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A BRIEF SURVEY ON STOCHASTIC CALCULUS IN MARKOV PROCESSES

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1 From A.N. Kolmogorov to P. Lévy and K. Itô

[K] A. Kolmogorov, Über die analytischen Methoden in der Wahrscheilichkeitsrechnung, *Math. Ann.* **104** (1931), 415-458

For a Markov process (X_t, P_x), define $P_t f(x) = E_x[f(X_t)]$.
Then $u(t,x) = P_t f(x)$ satisfies the *Kolmogorov equation*

$$\frac{\partial}{\partial t}u(t,x) = \mathcal{G}_x u(t,x),$$

where $\mathcal{G}$ is the it generator of the transition semigroup $\{P_t; t \geq 0\}$:

$$\mathcal{G}v(x) = \lim_{t\downarrow 0} \frac{P_t v(x) - v(x)}{t}.$$

In a special case, $\mathcal{G}$ is a second order differential operator:

$$\mathcal{G}v(x) = \frac{1}{2}a(x)v''(x) + b(x)v'(x).$$

[L] P. Lévy, *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937
[I.1] K. Itô, On stochastic processes (infinitely divisible laws of probability), *Japan. Journ. Math.* **XVIII** (1942), 261-301

Lévy-Itô decomposition of the sample path X_t of a Lévy process as a sum of a Gaussian process $X_t^{(1)}$ and an independent process $X_t^{(2)}$ expressed using a Poisson random measure J with intensity $ds\nu(dx)$ by

$$X_t^{(2)} = \lim_{n\to\infty}\left(\int_{(0,t)\times(1/n,1)} xJ(d\xi) - t\int_{(1/n,1)} x\nu(dx)\right) + \int_{(0,t)\times(1,\infty)} xJ(d\xi)$$

$\Longrightarrow$ *Lévy-Khinchin formula* of $\varphi(z) = \log E[e^{izX_1}]$:

$$-\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{|x|<1}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu(dx) + \int_{|x|\geq 1}(e^{i\langle z,x\rangle} - 1)\nu(dx).$$

SDE and Itô's formula

[I.2] K. Itô, Differential equations determining a Markoff process (in Japanese), *Journ. Pan-Japan Math. Coll.* No. 1077(1942), 1352-1440;
(in English) in *Kiyosi Itô Selected Papers*, 42-75, Springer-Verlag, 1986
[I.3] K. Itô, On stochastic differential equations, *Mem. Amer. Math. Soc.* 4(1951), 1-51
[I.4] K. Itô, On a formula concerning stochastic differentials, Nagoya Math. J. 3 (1951), 55-65

Solution X_t of SDE $dX_t = \sqrt{a(X_t)}dB_t + b(X_t)dt$ and Itô's formula

$$v(X_t) - v(X_0) = \int_0^t v'(X_s)\sqrt{a(X_s)}dB_s + \int_0^t \left(\frac{1}{2}av'' + bv'\right)(X_s)ds$$

yields $\mathcal{G}v = \frac{1}{2}av'' + bv'$.

2 Books on SDE

[I.5] K. Itô, *Probability Theory,* Iwanami (in Japanese), 1952

[I.7] K. Itô, *Lectures on Stochastic Processes,* Tata Institute of Fundamental Research, Bombay 1960

[C] Ju-Gyoung Cho, *Stochastic Integral Equations,* Pyongyang, 1963

[M] H.P. McKean, Jr., *Stochastic Integrals,* Academic Press, New York and London, 1969

[IW] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes,* North-Holland/Kodansha, 1980

During the period 1955-1965, the Japanese probability school led by Itô was mostly concerned with the study of the one dimensional diffusion processes and its possible extensions to more general Markov processes, while the Russian probability school led by Dynkin was equally concerned with the theory of SDE initiated by Itô and Gihman-Skorohod. Notably the drift transformation by G.Maruyama 1954, M. Motoo 1960, I.V. Girsanov 1960

Contents of Cho's book

Chap.1 Basic concepts (measurable functions, conditional expectations, independence)

Chap.2 Stochastic integrals (based on Lévy processes)

Chap.3 Transformation formula for stochastic integrals

Chap.4 Existence theorems of the solutions of stochastic integral equations

(Uniqueness and existence, Markov property of the solution and its generator, *continuity and differentiability of the solution with respect to the initial position*)

Chap.5 Linear stochastic integral equations

Chap.6 Stability in stochastic equations

Excerpt from Preface of Cho's book(translated by Daehong Kim)

In dealing with random phenomena, a principal feature of stochastic equation is in that it describes the states of phenomena directly rather than describing them by means of probability distributions indirectly. This resembles the classical differential equations which give direct expressions of the changes of states of deterministic phenomena. Due to this common feature, stochastic equations can be actually regarded as extensions of differential equations.

In order to build up a theory of stochastic equations, we first need to introduce the concept of the stochastic integral and study its properties, which will be the basic contents of Chapters 2 and 3. In particular, the transformation formula of stochastic integrals derived in Chapter 3 is a new formula that can not be found in any other ordinary integration theory and will play very important roles throughout the first half of the present volume.

3 One dimensional diffusions and general Markov processes

[I.6] K. Itô, *Stochastic Processes* **I, II** (in Japanese), Iwanami-Shoten, Tokyo, 1957; (Russian translation by A.D. Wentzell), Moscow, 1960, 1963

(English translation by Yuji Ito) *Essentials of Stochastic Processes*, Translations of Mathematical

Monographs Vol.231, AMS, Providence, 2006
[I.7] K. Itô, *Lectures on Stochastic Processes,* Tata Institute, Bombay 1960
[D] E.B. Dynkin, *Foundations of the theory of Markov Processes* (in Russioan), Moscow 1959
[D] E.B. Dynkin, *Markov Processes*, Moscow 1962; English translation, Springer-Verlag, 1965
[IM] K. Itô and H. P. McKean, Jr., *Diffusion Processes and Their Sample Paths,* Springer-Verlag, 1965

Intrinsic structures of the one-dimensional diffusion revealed by W. Feller

a general theory on a Markov process and the *structures of its additive functionals*

Intrinsic generator of a one dimensional diffusion

$X = (X_t, P_x)$: a diffusion (a continuous strong Markov process) on a one dimensional regular open interval I, quasi left continuous and of no killing inside. The generator $\mathcal{G}$ of X admits the expression

$$\mathcal{G}u = \frac{d}{dm}\frac{d}{ds}u$$

where s is a strictly increasing continuous function on I and m is a strictly increasing function on I given by

$$P_x(\sigma_a < \sigma_b) = \frac{s(x) - s(a)}{s(b) - s(a)}, \ a < x < b, \quad m_J(x) = -\frac{dE_x[\tau_J]}{ds(x)}, \ x \in J, \ \overline{J} \subset I.$$

Feller's saying: s indicates the *road map* and m indicates the *speed* of the diffusion traveller X.

Itô-McKean [IM] legitimates Feller's saying in the following fashion:

Let $X = (X_t, P_x)$ be the Brownian motion on R. In this case,
$s(x) = 2x, m(x) = x$. For a given strictly increasing function m_0, define

$$A_t = \int_R \ell(t, x)dm_0(x), \qquad \tau_t = A_t^{-1} \tag{1}$$

where $\ell(t, x)$ is Lévy's local time of X at $x \in R$. Then the *time changed process* $Y = (X_{\tau_t}, P_x)$ is a diffusion process on R corresponding to
$s(x) = 2x, m(x) = m_0(x)$, that is to say, a time change of X amounts a replacement of the measure m keeping the road map s invariant.

The above defined functional $\{A_t; t \geq 0\}$ is the most general expression of the *positive continuous additive functional* (*PCAF* in abbreviation) of the one-dimensional Brownian motion. But such expression does not hold in general.

Symmetry and Dirichlet form of a one dimensional diffusion

Let X be a diffusion on a regular open interval $I = (r_1, r_2)$ as before.
Then its generator is given by (1).
Since the resolvent $\{G_\alpha\}$ of X is known to have a symmetric density kernel with respect to the speed measure m ([I.6]), X is m-symmetric. Define

$$\mathcal{E}^{(s)}(u, v) = \int_I \frac{du}{ds}\frac{dv}{ds}ds.$$

By making use of a detailed boundary behaviors of the resolvent proved in [I.6], an integration by parts gives

$$-\int_I \mathcal{G}u \cdot v dm = \mathcal{E}^{(s)}(u,v), \quad u = G_\alpha f, \ v = G_\alpha g, \ f, g \in C_0(I)$$

Actually the *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ of X on $L^2(I;m)$ is given by

$$\begin{aligned} \mathcal{F} &= \{u \in L^2(I;m) : \text{ absolutely continuous in } s, \\ &\qquad \mathcal{E}^{(s)}(u,u) < \infty, \ u(r_i) = 0, \text{ if } r_i \text{ is approachable}\}, \\ &\qquad \mathcal{E}(u,v) = \mathcal{E}^{(s)}(u,v), \qquad u, v \in \mathcal{F}, \end{aligned} \tag{2}$$

where r_i is said to be *approachable* if s has a finite limit there. $(\mathcal{E}, \mathcal{F})$ is a regular, strongly local Dirichlet form on $L^2(I;m)$. This identification is proved only recently in

M. Fukushima, From one dimensional diffusions to symmetric Markov processes, a volume 'Tribute to Professor Kiyosi Itô' of SPA, to appear.

Thus the one dimensional absorbing diffusion on a regular open interval, its possible stochastic transformation and its possible symmetric extensions can be handled entirely in the framework of Dirichlet forms.

4 From one dimensional diffusions to symmetric Hunt processes

E: a locally compact separable metric space

$X = (X_t, P_x)$: a Hunt process (a right continuous strong Markov process with quasi left continuity) on E

An extended real valued function $A_t(\omega)$ of $t \geq 0, \omega \in \Omega$, is called an *additive functional* (*AF*in abbreviation) of X if it it right continuous in t, has a left limit and $A_{s+t}(\omega) = A_s(\omega) + A_t(\theta_s\omega)$. A $[0, \infty]$-valued continuous AF is called a *PCAF*. The totality of PCAF's of X is denoted by $\mathbf{A}_c^+$

Let X be symmetric with respect to a positive Radon measure m on E of full support with the associated Dirichlet from $(\mathcal{E}, \mathcal{F})$ ($\mathcal{F}$ is the domain of the form $\mathcal{E}$) being regular. The totality of the *smooth measures* (σ-finite positive measures charging no set of zero capacity) is denoted by S.

There is a one-to-one correspondence between S and $\mathbf{A}_c^+$ characterized by the relation that $\mu \in S$ is the *Revuz measure* of $A \in \mathbf{A}_c^+$ in the sense that

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_s) dA_s \right] = \int_E f d\mu, \quad \forall f \in \mathcal{B}_+(E)$$

The Revuz measure of $A \in \mathbf{A}_c^+$ will be denoted by μ_A.

Under the above setting, it was shown in

[F] M. Fukushima, *Dirichlet Forms and Markov Processes*, North-Holland/Kodansha, 1980

that a time change of X by a fully supported $A \in \mathbf{A}_c^+$ amounts to a replacement of m with μ_A by keeping the (extended) Dirichlet form $\mathcal{E}$ invariant, generalizing the Itô-McKean theorem for the one-dimensional diffusion.

5 Motoo-Watanabe theory on MAF of a Hunt process

For a general Hunt process X on E, an AF M_t of X is said to be a *martingale AF* (*MAF* in abbreviation) if

$$E_x[M_t^2] < \infty, \quad E_x[M_t] = 0, \quad \forall t \geq 0,\ x \in E.$$

The totality of MAF's is denoted by $\mathcal{M}$. The structure of the space $\mathcal{M}$ was explored in two papers, which marked a starting point of the modern theory of stochastic calculus in Markov processes:

[MW] M. Motoo and S. Watanabe, On a class of additive functionals of Markov processes, *J. Math. Kyoto Univ.* **4**(1965), 429-469

[W] S. Watanabe, On discontinuous additive functionals and Lévy measures of a Markov process, Japanese J. Math. 34 (1964), 53-70

1. For any $M \in \mathcal{M}$, there exists a unique PCAF $\langle M\rangle \in \mathbf{A}_c^+$ such that $E_x[M_t^2] = E_x[\langle M\rangle_t]$ for any $t \geq 0, x \in E$. Let $\langle M, L\rangle = \frac{1}{4}\{\langle M+L\rangle - \langle M-L\rangle\}$ for $M, L \in \mathcal{M}$.

2. For any $M \in \mathcal{M}$ and any function f on E with $E_x\left[\int_0^t |f(X_s)| d\langle M\rangle_s\right] < \infty$, there exists a unique $f \cdot M \in \mathcal{M}$ such that

$$\langle f \cdot M, L\rangle_t = \int_0^t f(X_s) d\langle M, L\rangle_s \quad \forall t \geq 0. \tag{3}$$

$f \cdot M$ is called the *stochastic integral.*

3. Any $M \in \mathcal{M}$ adimts a unique decomposition $M = M^c + M^d$, $M^c \in \mathcal{M}_c$, $M^d \in \mathcal{M}_d$, where

$$\mathcal{M}_c = \{M \in \mathcal{M} : M_t \text{ is continuous }\}, \ \mathcal{M}_d = \{M \in \mathcal{M} : \langle M, L\rangle = 0\ \forall L \in \mathcal{M}_c\}$$

4. A *Lévy system* for X is a pair (N, H) of a kernel $N(x, dy)$ on $(E_\Delta, \mathcal{B}(E_\Delta))$ and a PCAF $H \in \mathbf{A}_c^+$ such that

$$E_x\left[\sum_{s \leq t} f(X_{s-}, X_s)\right] = E_x\left[\int_0^t \left(\int_{E_\Delta} f(X_s, y) N(X_s, dy)\right) dH_s\right] \tag{4}$$

for any non-negative Borel function f on $E \times E$ vanishing on the diogonal.

A Lévy system exists.

Any $M \in \mathcal{M}_d$ can be represented using the Lévy system as a difference of functionals appearing in both sides of (4).

6 Stochastic calculus for semi-martingales

[KW] H. Kunita and S. Watanabe, On square integrable martingales, *Nagoya Math. J.* **30**(1967), 209-245

[M] P. A. Meyer, Intégrals stochastiques (4 exposés), in *Séminaire de Probabilités I*, LNM 39, Springer, 1967, pp 72-162

The space $\mathcal{M}$ of MAF's of a Hunt process is replaced by the space of general *square integrable martingales* M. Due to the Doob-Meyer decomposition theorem of a submartingale, $\langle M \rangle$ is well defined as a predictable increasing process and the stochastic integral is defined analogously to (3).
Itô's formula is established for a general semimartingale (local martingale + process of bounded variation). Semimaringale theory is further developed in

[DM] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiel*, *Chap. I-IV*, 1975; *Chap. V-VIII*, 1980; *Chap. IX-XI*, 1983; *Chap. XII-XVI*, 1987, Hermann, Paris.

7 Decomposition of AF of a symmetric Hunt process

Let $X = (X_t, P_x)$ be a Hunt process on E symmetric with respect to a fully supported positive Radon measure m on E with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ being regular.

For $u \in \mathcal{F}$, the composite process $u(X_t)$ is not necesarily a semimartingale. Nevertheless, it admits a unique decomposition ([F])

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0$$

where $M^{[u]}$ is a MAF of finite energy and $N^{[u]}$ is a continuous AF of zero energy. A *energy* $\mathbf{e}(A)$ of an AF is defined by $\mathbf{e}(A) = \lim_{t \downarrow 0} \frac{1}{t} E_m[A_t^2]$. A CAF of zero energy is not necessarily of bounded variation but its quadratic variation is zero with respect to P_m.

8 Stochastic derivation of the Beurling-Deny formula

[FOT] M. Fukushima, Y, Oshima and M. Takeda, *Symmetric Markov processes and Dirichlet forms*, Walter de Gruyter, 1994

[FT] M. Fukushima and M. Takeda, *Markov Processes*, (in Japanese) Baifukan, 2008

[CF] Z.Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Changes and Boundary Theory*, a Book Manuscript

Any $A \in \mathbf{A}_c^+$ admits its Revuz measure $\mu_A \in S$.

Let(N, H) be a Lévy system of X. Using the Revuz measure μ_H of $H \in \mathbf{A}_c^+$, define the *jumping measure* and the *killing measure* of X by

$$J(dx, dy) = N(x, dy)\mu_H(dx) \qquad \kappa(dx) = N(x, \{\Delta\})\mu_H(dx)$$

For any $u \in \mathcal{F}$, the following identity holds:

$$\mathcal{E}(u, u) = \frac{1}{2}\mu_{\langle M^{[u]} \rangle}(E) + \frac{1}{2}\int_E u(x)^2 \kappa(dx).$$

Let $M^{[u]} = M^{[u],c} + M^{[u],d}$ be the Motoo-Watanabe decompostion of $M^{[u]} \in \mathcal{M}$ and $\mathcal{E}^{(c)}(u, u) = \frac{1}{2}\mu_{\langle M^{[u],c} \rangle}(E)$.

A computation of $\frac{1}{2}\mu_{\langle M^{[u]},d\rangle}(E)$ using the Lévy system formula (5) then yield

$$\mathcal{E}(u,u) = \mathcal{E}^{(c)}(u,u) + \frac{1}{2}\int_{E\times E}(u(x)-u(y))^2 J(dx,dy) + \int_E u(x)^2\kappa(dx). \tag{5}$$

$\mathcal{E}^{(c)}(u,v) = \frac{1}{4}\left\{\mathcal{E}^{(c)}(u+v,u+v) - \mathcal{E}^{(c)}(u-v,u-v)\right\}$ has the strongly local property: if $u\cdot m$ has a compact support and v is constant on a neighbourhood of it, then $\mathcal{E}(u,v)=0$.

The formula (5) was first announced by

[BD] A. Beurling and J. Deny, Dirichlet spaces, *Proc. Nat. Acad. Sci. U.S.A* **45** (1959), 208-215

and its analytic proof was given in 1974.

Lévy system is well defined for any Hunt process and special standard process as well. Revuz measure of a PCAF is well defined for any right process relative to any excessive measure. Any right process admits a weak dual moderate Markov process with respect to a given excessive measure.

[FG] P. J. Fitzsimmons and R. K. Getoor, Smooth measures and continuous additive functionals of right Markov processes. In *Itô's stochastic calculus and probobability theory.* Springer, Tokyo, 1996, pp 31-49

has established the one-to-one Revuz correspondence between PCAF's and smooth measures (σ-finite measures charging no semi-polar sets).

Those suggest some possibility to extend the above calculus to non-symmetric Markov processes (Z. M. Ma et.al., G. Trutnau).