+\hat{C}} \to +\infty
\end{equation}
as $k \to \infty$.

Here, we use $h_k \in C^2(B_1)$, $y_k \in B_{1/2}$, $w_k = w_k(y)$, $\sigma_k$, $d_k$ and $\mu_k$ that are taken in the proof of Lemma 2, see (2.6) and (2.8)-(2.9). Then it holds that
\begin{equation}
d_k \geq (v_k(0))^{1/q} \to +\infty.
\end{equation}

by (3.5). We have also (2.10) for all $y \in B_{d_k/2}(y_k)$, and so
\begin{equation}
w_k \leq 2^q & \text{ in } B_{d_k/2}(y_k).
\end{equation}

Similarly to the proof of Lemma 2, we deduce
\begin{equation}
\begin{cases}
-\Delta w_k = (w_k)_+^\gamma & \text{in } B_{d_k/2} \\
\int_{B_{d_k/2}} (w_k)_+\frac{\gamma(\gamma-1)}{2}\,dx = \int_{B_{d_k/2}(y_k)} (v_k)_+\frac{\gamma(\gamma-1)}{2}\,dx \leq T \\
w_k(0) = 1 \\
w_k \leq 2^q & \text{ in } B_{d_k/2}
\end{cases}
\end{equation}
from (3.4) and (3.7). Therefore, we can extract a subsequence, still denoted by $\{w_k\}$, and a function $\tilde{w} \in C^2(\mathbb{R}^n)$ such that
\begin{equation}
w_k \to \tilde{w} & \text{ in } C^2_{loc}(\mathbb{R}^n),
\end{equation}

\begin{equation}
\begin{cases}
-\Delta \tilde{w} = 0 & \text{in } \mathbb{R}^n \\
\int_{\mathbb{R}^n} \tilde{w}_+\frac{\gamma(\gamma-1)}{2}\,dx \leq T \\
\tilde{w}(0) = 1 \\
\tilde{w} \leq 2^q & \text{ in } \mathbb{R}^n,
\end{cases}
\end{equation}

where we have used (3.6), Lemma 1 and the elliptic regularity.

We may assume $T \geq \lambda_\gamma^*$ thanks to Theorem 1. Noting the third and fourth properties of (3.9), we have (1.4) for some $x_0 \in \mathbb{R}^n$ and $\mu = \mu_0 \in [1,2]$. In particular, it holds that
\begin{equation}
w(0) = 1, \quad \lim_{|x| \to \infty} w(x) \leq -C_3
\end{equation}
for some $C_3 = C_3(n, \gamma) > 0$. Consequently, there exist $C_4 = C_4(n, \gamma) > 0$ and $R = R(n, \gamma) \gg 1$ such that

$$w(0) + C_4 \inf_{\partial B_R} w < 0.$$  
(3.10)

Hence it follows from (3.8) and (3.10) that

$$w_k(0) + C_4 \inf_{\partial B_R} w_k < 0.$$  
(3.11)

for $k \gg 1$.

Noting that $v_k$ is super-harmonic, and that $B(y_k, \mu_k R) \subset B_1$ for $k \gg 1$ by (3.6). Then we obtain

$$v_k(0) + C_4 \inf_{B_1} v_k \leq v_k(y_k) + C_4 \inf_{\partial B(y_k, \mu_k R)} v_k$$

$$= \mu_k^{q-q} \left( w_k(0) + C_4 \inf_{\partial B_R} w_k \right) < 0$$

for $k \gg 1$ by virtue of the scale invariance and (3.11). However, this is contrary to (3.4) if $\hat{C} \geq C_4$, since $v_k(0) > 0$ by (3.4).

**Proof of Theorem 2:** Let $\Omega$ be a bounded domain, fix any positive number $T$ and compact set $K \subset \Omega$, and suppose that $v = v(x)$ is a classical solution to (1.1) and satisfies (1.9). Then we have $\mu_0 = \mu_0(K) > 0$ and $x_0 \in K$ such that

$$\bigcup_{x \in K} B(x, \mu_0) \subset \Omega.$$  

We introduce the function

$$w(x) = \mu_0^q v(x_0 + \mu_0 x)$$

for $x \in B_1$ and $q = \frac{2}{\gamma-1}$. By the scale invariance, it holds that

$$v(x_0) + C \inf_{\Omega} v \leq v(x_0) + C \inf_{B(\mu_0, \mu_0)} v = \mu_0^{-q}(w(0) + C \inf_{B_1} w),$$  
(3.12)

for any $C > 0$, and that $w = w(x)$ satisfies (3.3). Hence Lemma 5 yields $C_5 = C_5(n, \gamma) > 0$ and $C_6 = C_6(n, \gamma, T)$ such that

$$w(0) + C_5 \inf_{B_1} w \leq C_6.$$  
(3.13)

Inequality (1.8) follows from (3.12) and (3.13) as $C_1 = C_5$ and $C_2 = \mu_0^{-q}C_6$.

### 4 Proof of Theorem 3 (Sketch)

In this section, we shall assume that $\gamma \in \left( \frac{n}{n-2} - \frac{\alpha+2}{n-2} \right)$ and $n \geq 3$. Also, we shall denote a subsequence of the sequence by the same notation without notice.

Proof of Theorem 3 is reduced to those of the following two propositions:
Proposition 2  Assume that $\gamma \in \left[ \frac{n-2}{n-2}, \frac{1+2}{n-2} \right)$ and $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$ and $\{v_k\}$ be a sequence of the classical solutions satisfying (1.10) for some $T > 0$. Then there exists a subsequence, still denoted by the same symbol $\{v_k\}$, such that the following alternatives occur:

(i) $\{v_k\}$ is locally uniformly bounded.

(ii) $v_k \to -\infty$ locally uniformly in $\Omega$.

(iii) There exists a finite set $S = \{x_i\}_{i=1}^m$ such that $v_k \to -\infty$ locally uniformly in $\Omega \setminus S$ and that

$$
(v_k)^{\frac{n(n-1)}{2}}_+ \, dx \to \sum_{i=1}^m \alpha_*(x_i)\delta_{x_i}(dx)
$$

in $\mathcal{M}(\Omega)$ with $\alpha_*(x_i) \geq \lambda^*_{\gamma}$ for all $i = 1, \ldots, m$.

Proposition 3  In the alternative (iii) of Proposition 2, it holds that $\alpha_*(x_i) = l_i\lambda^*_{\gamma}$ for some $l_i \in \mathbb{N}$ and for all $i = 1, \ldots, m$.

Proof of Proposition 2: Since $\{(v_k)^{\frac{n(n-1)}{2}}_+\}$ is bounded in $L^1(\Omega)$, there exist a subsequence $\{v_k\}$ and a bounded non-negative measure $\mu$ such that

$$
(v_k)^{\frac{n(n-1)}{2}}_+ \, dx \to \mu \quad \text{in } \mathcal{M}(\Omega),
$$

where $\mathcal{M}(\Omega)$ stands for the space of measure. Set

$$
\Sigma = \{ x \in \Omega \mid \mu(\{x\}) \geq \lambda^*_{\gamma} \}
$$

$$
S = \{ x \in \Omega \mid \text{there exists } x_k \subset \Omega \text{ such that } x_k \to x \text{ and } v_k(x_k) \to +\infty \}.
$$

First, we claim

$$
\Sigma = S. \quad (4.2)
$$

Suppose that $x_0 \notin \Sigma$. Then there exists $0 < r_0 < 1$ such that

$$
\mu(B(x_0, r_0)) < \lambda^*_{\gamma} \quad (4.3)
$$

because of the property of the bounded non-negative measure. Hence we obtain $\delta_0 \in (0, \lambda^*_{\gamma})$ such that

$$
\int_{B(x_0, r_0)} (v_k)^{\frac{n(n-1)}{2}}_+ \, dx \leq \delta_0
$$

for $k \gg 1$ by (4.1) and (4.3). Putting

$$
w_k(x) = r_0^n v_k(x_0 + r_0 x)
$$

for $x \in B_1$ and $q = \frac{2}{\gamma-1}$, we see that $w_k$ satisfies

$$
\begin{cases}
-\Delta w_k = (w_k)^+_q & \text{in } B_1 \\
\int_{B_1} (w_k)^{\frac{n(n-1)}{2}}_+ \, dx \leq \delta_0
\end{cases}
$$
for $k \gg 1$. Consequently, Lemma 4 assures that there exists $C_{b_0} = C_{b_0}(n, \gamma, \delta_0) > 0$ such that
\[ \max_{B(x, r)} u_k \leq C_{b_0} \]
for $k \gg 1$, which implies
\[ \max_{B(x, r_{n/4})} v_k \leq r_0^{-q} C_{b_0} \]
for $k \gg 1$. Thus we have $\mathcal{S} \subseteq \Sigma$. In turn, suppose that $x_0 \notin \mathcal{S}$. From the definition of $\mathcal{S}$, it is clear that there exists $0 < r_0 \ll 1$ such that
\[ \sup_k \| (v_k)_+ \|_{L^\infty(B(x_0, r_0))} < +\infty \]
for some subsequence $\{v_k\}$. Hence we obtain
\[ \lim_{r \to 0} \limsup_{k \to \infty} \int_{B(x_0, r_0)} (v_k)_+ \frac{v_k^{\frac{n+1}{2}}}{2} \, dx = 0. \tag{4.4} \]
We deduce from (4.1) and (4.4) that $\mu(\{x_0\}) = 0$, and therefore $x_0 \notin \Sigma$. Thus we have $\Sigma \subseteq \mathcal{S}$, and hence (4.2).

Next, we shall show that $\mathcal{S} = \emptyset$ implies (i) or (ii). Assume that $\mathcal{S} = \emptyset$ and fix an open set $\omega$ satisfying $\overline{\omega} \subseteq \Omega$. Similarly to the proof of (4.2), we deduce that there exists $C_1 = C_1(n, \gamma, \omega) > 0$ such that
\[ \sup_k \| (v_k)_+ \|_{L^\infty(\omega)} \leq C_1. \tag{4.5} \]
Let $v_{1,k}$ be a solution to
\[
\begin{cases}
-\Delta v_{1,k} = (v_k)_+^2 & \text{in } \omega \\
v_{1,k} = 0 & \text{on } \partial \omega.
\end{cases}
\]
It holds that $v_{1,k} \geq 0$ in $\omega$ by the maximum principle, and that $\{v_{1,k}\}$ is uniformly bounded in $\omega$ because of (4.5) and the elliptic regularity. In other words, there exists $C_2 = C_2(n, \gamma, \omega) > 0$ such that
\[ 0 \leq v_{1,k} \leq C_2 \quad \text{in } \omega. \tag{4.6} \]
Hence $\tilde{v}_k = v_k - v_{1,k}$ is harmonic and bounded from above in $\omega$. Since $\omega$ is arbitrary, we use the Harnack principle to the harmonic function and find that $\{\tilde{v}_k\}$ is locally uniform bounded in $\Omega$, or otherwise $\tilde{v}_k \to -\infty$ locally uniformly in $\Omega$. Noting inequality (4.6), we have (i) or (ii) in each cases.

Finally, we shall show that $\mathcal{S} \neq \emptyset$ implies (iii). Since $\mathcal{S} = \{x_i\}_{i=1}^m$ is finite, we perform the argument similar to above and find that $\{v_k\}$ is bounded in $L^\infty_{\text{loc}}(\Omega \setminus \mathcal{S})$, or otherwise $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$. We now claim that the former does not hold. To show this claim, we suppose the contrary and take $r_1 > 0$ such that $B(x_1, r_1) \cap \mathcal{S} = \{x_1\}$ which is possible by the finiteness of $\mathcal{S}$. Then there exists $C_3 = C_3(n, \gamma, x_1, r_1) > 0$ such that
\[ v_k \geq -C_3 \quad \text{on } \partial B(x_1, r_1). \tag{4.7} \]
Let $z_k$ be a solution to
\[
\begin{cases}
-\Delta z_k = (v_k)_+^{\gamma} & \text{in } B(x_1, r_1) \\
z_k = -C_3 & \text{on } \partial B(x_1, r_1).
\end{cases}
\]

We obtain $z_k \leq v_k$ in $B(x_1, r_1)$, and
\[
z_k(x) dx \to \alpha \delta_1(dx) + f(x) dx
\]
in $\mathcal{M}(\overline{B(x_1, r_1)})$ with
\[
\alpha \geq \lambda_\gamma^* \quad \text{and} \quad 0 \leq f \in L^1(B(x_1, r_1)),
\]
and therefore $z_k \to z$ locally uniformly in $B(x_1, r_1) \setminus \{x_1\}$ with
\[
z(x) \geq \frac{\lambda_\gamma^*}{\omega_{n-1}(n-2)|x-x_1|^{n-2}} - O(1)
\]
for $x \in \overline{B(x_1, r_1)} \setminus \{x_1\}$. Then Fatou's lemma assures
\[
+\infty = \int_{B(x_1, r_1)} z_{k+}^\frac{n(n-1)}{2} dx \leq \lim\inf_k \int_{B(x_1, r_1)} (z_k)_{+}^\frac{n(n-1)}{2} dx
\]
\[
\leq \lim\inf_k \int_{B(x_1, r_1)} (v_k)_{+}^\frac{n(n-1)}{2} dx < +\infty
\]
because of the assumption $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ and the constraint of (1.10). This inequality is a contradiction. Thus we obtain $v_k \to -\infty$ locally uniformly in $\Omega \setminus S$. The proof is complete.

Proof of Proposition 3 is done similarly to [13]. More precisely, it is reduced to the following lemmas.

**Lemma 6** Given $R > 0$, we assume that $v_k = v_k(x)$ satisfies
\[
\begin{align*}
-\Delta v_k &= (v_k)_+^{\gamma} \quad &\text{in } B_R, \\
\max_{B_R} v_k &\to +\infty \quad \text{and} \quad \max_{\partial B_R} v_k &\to -\infty \quad \text{for any } r \in (0, R), \\
\lim_{k \to \infty} \int_{B_R} (v_k)_{+}^{\frac{n(n-1)}{2}} dx &= \alpha \quad \text{for some } \alpha > 0, \\
\sup_k \sup_{x \in B_R} v_k(x) |x|^q &\leq C_4 \quad \text{for some } C_4 > 0.
\end{align*}
\]
where $q = \frac{2}{\gamma-1}$. Then, $\alpha = \lambda_\gamma^*$ and there exist $C_5 = C_5(\cdots) > 0$ and $k_0 \in \mathbb{N}$ such that $v_k \leq 0$ in $\Omega \setminus B_{C_5\delta_k}$ for all $k \geq k_0$ with $\delta_k = \max_{B_R} v_k$.

**Lemma 7** Given $R > 0$, we assume that $v_k = v_k(x)$ satisfies (4.8)-(4.10) and there is $T > 0$, independent of $k$, such that
\[
\int_{B_R} (v_k)_{+}^{\frac{n(n-1)}{2}} dx \leq T
\]
for all $k$. Then, passing to a subsequence, we have $\{x_k^{(j)}\}_{j=0}^{m-1} \subset B_R$. Let $\{\ell_k^{(j)}\}_{j=0}^{m-1} \subset \mathbb{N}$ and $m \in \mathbb{N}$ with $x_k^{(j)} \to 0$, $\ell_k^{(j)} \to \infty$ and $1 \leq m \leq T/\lambda_r^*$ such that the following (4.13)-(4.17) hold:

$$v_k(x_k^{(j)}) = \max_{|x-x_k^{(j)}| \leq \ell_k^{(j)} \delta_k^{(j)}} v_k(x) \to +\infty \quad (4.13)$$

for all $0 \leq j \leq m-1$, 

$$B(x_k^{(j)}, 2\ell_k^{(j)} \delta_k^{(j)}) \cap B(x_k^{(j)} \cdot 2\ell_k^{(j)} \delta_k^{(j)}) = \emptyset \quad (4.14)$$

for all $k$ and $0 \leq i, j \leq m-1$ satisfying $i \neq j$.

$$\frac{\partial}{\partial t} v_k(ty + x_k^{(j)}) \bigg|_{t=1} < 0 \quad (4.15)$$

for all $k$, $0 \leq j \leq m-1$ and $y$ satisfying $2r_k^* \delta_k^{(j)} \leq |y| \leq 2\ell_k^{(j)} \delta_k^{(j)}$.

$$\lim_{k \to \infty} \int_{B(x_k^{(j)}, \ell_k^{(j)} \delta_k^{(j)})} (v_k)_+^{(n-1)} dx = \int_{B(x_k^{(j)}, \ell_k^{(j)} \delta_k^{(j)})} (v_k)_+^{(n-1)} dx = \lambda_r^* \quad (4.16)$$

for all $0 \leq j \leq m-1$. and 

$$\max_{B_R} \left\{ v_k(x) \right\}_{0 \leq j \leq m-1} \min_{m-1} |x-x_k^{(j)}|^q \leq C_6 \quad (4.17)$$

for all $k$ and for some $C_6 > 0$ independent of $k$, where $(\delta_k^{(j)})^q = v_k(x_k^{(j)})$, $q = \frac{2}{\gamma-1}$, and $r_k^*$ is as in Theorem 1.

**Lemma 8** Given $R > 0$, we assume that $v_k = v_k(x)$ satisfies (4.8)-(4.10), (4.12), and that there exist $\{x_k^{(j)}\}_{j=0}^{m-1}$ and $\{\ell_k^{(j)}\}_{j=0}^{m-1}$, $m \geq 1$, $r_k^{(j)} > 0$, such that the following (4.18)-(4.22) hold:

$$v_k(x_k^{(j)}) \to +\infty \quad (4.18)$$

for all $0 \leq j \leq m-1$.

$$\lim_{k \to \infty} \frac{r_k^{(i)}}{\delta_k^{(i)}} = +\infty \quad (4.19)$$

for all $0 \leq j \leq m-1$.

$$B(x_k^{(i)}, r_k^{(i)}) \cap B(x_k^{(j)}, r_k^{(j)}) = \emptyset \quad (4.20)$$

for all $k$ and $0 \leq i, j \leq m-1$ satisfying $i \neq j$.

$$\max_{B_R \setminus \bigcup_{j=0}^{m-1} B(x_k^{(i)}, r_k^{(i)})} \left\{ v_k(x) \right\}_{0 \leq j \leq m-1} \min_{m-1} |x-x_k^{(j)}|^q \leq C_7 \quad (4.21)$$

for all $k$ and for some $C_7 > 0$ independent of $k$, and

$$\lim_{k \to \infty} \int_{B(x_k^{(i)}, 2r_k^{(i)})} (v_k)_+^{(n-1)} dx = \lim_{k \to \infty} \int_{B(x_k^{(i)}, 2r_k^{(i)})} (v_k)_+^{(n-1)} dx = \beta_j \quad (4.22)$$

for some $\beta_j > 0$, $0 \leq j \leq m-1$. Then it holds that

$$\lim_{k \to \infty} \int_{B_R} (v_k)_+^{(n-1)} dx = \sum_{j=0}^{m-1} \beta_j. \quad (4.23)$$
Proposition 3 is obtained by combining Lemmas 6-8. We will be able to find their rigorous proofs in the forthcoming paper.

References


