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<th>Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent (Variational Problems and Related Topics)</th>
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<tr>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1671: 91-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141165">http://hdl.handle.net/2433/141165</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

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Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary \( \partial \Omega \) with \( N \geq 3 \). We consider the existence of multiple positive solutions of the following semilinear elliptic equations

\[
\begin{cases}
-\Delta u + \kappa u = u^p + \lambda f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \kappa \in \mathbb{R}, \lambda > 0 \) are parameters, \( p \) is the critical Sobolev exponent \( p = (N+2)/(N-2) \), and \( f(x) \) is a non-homogeneous perturbation satisfying

\[
f \in H^{-1}(\Omega), \quad f \geq 0, \quad f \not\equiv 0 \quad \text{a.e. in } \Omega.
\]

Since \( p \) is a critical Sobolev exponent for which the embedding \( W^{1,2}(\Omega) \subset L^{2N/(N-2)}(\Omega) \) is not compact, we encounter serious difficulties in applying variational methods to the problem (1.1).

Let us recall the results for the case \( f \equiv 0 \):

\[
\begin{cases}
-\Delta u + \kappa u = u^p & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

In this case, by using the Pohozaev identity, it can be shown that (1.3) admits no nontrivial solutions for each \( \kappa \geq 0 \), provided that \( \Omega \) is star-shaped. On the other hand, Brezis and Nirenberg \cite{1} obtained the following results when \( \kappa < 0 \): let \( \kappa_1 \) be the first eigenvalue of \( -\Delta \) with zero Dirichlet condition on \( \Omega \); then

(i) if \( N \geq 4 \), then for every \( \kappa \in (-\kappa_1, 0) \), there exists a positive solution;

(ii) if \( N = 3 \) and \( \Omega \) is a ball, then there exists a positive solution if and only if \( \kappa \in (-\kappa_1, -\kappa_1/4) \).
Let us consider the case where \( f \) satisfies (1.2). Tarantello [6] considered the problem with \( \kappa = 0; \)

\[
\begin{cases}
-\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{1.4}
\]

and showed that (1.4) has at least two positive solution if \( \lambda \) is small enough. The main idea is to divide the Nehari manifold \( \Lambda = \{u \in H_0^1(\Omega) : \langle I'(u), u \rangle = 0 \} \) into three parts \( \Lambda^+, \Lambda^- \) and \( \Lambda_0 \), and to use the Ekeland principle to get one solution for \( \Lambda^+ \) and another solution for \( \Lambda^- \). We note here that no positive solution exists if \( \lambda \) is sufficiently large.

The existence of two nontrivial solutions for more general problem

\[
\begin{cases}
-\Delta u = u^p + g(x, u) + \lambda f(x) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( g(x, u) \) is a suitable lower-order perturbation of \( u^p \), was proved by Cao and Zhou [2]. These achievements have been extended to the \( p \)-Laplace equation by Chabrouski [3] and Zhou [7], and to more general problems by Squassina [5].

In this paper we will consider the problem (1.1) with \( \kappa \in \mathbb{R} \) in the case where \( f \) satisfies (1.2), and show that, when \( \kappa > 0 \), the situation is drastically different between the cases \( N = 3, 4, 5 \) and \( N \geq 6 \).

We call a positive minimal solution \( u_\lambda \) of (1.1), if \( u_\lambda \) satisfies \( u_\lambda \leq u \) in \( \Omega \) for any positive solution \( u \) of (1.1). Our main results are stated as following theorems.

**Theorem 1.** Assume that \( \kappa > -\kappa_1 \). Then there exists \( \overline{\lambda} \in (0, \infty) \) such that

(i) if \( 0 < \lambda < \overline{\lambda} \) then the problem (1.1) has a positive minimal solution \( u_\lambda \in H_0^1(\Omega) \).

Furthermore, if \( 0 < \lambda < \lambda < \overline{\lambda} \) then \( u_\lambda < u_\lambda \) a.e. in \( \Omega \);

(ii) if \( \lambda > \overline{\lambda} \) then the problem (1.1) has no positive solution \( u \in H_0^1(\Omega) \).

**Remark.** There is no positive solution of (1.1) with \( \kappa \leq -\kappa_1 \). Assume to the contrary that there exists a positive solution \( u \) of (1.1) with \( \kappa \leq -\kappa_1 \). Let \( \phi_1 \) be the eigenfunction of \( -\Delta \) corresponding to \( \kappa_1 \) with \( \phi_1 > 0 \) on \( \Omega \). Then we have

\[
0 = \int_\Omega \nabla u \cdot \nabla \phi_1 - \kappa_1 u \phi_1 dx \geq \int_\Omega \nabla u \cdot \nabla \phi_1 + \kappa u \phi_1 dx = \int_\Omega u^p \phi_1 + \lambda f \phi_1 dx > 0.
\]

This is a contradiction.

We consider the existence of the solutions of (1.1) at the extremal value \( \lambda = \overline{\lambda} \), so called extremal solutions.
Theorem 2. Let $\kappa > -\kappa_1$. If $\lambda = \overline{\lambda}$ then the problem (1.1) has a unique positive solution in $H_0^1(\Omega)$.

Next, let us consider the existence and nonexistence of second positive solutions to (1.1) for $0 < \lambda < \overline{\lambda}$.

Theorem 3. Assume that either (i) or (ii) holds.

(i) $\kappa \in (-\kappa_1, 0]$ and $N \geq 3$; (ii) $\kappa > 0$ and $N = 3, 4, 5$.

If $0 < \lambda < \overline{\lambda}$ then (1.1) has a unique positive solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > u_\lambda$.

Theorem 4. Assume that $\kappa > 0$ and $N \geq 6$.

(i) There exists $\lambda^* = \lambda^*(\kappa) \in (0, \overline{\lambda})$ such that if $\lambda^* < \lambda < \overline{\lambda}$ then the problem (1.1) has a positive solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > u_\lambda$.

(ii) Let $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$ with some $R > 0$, and let $f = f(|x|)$ be radially symmetric about the origin. Assume that $f \in C^\alpha([0, R])$ with some $0 < \alpha < 1$, and $f(r)$ is nonincreasing in $r \in (0, R)$. Then there exists $\lambda_0 \in (0, \lambda^*)$ such that (1.1) has a unique positive solution $u_\lambda$ for $\lambda \in (0, \lambda_0]$.

In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of (1.1) to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of (1.1) at $\lambda = \overline{\lambda}$.

In order to find a second positive solution of (1.1), we introduce the problem

(1.5) \[-\Delta v + \kappa v = (v + u_\lambda)^p - \overline{u}_\lambda^p \text{ in } \Omega, \quad v \in H_0^1(\Omega),\]

where $u_\lambda$ is the minimal positive solution of (1.1) for $\lambda \in (0, \overline{\lambda})$ obtained in Theorem 1. In fact, assume that (1.5) has a positive solution $v$, and put $\overline{u}_\lambda = v + u_\lambda$. Then $\overline{u}_\lambda \in H_0^1(\Omega)$ and solves (1.1) and satisfies $\overline{u}_\lambda > u_\lambda$ in $\Omega$. In the proof of Theorem 3, we will show the existence of solutions of (1.5) by using a variational method. To this end we define the corresponding variational functional of (1.5) by

$$I_\kappa(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + \kappa v^2) dx - \int_{\mathbb{R}^N} G(v, u_\lambda) dx$$

for $v \in H_0^1(\Omega)$, where

$$G(t, s) = \frac{1}{p+1}(t_+ + s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t_+.$$

It is easy to see that $I_\kappa : H_0^1(\Omega) \to \mathbb{R}$ is $C^1$ and the critical point $v_0 \in H_0^1(\Omega)$ satisfies

$$\int_\Omega (\nabla v_0 \cdot \nabla \psi + \kappa v_0 \psi + g(v_0, u_\lambda) \psi) dx = 0.$$
for any \( \psi \in H^1_0(\Omega) \), where
\[ g(t, s) = (t_+ + s)^p - s^p. \]

Denote by \( S \) the best Sobolev constant of the embedding \( H^1_0(\Omega) \subset L^{p+1}(\Omega) \), which is given by
\[ S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\left( \int_{\Omega} |u|^{p+1} \, dx \right)^{2/(p+1)}}. \]

We will obtain Theorem 3 as a consequence of the following two propositions.

**Proposition 5.** Let \( \lambda \in (0, \lambda^*) \). Assume that there exists \( \nu_0 \in H^1_0(\Omega) \) with \( \nu_0 \geq 0 \), \( \nu_0 \neq 0 \) such that
\[ \sup_{t > 0} I_\kappa(t \nu_0) < \frac{1}{N} S^{N/2}. \]
Then there exists a positive solution \( \nu \in H^1_0(\Omega) \) of (1.5).

**Proposition 6.** Assume that either (i) or (ii) holds.

(i) \( \kappa \in (-\kappa_1, 0] \) and \( N \geq 3 \);   
(ii) \( \kappa > 0 \) and \( N = 3, 4, 5 \).

Then there exists a positive function \( \nu_0 \in H^1_0(\Omega) \) such that (1.6) holds.

In the proof of Proposition 5, we will derive some estimates to establish inequalities relating certain minimizing sequences. In order to prove Proposition 6, for \( \varepsilon > 0 \), we will set
\[ u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}}. \]
where \( \phi \in C^\infty_0(\mathbb{R}^N) \), \( 0 \leq \phi \leq 1 \), is a cut off function, and will show that (1.6) holds with \( \nu_0 = u_\varepsilon \) for sufficiently small \( \varepsilon > 0 \).

In the proof of Theorem 4 (ii), we will verify the nonexistence of positive solutions of (1.5) in the radial case by the Pohozaev type argument for the associated ODE. In fact, by [4], the solution \( \nu \) of (1.5) must be radially symmetric, and \( \nu = \nu(r) \), \( r = |x| \), satisfies the problem of the following ordinary differential equation
\[ \begin{cases} 
(r^{N-1}\nu_r)_r - \kappa r^{N-1}\nu + r^{N-1}g(\nu, u_\lambda) = 0, & 0 < r < R, \\
\nu_r(0) = \nu(R) = 0.
\end{cases} \]

(1.7)

For the solution \( \nu \) to (1.7), we will obtain the following Pohozaev type identity:
\[ \int_0^R r^{N-1} \left[ \frac{2N}{N-2} G(u, u_\lambda) - g(u, u_\lambda) u \right] \, dr + \frac{2}{N-2} \int_0^R r^N G_s(u, u_\lambda) u'_\lambda \, dr \\
+ \frac{2\kappa}{N-2} \int_0^\infty r^{N-1} u^2 \, dr = \frac{1}{N-2} R^N \nu_r(R)^2. \]
In the proofs of Theorems 2, 3 and 4, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions

$$-\Delta \phi + \phi = \mu p(u_\lambda)^{p-1}\phi \text{ in } \Omega. \quad \phi \in H^1_0(\Omega).$$

play a crucial role.

References


