<table>
<thead>
<tr>
<th>Title</th>
<th>Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent (Variational Problems and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Naito, Yuki; Sato, Tokushi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1671: 91-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141165">http://hdl.handle.net/2433/141165</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

Yūki Naito\textsuperscript{a} and Tokushi Sato\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Ehime University, Matsuyama 790-8577, Japan
\textsuperscript{b} Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$ with $N \geq 3$. We consider the existence of multiple positive solutions of the following semilinear elliptic equations

\begin{equation}
\begin{cases}
-\Delta u + \kappa u = u^p + \lambda f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\kappa \in \mathbb{R}$, $\lambda > 0$ are parameters, $p$ is the critical Sobolev exponent $p = (N+2)/(N-2)$, and $f(x)$ is a non-homogeneous perturbation satisfying

\begin{equation}
f \in H^{-1}(\Omega), \quad f \geq 0, \quad f \not\equiv 0 \quad \text{a.e. in } \Omega.
\end{equation}

Since $p$ is a critical Sobolev exponent for which the embedding $W^{1,2}(\Omega) \subset L^{2N/(N-2)}(\Omega)$ is not compact, we encounter serious difficulties in applying variational methods to the problem (1.1).

Let us recall the results for the case $f \equiv 0$:

\begin{equation}
\begin{cases}
-\Delta u + \kappa u = u^p & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

In this case, by using the Pohozaev identity, it can be shown that (1.3) admits no nontrivial solutions for each $\kappa \geq 0$, provided that $\Omega$ is star-shaped. On the other hand, Brezis and Nirenberg [1] obtained the following results when $\kappa < 0$: let $\kappa_1$ be the first eigenvalue of $-\Delta$ with zero Dirichlet condition on $\Omega$; then

(i) if $N \geq 4$, then for every $\kappa \in (-\kappa_1, 0)$, there exists a positive solution;

(ii) if $N = 3$ and $\Omega$ is a ball, then there exists a positive solution if and only if $\kappa \in (-\kappa_1, -\kappa_1/4)$. 

Let us consider the case where $f$ satisfies (1.2). Tarantello [6] considered the problem with $\kappa = 0$;

$$
\begin{cases}
-\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

(1.4)

and showed that (1.4) has at least two positive solution if $\lambda$ is small enough. The main idea is to divide the Nehari manifold $\Lambda = \{u \in H_0^1(\Omega) : \langle I'(u), u \rangle = 0\}$ into three parts $\Lambda^+, \Lambda^-$ and $\Lambda_0$, and to use the Ekeland principle to get one solution for $\Lambda^+$ and another solution for $\Lambda^-$. We note here that no positive solution exists if $\lambda$ is sufficiently large.

The existence of two nontrivial solutions for more general problem

$$
\begin{cases}
-\Delta u = u^p + g(x, u) + \lambda f(x) & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $g(x, u)$ is a suitable lower-order perturbation of $u^p$, was proved by Cao and Zhou [2]. These achievements have been extended to the $p$-Laplace equation by Chabrouski [3] and Zhou [7], and to more general problems by Squassina [5].

In this paper we will consider the problem (1.1) with $\kappa \in \mathbb{R}$ in the case where $f$ satisfies (1.2), and show that, when $\kappa > 0$, the situation is drastically different between the cases $N = 3, 4, 5$ and $N \geq 6$.

We call a positive minimal solution $u_\lambda$ of (1.1)$_\lambda$, if $u_\lambda$ satisfies $u_\lambda \leq u$ in $\Omega$ for any positive solution $u$ of (1.1)$_\lambda$. Our main results are stated as following theorems.

**Theorem 1.** Assume that $\kappa > -\kappa_1$. Then there exists $\overline{\lambda} \in (0, \infty)$ such that

(i) if $0 < \lambda < \overline{\lambda}$ then the problem (1.1)$_\lambda$ has a positive minimal solution $u_\lambda \in H_0^1(\Omega)$.

Furthermore, if $0 < \lambda < \overline{\lambda}$ then $u_\lambda < u$ a.e. in $\Omega$.

(ii) if $\lambda > \overline{\lambda}$ then the problem (1.1) has no positive solution $u \in H_0^1(\Omega)$.

**Remark.** There is no positive solution of (1.1) with $\kappa \leq -\kappa_1$. Assume to the contrary that there exists a positive solution $u$ of (1.1) with $\kappa \leq -\kappa_1$. Let $\phi_1$ be the eigenfunction of $-\Delta$ corresponding to $\kappa_1$ with $\phi_1 > 0$ on $\Omega$. Then we have

$$
0 = \int_{\Omega} \nabla u \cdot \nabla \phi_1 - \kappa_1 u \phi_1 dx \geq \int_{\Omega} \nabla u \cdot \nabla \phi_1 + \kappa u \phi_1 dx = \int_{\Omega} u^p \phi_1 + \lambda f \phi_1 dx > 0.
$$

This is a contradiction.

We consider the existence of the solutions of (1.1) at the extremal value $\lambda = \overline{\lambda}$, so called extremal solutions.
Theorem 2. Let $\kappa > -\kappa_1$. If $\lambda = \overline{\lambda}$ then the problem (1.1) has a unique positive solution in $H_0^1(\Omega)$.

Next, let us consider the existence and nonexistence of second positive solutions to (1.1) for $0 < \lambda < \overline{\lambda}$.

Theorem 3. Assume that either (i) or (ii) holds.

(i) $\kappa \in (-\kappa_1, 0]$ and $N \geq 3$;  
(ii) $\kappa > 0$ and $N = 3, 4, 5$.

If $0 < \lambda < \overline{\lambda}$ then (1.1) has a unique positive solution $u_{\lambda} \in H_0^1(\Omega)$ satisfying $u_{\lambda} > u_\lambda$.

Theorem 4. Assume that $\kappa > 0$ and $N \geq 6$.

(i) There exists $\lambda^* = \lambda^*(\kappa) \in (0, \overline{\lambda})$ such that if $\lambda^* < \lambda < \overline{\lambda}$ then the problem (1.1) has a positive solution $\overline{u}_{\lambda} \in H_0^1(\Omega)$ satisfying $\overline{u}_{\lambda} > u_\lambda$.

(ii) Let $\Omega = \{x \in \mathbb{R}^N : |x| < B\}$ with some $B > 0$, and let $f = f(|x|)$ be radially symmetric about the origin. Assume that $f \in C^{\alpha}([0, B])$ with some $0 < \alpha < 1$, and $f(r)$ is nonincreasing in $r \in (0, R)$. Then there exists $\lambda^*_1 \in (0, \lambda^*)$ such that (1.1) has a unique positive solution $u_{\lambda}$ for $\lambda \in (0, \lambda^*_1]$.

In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of (1.1) to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of (1.1) at $\lambda = \overline{\lambda}$.

In order to find a second positive solution of (1.1), we introduce the problem

(1.5) \[-\Delta \psi + \kappa \psi = (\psi + \overline{u}_{\lambda})^p - \overline{u}_{\lambda}^p \quad \text{in} \quad \Omega, \quad \psi \in H_0^1(\Omega),\]

where $u_{\lambda}$ is the minimal positive solution of (1.1) for $\lambda \in (0, \overline{\lambda})$ obtained in Theorem 1. In fact, assume that (1.5) has a positive solution $\psi$, and put $\overline{u}_{\lambda} = \psi + u_{\lambda}$. Then $\overline{u}_{\lambda} \in H_0^1(\Omega)$ and solves (1.1) and satisfies $\overline{u}_{\lambda} > u_{\lambda}$ in $\Omega$. In the proof of Theorem 3, we will show the existence of solutions of (1.5) by using a variational method. To this end we define the corresponding variational functional of (1.5) by

$$I_{\kappa}(v) = \frac{1}{2} \int_{\Omega} (|
abla v|^2 + \kappa v^2) \, dx - \int_{\mathbb{R}^N} G(v, u_{\lambda}) \, dx$$

for $v \in H_0^1(\Omega)$, where

$$G(t, s) = \frac{1}{p+1} (t_+ + s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t_+.$$ 

It is easy to see that $I_{\kappa} : H_0^1(\Omega) \to \mathbb{R}$ is $C^1$ and the critical point $v_0 \in H_0^1(\Omega)$ satisfies

$$\int_{\Omega} (\nabla v_0 \cdot \nabla \psi + \kappa v_0 \psi + g(v_0, u_{\lambda}) \psi) \, dx = 0$$
for any $\psi \in H^1_0(\Omega)$, where
\[ g(t,s) = (t_+ + s)^p - s^p. \]
Denote by $S$ the best Sobolev constant of the embedding $H^1_0(\Omega) \subset L^{p+1}(\Omega)$, which is given by
\[ S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{(\int_{\Omega} |u|^{p+1} \, dx)^{2/(p+1)}}. \]
We will obtain Theorem 3 as a consequence of the following two propositions.

**Proposition 5.** Let $\lambda \in (0, \lambda^*)$. Assume that there exists $v_0 \in H^1_0(\Omega)$ with $v_0 \geq 0$, $v_0 \neq 0$ such that
\[ \sup_{t > 0} I_\kappa(tv_0) < \frac{1}{N} S^{N/2}. \]
Then there exists a positive solution $\nu \in H^1_0(\Omega)$ of (1.5).

**Proposition 6.** Assume that either (i) or (ii) holds.

(i) $\kappa \in (-\kappa_1, 0]$ and $N \geq 3$;  
(ii) $\kappa > 0$ and $N = 3, 4, 5$.

Then there exists a positive function $v_0 \in H^1_0(\Omega)$ such that (1.6) holds.

In the proof of Proposition 5, we will derive some estimates to establish inequalities relating certain minimizing sequences. In order to prove Proposition 6, for $\epsilon > 0$, we will set
\[ u_\epsilon(x) = \frac{\phi(x)}{(\epsilon + |x|^2)^{(N-2)/2}}. \]
where $\phi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, is a cut off function, and will show that (1.6) holds with $v_0 = u_\epsilon$ for sufficiently small $\epsilon > 0$.

In the proof of Theorem 4 (ii), we will verify the nonexistence of positive solutions of (1.5) in the radial case by the Pohozaev type argument for the associated ODE. In fact, by [4], the solution $v$ of (1.5) must be radially symmetric, and $v = v(r)$, $r = |x|$, satisfies the problem of the following ordinary differential equation
\[
\begin{cases}
(r^{N-1}v_r)_r - \kappa r^{N-1}v + r^{N-1}g(v, u_\lambda) = 0, & 0 < r < R, \\
v_r(0) = v(R) = 0.
\end{cases}
\tag{1.7}
\]
For the solution $v$ to (1.7), we will obtain the following Pohozaev type identity:
\[
\int_0^R r^{N-1} \left[ \frac{2N}{N-2} G(u, u_\lambda) - g(u, u_\lambda) u \right] \, dr + \frac{2}{N-2} \int_0^R r^N G_s(u, u_\lambda) u_\lambda \, dr \\
+ \frac{2\kappa}{N-2} \int_0^\infty r^{N-1} u^2 \, dr = \frac{1}{N-2} R^N v_r(R)^2.
\]
In the proofs of Theorems 2, 3 and 4, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions

$$-\Delta \phi + \phi = \mu p(u_{\lambda})^{p-1}\phi \quad \text{in} \quad \Omega, \quad \phi \in H^{1}_{0}(\Omega).$$

play a crucial role.

References


