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Concentration and Stability of standing waves of nonlinear Schrödinger equation with inhomogeneous nonlinearity

Masaya Maeda

Department of Mathematics, Graduate School of Science, Kyoto University, Sakyo-ku Kyoto, 606-8502, Japan

1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation with inhomogeneous nonlinearity.

\[ iu_t = -\Delta u - b(x)|u|^{p-1}u, \quad (x, t) \in \mathbb{R}^{N+1}, \tag{1.1} \]

where \( N \geq 1, \) \( u : \mathbb{R}^{N+1} \to \mathbb{C} \) is an unknown function, \( p \in (1, 1 + 4/N) \) and \( b(x) \) is a smooth function which satisfies

\[ 0 < \inf_{x \in \mathbb{R}^N} b(x) = \lim_{|x| \to \infty} b(x) \leq \sup_{x \in \mathbb{R}^N} b(x) = 1. \]

A standing wave is a solution of equation (1.1) with the form \( u(x, t) = e^{i\omega t}\phi(x) \). In this case, \( \phi \) satisfies the following partial differential equation.

\[ -\Delta \phi + \omega \phi - b(x)|\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N. \tag{1.2} \]

The flow of equation (1.1) conserves the \( L^2 \)-norm and the following functional, which we call the energy.

\[ \mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}} |\nabla u|^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}} b(x)|u|^{p+1} \, dx. \]

The well-posedness of equation (1.1) is well known. See for example [2].

**Proposition 1.** For every \( u_0 \in H^1(\mathbb{R}^N) \), there exists a solution \( u \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \) of (1.1) such that

(a) \( u(x, 0) = u_0(x) \) for \( x \in \mathbb{R}^N \).

(b) \( \mathcal{E}(u(t)) = \mathcal{E}(u_0), \quad ||u(t)||_{L^2} = ||u_0||_{L^2} \) for \( t \in \mathbb{R} \).

Equation (1.1) appears in various regions of physics such as nonlinear optics, plasma physics and Bose-Einstein condensation (BEC). In the context of BEC, the ground states are considered to describe the physical properties of Bose gas in low temperature. Here, a ground state is a standing wave which minimizes the energy functional \( \mathcal{E} \) under the constraint of the \( L^2 \)-norm. Note that by the
Lagrange multiplier method, the ground state satisfies (1.2) for some $\omega \in \mathbb{R}$.
For the case $b \equiv 1$, it is known that the ground state is unique ([5, 9]), and if $1 < p < 1 + 4/N$, it is stable ([1]). For the case $b = |x|^{-\beta}$, $\beta \in (0, 2)$, $N \geq 3$, it is proved that the ground state is stable ([4]).

We now state the notations.

**Definition 1.** Set

\[ G_\alpha := \{ u \in H^1(\mathbb{R}^N) \mid ||u||_{L^2} = \alpha, \ E(u) = E_\alpha \}, \]

where

\[ E_\alpha = \inf \{ E(v) \mid v \in H^1(\mathbb{R}^N), \ ||v||_{L^2} = \alpha \}. \]

In this paper, we call the elements of $G_\alpha$, the ground states.

For the case, $b$ is a radial symmetric function, we can consider a minimizer of $E$ under the constraint $u \in H^1_r(\mathbb{R}^N)$ and $||u||_{L^2} = \alpha$, where

\[ H^1_r(\mathbb{R}^N) := \{ u \in H^1(\mathbb{R}^N) \mid u \text{ is radially symmetric} \}. \]

**Definition 2.** Set

\[ G_{\alpha,r} := \{ u \in H^1_r(\mathbb{R}^N) \mid ||u||_{L^2} = \alpha, \ E(u) = E_{\alpha,r} \}, \]

where

\[ E_{\alpha,r} = \inf \{ E(v) \mid v \in H^1_r(\mathbb{R}^N), \ ||v||_{L^2} = \alpha \}. \]

In this paper, we call the elements of $G_{\alpha,r}$, the radial minimizers.

We investigate the concentration and stability of ground states and radial minimizers.

**Definition 3.** We say that the $G_\alpha$ (resp. $G_{\alpha,r}$) concentrates for sufficiently large $\alpha$ if the elements of $G_\alpha$ ($G_{\alpha,r}$) satisfies the following: For arbitrary $\varepsilon > 0$, there exists an $\alpha_\varepsilon > 0$ such that for every $\alpha > \alpha_\varepsilon$ and every $\phi \in G_\alpha$ ($G_{\alpha,r}$), there exists $y_{\alpha,\phi} \in \mathbb{R}^N$ such that

\[ \int_{|x-y_{\alpha,\phi}| > \varepsilon} |\phi|^2 \ dx < \varepsilon \int_{\mathbb{R}^N} |\phi|^2 \ dx = \varepsilon \alpha^2. \]

We call $y_{\alpha,\phi} \in \mathbb{R}^N$, the concentration center.

**Definition 4.** We say that $G_\alpha$ (resp. $G_{\alpha,r}$) is stable if the following property is satisfied: For arbitrary $\varepsilon > 0$, there exists an $\delta_\varepsilon > 0$ such that for every $u_0 \in H^1$ with

\[ \inf_{v \in G_\alpha(G_{\alpha,r})} ||u_0 - v||_{H^1} < \delta_\varepsilon, \]

the solution of equation (1.1) with $u(0) = u_0$ satisfies

\[ \sup_{t > 0} \inf_{v \in G_\alpha(G_{\alpha,r})} ||u(t) - v||_{H^1} < \varepsilon. \]

If $G_\alpha$ ($G_{\alpha,r}$) is not stable, we say $G_\alpha$ ($G_{\alpha,r}$) is unstable.
The existence, concentration and stability of $G_\alpha$ is well known.

**Proposition 2.** For $\alpha > 0$, $G_\alpha \neq \emptyset$ and $G_\alpha$ is stable. Further, $G_\alpha$ concentrates for sufficiently large $\alpha$ and the concentration center converges to some maximum point of $b$.

**Remark 1.** For the existence of ground states, see Proposition 8.3.6 of [2]. For the stability result, see [1] and for the concentration result, see [13].

The purpose of this paper is to investigate the stability and concentration for the elements of $G_{\alpha,r}$.

**Proposition 3.** Let $b$ radially symmetric. Then for $\alpha > 0$, we have $G_\alpha \neq \emptyset$.

**Remark 2.** Proposition 3 can be proved as the existence of ground states.

We first study the case $N \geq 2$.

**Theorem 1.** Let $N \geq 2$. Then $G_\alpha$ concentrates for sufficiently large $\alpha$ and the concentration center is 0. Further, if $0$ is a nondegenerate minimum point (resp. maximum point), then for sufficiently large $\alpha > 0$, $G_{\alpha,r}$ is stable (unstable).

Thus, we see that the concentration result holds but the stability result some times fails for the case of radial minimizers. For the case $N = 1$, we see that also the concentration result sometimes fails.

**Theorem 2.** Let $N = 1$.

(i) If $1 \geq b(0) > 2^{-(p-1)/2}$, then $G_{\alpha,1}$ concentrates for sufficiently large $\alpha$ and the concentration center is 0. Further, if $0$ is a nondegenerate minimum point (resp. maximum point), then for sufficiently large $\alpha > 0$, $G_{\alpha,r}$ is stable (unstable).

(ii) If $0 < b(0) < 2^{-(p-1)/2}$, then $G_\alpha$ is unstable and does not concentrate for sufficiently large $\alpha$.

The plan of this paper is as follows. In section 2, we rescale our problem. In section 3 and 4, we prove Theorems 1 and 2 respectively. The proof of the concentration result of Theorem 1 relies on the radial lemma due to Strauss [14]. For the proof of the concentration result of Theorem 2, we use the concentration compactness method due to Lions [10, 11]. For the stability result, we use the abstract theory developed by Grillakis, Shatah and Strauss [7] and for the instability result, we use the result of [12] for $N \geq 2$ and [6] for the case $N = 1$.

## 2 Preliminary

We rescale our problem. Take $\phi \in H^1_0(\mathbb{R}^N)$ with $||\phi||_{L^2} = 1$. Then, we have

$$\mathcal{E}(\alpha \phi) = \alpha^2 \left( \frac{1}{2} \int_{\mathbb{R}} |\nabla \phi|^2 \, dx - \frac{\alpha^{p-1}}{p+1} \int_{\mathbb{R}} b(x)|\phi|^{p+1} \, dx \right).$$

Next, set $\phi_\alpha(x) = \alpha^{AN/2} \phi(\alpha^A x)$, where $A = \frac{2(p-1)}{4-N(p-1)}$. Then, we have

$$\mathcal{E}_\alpha(\alpha \phi_\alpha) = \alpha^{2+2A} \left( \frac{1}{2} \int_{\mathbb{R}} |\nabla \phi|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}} b(\alpha^{-A} x)|\phi|^{p+1} \, dx \right).$$
Therefore, we set
\[ I_{\alpha}(\phi) := \frac{1}{2} \int_{\mathbb{R}} |\nabla \phi|^{2} \, dx - \frac{1}{p+1} \int_{\mathbb{R}} b(\alpha^{-A} x) |\phi|^{p+1} \, dx, \]
and
\[ I_{\alpha,r} := \{ \phi \in H^{1}_{r}(\mathbb{R}^{N}) \mid ||\phi||_{L^{2}} = 1, \, I_{\alpha}(\phi) = \inf_{\|\psi\|_{L^{2}} = 1, \psi \in H^{1}_{r}(\mathbb{R}^{N})} I_{\alpha}(\psi) \}. \]
Thus, we obtain
\[ G_{\alpha,r} = \{ \alpha \phi_{\alpha} \mid \phi \in I_{\alpha,r} \}. \]
We also define the following functional:
\[ I_{\infty,b}(\phi) := \frac{1}{2} \int_{\mathbb{R}} |\nabla \phi|^{2} \, dx - \frac{b}{p+1} \int_{\mathbb{R}} |\phi|^{p+1} \, dx. \]
Then, it is well known that there exists a unique positive radial minimizer \( \psi_{b,\beta} \) of \( I_{\infty,b} \) under the constraint \( ||\phi||_{L^{2}}^{2} = \beta \). That is
\[ I_{\infty,r,b,\beta} := \left\{ \phi \in H^{1}_{r}(\mathbb{R}^{N}) \mid ||\phi||_{L^{2}}^{2} = \beta, \, I_{\infty,b}(\phi) = \inf_{\|\psi\|_{L^{2}} = \beta, \psi \in H^{1}_{r}} I_{\infty,b}(\psi) \right\} = \{ c\psi_{b,\beta} \mid c \in \mathbb{C}, \, |c| = 1 \}. \]

**Remark 3.** The uniqueness of positive radial solution of equation (1.2) in the case \( b(x) \equiv b > 0 \) is proved by Kwong [9]. Further, letting \( \phi_{b,\omega} \) be the unique positive radial solution of equation (1.2) in the case \( b(x) \equiv b > 0 \), we have \( \phi_{b,\omega}(x) = \omega^{1/2} \phi_{b}(\omega^{1/2} x) \), where \( \phi_{b} \) is the unique positive radial solution of
\[-\Delta \phi_{b} + \phi_{b} - b\phi_{b}^{p} = 0, \, x \in \mathbb{R}^{N}.\]
Therefore, we see \( \frac{d}{dx} ||\phi_{b,\omega}||_{L^{2}}^{2} > 0 \) for \( 1 < p < 1 + 4/N \). This implies the uniqueness of the radial minimizer up to constant phase.

We now calculate the value
\[ I_{\infty,b}(\psi_{b,\beta}) = \inf \{ I_{\infty,b}(\phi) \mid \phi \in H^{1}_{r}(\mathbb{R}^{N}), \, ||\phi||_{L^{2}}^{2} = \beta \}. \]

**Lemma 1.** Let
\[ J_{\infty} = \inf_{||u||_{L^{2}} = 1} I_{\infty,1}(u) = I_{\infty,1}(\psi_{1,1}) < 0. \]
Then
\[ I_{\infty,b}(\psi_{b,\beta}) = b^{2A_{\beta}+1} J_{\infty}, \]
where \( A = \frac{2(p-1)}{4-N(p-1)} > 0. \)
Proof.

\[ I_{\infty,b}(\psi_{b,\beta}) = \inf_{\phi \in H^{1}_{\infty}:||\phi||_{L^{2}}=1} \left( \frac{1}{2} \int_{\mathbb{R}} |\nabla \phi|^2 \, dx - \frac{b}{p+1} \int_{\mathbb{R}} |\phi|^{p+1} \, dx \right) \]

Now, setting \( \phi(x) = (b\beta^{\frac{p+1}{2}})^{1-N(\rho-1)} \varphi((b\beta^{\frac{p+1}{2}})^{1-N(\rho-1)} x) \), we have \( ||\varphi||_{L^{2}} = ||\psi||_{L^{2}} \) and

\[ \frac{1}{2} \int_{\mathbb{R}} |\nabla \varphi|^2 \, dx - \frac{b\beta^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}} |\varphi|^{p+1} \, dx = (b\beta^{\frac{p+1}{2}})^{1-N(\rho-1)} I_{\infty,1}(\varphi). \]

Thus, we have

\[ \inf_{||\varphi||_{L^{2}}=1} I_{\infty,b}(u) = b^{1-N(\rho-1)} \beta^{1+\frac{2(p-1)}{4-N(\rho-1)}} I_{\infty}. \]

We further prepare some compactness results. To show the concentration result of Theorem 1, we use the following lemma due to Strauss [14].

**Lemma 2.** Let \( N \geq 2 \). Then every \( u \in H^{1}_{\infty} \) is almost everywhere equal to a function \( U \), continuous for \( x \neq 0 \), such that

\[ |U(x)| \leq C_{N}|x|^{-\frac{N-1}{2}}||u||_{H^{1}} \text{ for } |x| \geq C_{N}, \]

where \( C_{N} \) depends only on the dimension \( N \).

To show Theorem 2, we prepare two concentration compactness lemmas, which are slight modifications of the concentration compactness lemma due to Lions [10, 11] (See also [2]).

**Lemma 3.** Let \( \{u_{n}\} \subset H^{1}_{\infty}(\mathbb{R}) \) be such that

\[ ||u_{n}||_{L^{2}} = 1, \quad \sup_{n \in \mathbb{N}} ||\nabla u_{n}||_{L^{2}} < \infty. \quad (2.1) \]

Set

\[ \tilde{\mu} = \lim_{l \to \infty} \liminf_{n \to \infty} \int_{|x|<l} |u_{n}|^{2} \, dx. \quad (2.2) \]

Then, there exists a subsequence \( \{u_{n_{k}}\} \) that satisfies the following.

(i) If \( \tilde{\mu} = 1 \), then there exists a \( u \in H^{1}_{\infty}(\mathbb{R}) \) such that \( u_{n_{k}} \to u \) in \( L^{p}(\mathbb{R}) \) for \( p \in [2, \infty] \).
There exist \( \{ \iota_k \} \) and \( \{ 1 \iota' k + \} \) such that
\[
\supp \iota_k \subset (0, \infty), \quad \supp w_{k,-} \subset (-\infty, 0),
\]
\[
\supp v_k \cap \supp w_{k,+} = \supp v_k \cap \supp w_{k,-} = \emptyset,
\]
\[
|v_k| + |w_{k,+}| + |w_{k,-}| \leq |u_n|
\]
\[
||v_k||_{H^1} + ||w_{k,+}||_{H^1} + ||w_{k,-}||_{H^1} \leq ||u_n||_{H^1}
\]
\[
||v_k||^2_{L^2} - \tilde{\mu}, \quad ||w_{k,+}||^2_{L^2} - \frac{1}{2} (1 - \tilde{\mu}) \quad ||w_{k,-}||^2_{L^2} - \frac{1}{2} (1 - \tilde{\mu})
\]
\[
\liminf_{k \to \infty} \int (|\nabla u_{n_k}|^2 - |\nabla v_k|^2 - |\nabla w_{k,+}|^2 - |\nabla w_{k,-}|^2) \geq 0
\]
\[
\left| \int \left( |u_{n_k}|^p - |v_k|^p - |w_{k,+}|^p - |w_{k,-}|^p \right) \right| \to 0, \quad (k \to \infty)
\]
for all \( 2 \leq p \leq \infty \).

**Lemma 4.** Let \( \{ u_n \} \) satisfy (2.1). Define \( \tilde{\mu} \) as (2.2) and
\[
\mu := \lim \liminf_{t \to -\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| < t} |u_n|^2 \, dx.
\]
Assume \( \tilde{\mu} = 0 \). Then, \( 0 \leq \mu \leq 1/2 \) and there exists a subsequence \( \{ u_{n_k} \} \) that satisfies the following.

(i) If \( \mu = 1/2 \), then there exist \( u \in H^1_r(\mathbb{R}) \) and \( y_k > 0 \) such that \( y_k \to \infty \) and \( \chi_+ (\cdot - y_k) u_{n_k} (\cdot - y_k) \rightharpoonup u \) in \( L^p(\mathbb{R}) \) for \( p \in [2, \infty) \), where \( \chi_+ \in C^\infty \) satisfies \( 0 \leq \chi_+ \leq 1 \), \( \supp \chi_+ \subset [0, \infty) \) and \( \chi_+(x) = 1 \) for \( x \geq 1 \).

(ii) If \( \mu = 0 \), then \( u_{n_k} \to 0 \) in \( L^p \) for \( p \in (2, \infty] \).

(iii) There exist \( \{ v_{k,+} \} \), \( \{ v_{k,-} \} \), \( \{ w_{k,+} \} \) and \( \{ w_{k,-} \} \subset H^1_r(\mathbb{R}) \) such that
\[
\supp v_{k,+}, \quad \supp w_{k,+} \subset (0, \infty), \quad \supp v_{k,-}, \quad \supp w_{k,-} \subset (-\infty, 0),
\]
\[
\supp v_{k,+} \cap \supp w_{k,+} = \supp v_{k,-} \cap \supp w_{k,-} = \emptyset,
\]
\[
||v_{k,+}||_{H^1} + ||v_{k,-}||_{H^1} + ||w_{k,+}||_{H^1} + ||w_{k,-}||_{H^1} \leq ||u_n||_{H^1}
\]
\[
||v_{k,+}||^2_{L^2} - \tilde{\mu}, \quad ||w_{k,+}||^2_{L^2} - \frac{1}{2} (1 - \tilde{\mu}) \quad ||w_{k,-}||^2_{L^2} - \frac{1}{2} (1 - \tilde{\mu})
\]
\[
\liminf_{k \to \infty} \int (|\nabla u_{n_k}|^2 - |\nabla v_{k,+}|^2 - |\nabla v_{k,-}|^2 - |\nabla w_{k,+}|^2 - |\nabla w_{k,-}|^2) \geq 0
\]
\[
\left| \int \left( |u_{n_k}|^p - |v_{k,+}|^p - |v_{k,-}|^p - |w_{k,+}|^p - |w_{k,-}|^p \right) \right| \to 0, \quad (k \to \infty)
\]
for all \( 2 \leq p \leq \infty \).

### 3 Proof of Theorem 1

Let \( \psi_{b(0),1} \in \mathcal{I}_{\infty} \) and \( \psi_{b(0),1} > 0 \). We show that the rescaled radial minimizers converge to \( \psi_{b(0),1} \).
Lemma 5. Let $N \geq 2$ and $b$ radially symmetric. Let $\phi_n \in I_{\alpha_n}$ with $\phi_n > 0$, where $\alpha_n \to \infty$ as $n \to \infty$. Then $\{\phi_n\}$ is a minimizing sequence of $I_{\infty, b(0)}$ under the constraint $||\phi||_{L^2} = 1$. In particular, $\phi_n \rightharpoonup \psi_{b(0), 1}$.

Proof. We calculate $I_{\infty, b(0)}(\phi_n)$.

$$I_{\infty, b(0)}(\phi_n) = \frac{1}{2} \int_{\mathbb{R}} |\nabla \phi_n|^2 dx - \frac{b(0)}{p+1} \int_{\mathbb{R}} |\phi_n|^{p+1} dx$$

$$\leq I_{\alpha_n}(\phi_n) + \frac{1}{p+1} \int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\phi_n|^{p+1} dx$$

$$\leq I_{\alpha_n}(\psi_{b(0), 1}) + \frac{1}{p+1} \int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\phi_n|^{p+1} dx$$

$$\leq I_{\infty, b(0)}(\psi_{b(0), 1})$$

$$+ \frac{1}{p+1} \int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\phi_n|^{p+1} dx,$$

where $A = \frac{2(p-1)}{4-N(p-1)} > 0$. Now, for arbitrary $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $|b(x) - b(0)| < \varepsilon$ for $|x| < R_\varepsilon$. Therefore, we have

$$\int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\psi_{b(0), 1}|^{p+1} dx \leq \varepsilon \int_{\mathbb{R}} |\psi_{b(0), 1}|^{p+1} dx + \int_{|x| > \alpha_n R_\varepsilon} |\psi_{b(0), 1}|^{p+1} dx.$$ 

Further, for sufficiently large $\alpha_n$, we have

$$\frac{1}{p+1} \int_{|x| > \alpha_n R_\varepsilon} |\psi_{b(0), 1}|^{p+1} dx \leq \varepsilon.$$ 

Thus, we obtain

$$\frac{1}{p+1} \int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\phi_n|^{p+1} dx \to 0, \quad n \to \infty.$$ 

Next, using the fact that $\phi_n$ is a radial minimizer of $I_{\alpha_n}$, we see that $I_{\alpha_n}(\phi_n) < 0$. Combining this to Gagliardo-Nirenberg's inequality, we see that $||\phi_n||_{H^1}$ is uniformly bounded. Therefore, by Lemma 2, we have

$$\int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\phi_n|^{p+1} dx \leq \varepsilon \int_{\mathbb{R}} |\phi_n|^{p+1} dx + C \int_{|x| > \alpha_n R_\varepsilon} |x|^{-\frac{(N-1)(p+1)}{2}} dx$$

$$\leq C' + C' (\alpha_n R_\varepsilon)^{1-\frac{(N-1)(p+1)}{2}}.$$ 

Since $1 - \frac{(N-1)(p+1)}{2} < 0$, we see that

$$\frac{1}{p+1} \int_{\mathbb{R}} |b(\alpha_n^{-A} x) - b(0)||\phi_n|^{p+1} dx \to 0, \quad n \to \infty.$$ 

Therefore, we see that $\phi_n$ is a minimizing sequence of $I_{\infty, b(0)}$. \qed

We now prove Theorem 1.

Proof of Theorem 1. Let $u_n \in I_{\alpha_n}$ with $\alpha_n \to \infty$ as $n \to \infty$. Then, there exists $\phi_n \in I_{\alpha_n}$ such that

$$\alpha_n^{1+NA/2} \phi_n(\alpha_n^{-A} x) = \psi_{b(0), 1}.$$
where \( A = \frac{2(p-1)}{4-N(p-1)} \). We compute \( \left( \int_{|x| > \epsilon} |u_n|^2 \, dx \right)^{1/2} \).

\[
\left( \int_{|x| > \epsilon} |u_n|^2 \, dx \right)^{1/2} = \alpha \left( \int_{|x| > \epsilon} |\phi_n|^2 \, dx \right)^{1/2} \\
\leq \alpha \left( \int_{\mathbb{R}^N} |\psi_{b(0),1} - \phi_n|^2 \, dx \right)^{1/2} + \alpha \left( \int_{|x| > \epsilon} |\psi_{b(0),1}|^2 \, dx \right)^{1/2},
\]

where \( \psi_{b(0),1} \) is the positive radial minimizer of \( I_{\infty,b(0)} \) under the constraint \( ||\phi||_{L^2} = 1 \). Since \( \phi_n \rightharpoonup \psi_{b(0),1} \) in \( L^2(\mathbb{R}^N) \), we have

\[
\left( \int_{\mathbb{R}} |\psi - \phi_n|^2 \, dx \right)^{1/2} < \frac{1}{2} \epsilon^{1/2}
\]

for sufficiently large \( n \). Further, since \( \frac{2(p-1)}{4-N(p-1)} > 0 \) and \( \alpha_n \to \infty \), we see

\[
\left( \int_{|x| > \epsilon} |\psi|^2 \, dx \right)^{1/2} < \frac{1}{2} \epsilon^{1/2},
\]

for sufficiently large \( n \). Therefore, we have the concentration result.

We next show the stability for the case 0 is a nondegenerate minimum point of \( b \). For this case, modifying the result of Grossi [8], we see that for large \( \alpha > 0 \), the radial minimizer is unique up to constant phase. Therefore, the radial minimizer must correspond to the ground state with a penalizer which was introduced in [3]. Since this ground state is stable, we see that also the radial minimizer is stable.

Finally for the proof of the instability for the case 0 is a nondegenerate maximum point of \( b \), see [12]. \( \square \)

4 Proof of Theorem 2

Proof of Theorem 2 (i). Let \( u_n \in \mathcal{G}_{\alpha_n} \) with \( u_n > 0 \) and \( \alpha_n \to \infty \) as \( n \to \infty \). Then, there exists \( \phi_n \in \mathcal{I}_{\alpha_n} \) such that

\[
\alpha_n^{\frac{p-1}{p}} \phi_n (\alpha_n^{\frac{1}{p-1}} x) = u_n (x).
\]

Since \( ||\phi_n||_{L^2} = 1 \) and \( \sup_n ||\nabla \phi_n||_{L^2} < \infty \), we apply Lemma 3 to \( \{\phi_n\} \). As in the proof of Theorem 1, if we can show \( \phi_n \rightharpoonup \psi_{b(0),1} \) in \( H^1(\mathbb{R}) \), where \( \psi_{b(0),1} \) is the minimizer of \( I_{\infty,b(0)} = I_{\infty} \) under the constraint \( ||u||_{L^2} = 1 \), we have the concentration result. Further, the stability and instability follows as in the proof of Theorem 1.

Therefore, it suffices to show \( \phi_n \rightharpoonup \psi_{b(0),1} \) in \( H^1(\mathbb{R}) \). Now, let

\[
\tilde{\mu} = \lim_{t \to \infty} \liminf_{n \to \infty} \int_{|x| < t} |\phi_n|^2 \, dx.
\]
We show $\tilde{\mu} = 1$. If $\tilde{\mu} = 1$, we have a subsequence $\phi_{n_k}$ and $\phi$ such that $\phi_{n_k} \to \phi$ in $L^p$, $p \in [2, \infty]$. Thus, we have $||\phi||_{L^2} = 1$ and

$$I_{\infty, b(0)}(\phi) \leq \liminf_{k \to \infty} I_{\infty, b(0)}(\phi_{n_k})$$

$$\leq \liminf_{k \to \infty} \left( I_{\alpha_{n_k}}(\phi_{n_k}) + \int |b(0) - b(\alpha_{n_k}^{-A}x)| |\phi_{n_k}|^{p+1} dx \right)$$

$$\leq \liminf_{k \to \infty} \left( I_{\alpha_{n_k}}(\psi_{b(0)}) + \int |b(0) - b(\alpha_{n_k}^{-A}x)| |\phi_{n_k}|^{p+1} dx \right)$$

$$\leq I_{\infty, b(0)}(\psi_{b(0)}) + \liminf_{k \to \infty} \left( |b(0) - b(\alpha_{n_k}^{-A}x)| |\phi_{n_k}|^{p+1} + |\psi_{b(0)}|^{p+1} \right) dx$$

$$= I_{\infty, b(0)}(\psi_{b(0)}).$$

where $A = \frac{2(p-1)}{5-p}$. Therefore, from the definition of $\psi_{b(0)},1$ and the uniqueness of the radial minimizer of $I_{\infty, b(0)}$, we see that $\phi_{n_k} \to \psi_{b(0)},1$ in $H^1(\mathbb{R})$.

Therefore, it suffices to show $\tilde{\mu} = 1$. Suppose $\tilde{\mu} < 1$. Then, by Lemma 3, there exist $\{v_k\}, \{w_{k, +}\}$ and $\{w_{k, -}\}$ and we have

$$\liminf_{k \to \infty} I_{\alpha_{n_k}}(\phi_{n_k}) \geq \limsup_{k \to \infty} \left( I_{\alpha_{n_k}}(v_k) + I_{\infty, 1}(w_{k, +}) + I_{\infty, 1}(w_{k, -}) \right).$$

We claim $\limsup_{k \to \infty} I_{\alpha_{n_k}}(v_k) \geq b(0)\frac{2A}{p} \tilde{\mu}^{1+A} J_{\infty}$, where $A = \frac{2(p-1)}{5-p}$. Indeed, since $|v_k| \leq \alpha_{n_k}$, taking arbitrary $\varepsilon > 0$, there exists $R_{\varepsilon} > 0$ such that

$$\limsup_{k \to \infty} \int_{|x| > R_{\varepsilon}} |v_k|^2 dx < \varepsilon.$$

Therefore, we have

$$\limsup_{k \to \infty} I_{\alpha_{n_k}}(v_k) \geq \limsup_{k \to \infty} \left( I_{\infty, b(0)}(v_k) - \int_{|x| < R_{\varepsilon}} |b(\alpha_{n_k}^{-A}x) - b(0)||v_k|^{p+1} dx \right.$$

$$\left. - \int_{|x| > R_{\varepsilon}} |b(\alpha_{n_k}^{-A}x) - b(0)||v_k|^{p+1} dx \right).$$

Further, since $\sup_k ||v_k||_{L^\infty} \leq C_1 \sup_k ||v_k||_{H^1} \leq C_2 \sup_k ||\phi_{n_k}||_{H^1} < C_3$, we have

$$\int_{|x| > R_{\varepsilon}} |b(\alpha_{n_k}^{-A}x) - b(0)||v_k|^{p+1} dx \leq 2C_3^{p-1}\varepsilon,$$

and taking $\alpha_{n_k}$ sufficiently large, we have

$$\int_{|x| < R_{\varepsilon}} |b(\alpha_{n_k}^{-A}x) - b(0)||v_k|^{p+1} dx \leq \varepsilon \int_{\mathbb{R}} |v_k|^{p+1} dx \leq C\varepsilon.$$

Therefore, we obtain

$$\liminf_{k \to \infty} I_{\alpha_{n_k}}(\phi_{n_k}) \geq \left( b(0)\frac{2A}{p} \tilde{\mu}^{1+A} + 2 \left( \frac{1 - \tilde{\mu}}{2} \right)^{1+A} \right) J_{\infty}.$$
On the other hand, we have
\[
\liminf_{k \to \infty} I_{\alpha_n_k} (\phi_{n_k}) \leq \liminf_{k \to \infty} I_{\alpha_n_k} (\psi_{b(0)}) = b(0)^{\frac{2A}{p-1}} J_\infty.
\]
Therefore, since \( J_\infty < 0 \), we have
\[
b(0)^{\frac{2A}{p-1}} \leq \frac{(1 - \tilde{\mu})^{1 + A}}{2^A (1 - \tilde{\mu}^{1 + A})}.
\]
Since, \( \frac{(1 - \tilde{\mu})^{1 + A}}{1 - \tilde{\mu}^{1 + A}} \leq 1 \), we obtain
\[
b(0) \leq 2^{-\frac{8}{p-1}}.
\]
However we have assumed \( b(0) > 2^{-\frac{8}{p-1}} \). Therefore, this is a contradiction. \( \square \)

Proof of Theorem 2 (ii). Let \( u_n \in \mathcal{G}_{\alpha_n} \) with \( u_n > 0 \) and \( \alpha_n \to \infty \) as \( n \to \infty \). Then, there exists \( \phi_n \in \mathcal{I}_{\alpha_n} \) such that
\[
1 + \frac{\frac{\frac{2(p-1)}{5-p}}{\alpha_n}}{\tilde{\mu}\phi_n (\alpha_n^{\frac{2(p-1)}{5-p}} x) = u_n (x).
\]
We first show \( \tilde{\mu} = 0 \). Suppose \( \tilde{\mu} > 0 \). Then as in the proof of Theorem 2 (i), using Lemma 3, we have
\[
\lim_{k \to \infty} I_{\alpha_n_k} (\phi_{n_k}) \geq \left( b(0)^{\frac{2A}{p-1}} \tilde{\mu}^{1 + A} + 2 \left( \frac{1 - \tilde{\mu}}{2} \right)^{1 + A} \right) J_\infty,
\]
where \( A = \frac{2(p-1)}{5-p} \). On the other hand, take \( x_0 > 0 \) to satisfy \( b(x_0) = 1 \) and set
\[
\phi_k (x) = t_k \left( \psi_{1,1/2} (x - \alpha_n^{\frac{2(p-1)}{5-p}} x_0) + \psi_{1,1/2} (x + \alpha_n^{\frac{2(p-1)}{5-p}} x_0) \right),
\]
where \( \psi \) is the minimizer of \( I_{\infty,1} \) under the constraint \( \|u\|_{L^2} = 1/2 \) and \( t_k \to 1 \), \( t k \to 1 \) as \( k \to \infty \) is taken so that \( \|\psi_k\|_{L^2} = 1 \). By a simple calculation, we have
\[
\lim_{k \to \infty} I_{\alpha_n_k} (\phi_k) = 2^{-A} J_\infty.
\]
Since \( I_{\alpha_n_k} (\phi_{n_k}) \leq I_{\alpha_n_k} (\phi_k) \) and \( J_\infty < 0 \), we have
\[
b(0)^{\frac{2A}{p-1}} \tilde{\mu}^{1 + A} + 2 \left( \frac{1 - \tilde{\mu}}{2} \right)^{1 + A} \geq 2^{-A}.
\]
However, (4.2) implies
\[
b(0) \geq 2^{-\frac{8}{p-1}}.
\]
Thus, we have contradiction since we are assuming \( b(0) < 2^{-\frac{8}{p-1}} \).
Therefore, we have \( \tilde{\mu} = 0 \). We use Lemma 4. Suppose, \( \mu = 0 \). Then, by Lemma 4 (ii), we have \( \liminf_{k \to \infty} I_{\alpha_n_k} (\phi_{n_k}) \geq 0 \), so it contradicts to
\[
\liminf_{k \to \infty} I_{\alpha_n_k} (\phi_{n_k}) \leq \liminf_{k \to \infty} I_{\alpha_n_k} (\phi_k) < 0.
\]
Suppose $0 < \mu < 1/2$. Then calculating as the proof of Theorem 2 (i) and using Lemma 4 instead of Lemma 3, we obtain

$$\liminf_{k \to \infty} I_{\alpha_{nk}}(\phi_{nk}) \geq \left(2\mu^{1+\frac{1}{2}} + 2\left(\frac{1-2\mu}{2}\right)^{1+\frac{1}{2}}\right)J_{\infty}.$$ 

However, this implies $\liminf_{k \to \infty} I_{\alpha_{nk}}(\phi_{nk}) > \lim_{k \to \infty} I_{\alpha_{nk}}(\varphi_{l})$ and we have a contradiction. Therefore, we have $\mu = 1/2$.

By Lemma 4, there exist $\phi$ and $y_{k} > 0$ such that $\chi_{+}(\cdot-y)\phi_{nk}(\cdot-y) \to \phi$ in $L^{p}(\mathbb{R})$ for $p \in [2, \infty]$. Thus, we see that $||\chi_{+}(\cdot-y)\phi_{nk}(\cdot-y)||_{L^{2}} \to 1/2$. We claim $\chi_{+}(\cdot-y)\phi_{nk}(\cdot-y) \to \psi_{1,1/2}$ in $H^{1}(\mathbb{R})$, where $\psi_{1,1/2}$ is the positive radial minimizer of $I_{\infty,1}$ under the constraint $||\phi||_{L^{2}} = 1/2$. To show this, it suffices to show

$$I_{\infty,1}(\chi_{+}(\cdot-y)\phi_{nk}(\cdot-y)) \to I_{\infty,1}(\psi_{1,1/2}) = 2^{-(1+A)}J_{\infty}.$$ 

Now, suppose there exists $\varepsilon_{0} > 0$ such that

$$\frac{1}{p+1} \int_{\mathbb{R}} (1 - b(x/\alpha_{nk}))\phi_{nk}^{p+1} dx \geq \varepsilon_{0}.$$ 

Then, we have

$$\lim_{k \to \infty} I_{\infty,1}(\varphi_{l}) = \lim_{k \to \infty} I_{\alpha_{nk}}(\varphi_{l}) \geq \liminf_{k \to \infty} I_{\alpha_{nk}}(\phi_{nk}) = \liminf_{k \to \infty} \left( I_{\infty,1}(\phi_{nk}) + \frac{1}{p+1} \int_{\mathbb{R}} (1 - b(x/\alpha_{nk}))\phi_{nk} dx \right) \\
\geq 2I_{\infty,1}(\psi_{1,1/2}) + \varepsilon_{0} = \lim_{k \to \infty} I_{\infty,1}(\varphi_{l}) + \varepsilon_{0}.$$ 

Therefore, we have

$$\lim_{k \to \infty} \frac{1}{p+1} \int_{\mathbb{R}} (1 - b(x/\alpha_{nk}))\phi_{nk}^{p+1} dx = 0.$$ 

Thus, since $\tilde{\mu} = 0$, we have

$$\liminf_{k \to \infty} I_{\infty,1}(\chi_{+}(\cdot-y)\phi_{nk}(\cdot-y)) = \liminf_{k \to \infty} I_{\infty,1}(\chi \phi_{nk}) \\
= \liminf_{k \to \infty} \frac{1}{2} I_{\infty,1}(\phi_{nk}) \\
= \liminf_{k \to \infty} \frac{1}{2} \left( I_{\alpha_{nk}}(\phi_{nk}) + \frac{1}{p+1} \int_{\mathbb{R}} (1 - b(x/\alpha_{nk}))\phi_{nk}^{p+1} dx \right) \\
\leq \lim_{k \to \infty} \frac{1}{2} I_{\alpha_{nk}}(\varphi_{l}) = I_{\infty,1}(\psi_{1,1/2}).$$ 

Therefore, we see that $\chi_{+}(\cdot-y)\phi_{nk}(\cdot-y) \to \phi$ in $H^{1}$. Since $y_{k} \to \infty$, we see that $\phi_{nk}$ cannot concentrate around some point.
The instability follows from the fact that \( \phi_{n_k} \sim \psi_{1,1/2}(-y_k) + \psi_{1,1/2}(+y_k) \). We see that there exists two directions which is tangent to the hypersurface \( \{ \phi \in H^1(\mathbb{R}) \mid ||\phi||_{L^2} = \alpha \} \) and decreases the energy. Using this fact, by [6], we can show the linear instability of \( u_n \), and the instability follows from the linear instability.

\[ \square \]

References


