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Subharmonicity for symmetric Markov processes

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Abstract

We establish the equivalence of the analytic and probabilistic notions of subharmonicity in the framework of general symmetric Hunt processes on locally compact separable metric spaces, extending an earlier work of the first named author on the equivalence of the analytic and probabilistic notions of harmonicity. As a corollary, we prove a strong maximum principle for locally bounded finely continuous subharmonic functions in the space of functions locally in the domain of the Dirichlet form under some natural conditions.

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1 Introduction

This article is a summary of the paper [6] under preparation. It is known that a function being subharmonic in a domain $D \subset \mathbb{R}^d$ can be defined by $\Delta u \le 0$ on D in the distributional sense; that is, $u \in W^{1,2}_{\rm loc}(D) := \{u \in L^2_{\rm loc}(D) \mid \nabla u \in L^2_{\rm loc}(D)\}$ so that

$$\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x)\, dx \le 0 \qquad \text{for any non-negative } v \in C_c^\infty(D).$$

If u is continuous, then the above is equivalent to the following sub-averaging property by running a Brownian motion $X = (\Omega, X_t, \mathbf{P}_x)_{x\in\mathbb{R}^d}$: for every relatively compact open subset U of D:

$$u(X_{\tau_U}) \in L^1(\mathbf{P}_x) \quad \text{and} \quad u(x) \le \mathbf{E}_x[u(X_{\tau_U})] \quad \text{for every } x \in U.$$

Here $\tau_U := \inf\{t > 0 \mid X_t \notin U\}$ is the first exit time from U. A function u is said to be harmonic in D if both u and $-u$ are subharmonic in D. Recently, there have been interest from several areas of mathematics in determining whether the above two notions harmonicity and subharmonicity remain equivalent for general symmetric Hunt processes including symmetric Lévy processes. For instance, due to their importance in theory and applications, there has been intense interest recently in studying discontinuous processes

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and non-local (or integro-differential) operators by both analytical and probabilistic approaches. See, e.g. [4, 5] and the references therein. So it is important to identify the connection between the analytic and probabilistic notions of subharmonic functions. Very recently, in [3] the first named author established the equivalence between the analytic and probabilistic notions of harmonic functions for symmetric Markov processes. Subsequently, the above equivalence is extended in [18] to non-symmetric Markov processes associated with sectorial Dirichlet forms.

In this paper, we extend the previous work [3] to address the question of the equivalence of the analytic and probabilistic notions of subharmonicity in the context of symmetric Hunt processes on locally compact separable metric space (Theorem 2.7). As a byproduct of our result, we prove that strong maximum principle holds for locally bounded finely continuous $\mathcal{E}$-subharmonic functions under some conditions (Theorem 2.9). Strong maximum principles for subharmonic functions of second order elliptic operators have been powerful tools for various fields in analysis and geometry. In [15], the second named author established a strong maximum principle for finely continuous $\mathcal{E}$-subharmonic functions in the framework of irreducible local semi-Dirichlet forms whose Hunt processes satisfy the absolute continuity condition with respect to the underlying measure, which generalize the classical strong maximum principle for second order elliptic operators (for an extension of strong maximum principle for subharmonicity in the barrier sense, see also [16]). The strong maximum principle developed in [14, 15] can be applied to analysis or geometry for geometric singular spaces; Alexandrov spaces or spaces appeared in the Gromov-Hausdorff limit of Riemannian manifolds with uniform lower Ricci curvature bounds and so on. More concretely in [17], we establish splitting theorems for weighted Alexandrov spaces having measure contraction property, which are striking applications of the strong maximum principle treated in [14, 15] in terms of symmetric diffusion processes. The strong maximum principle established in this paper holds for symmetric Markov processes, which may possibly have discontinuous sample paths, on locally compact separable metric spaces, which should be useful in the study of non-local operator or jump type symmetric Markov processes.

Let X be be an m-symmetric Hunt process on a locally compact separable metric space E whose associated Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular on $L^2(E;m)$. Let D be an open subset of E and τ_D is the first exit time from D by X. Motivated by the example at the beginning of this section, loosely speaking (see next section for precise statements), there are two ways to define a function u being subharmonic in D with respect to X: (a) (probabilistically) $t\mapsto u(X_{t\wedge\tau_D})$ is a $\mathbf{P}_x$-uniformly integrable submartingale for quasi-every $x\in D$; (b) (analytically) $\mathcal{E}(u,g)\le 0$ for $g\in\mathcal{F}\cap C_c^+(D)$. We will show in Theorem 2.7 below that these two definitions are equivalent under some integrability conditions as imposed in the previous work [3] by the first author. Note that even in the Brownian motion case, a function u that is subharmonic in D is typically not in the domain $\mathcal{F}$ of the Dirichlet form. Denote by $\mathcal{F}_{D,\mathrm{loc}}$ the family of functions u on E such that for every relatively compact open subset D_1 of D, there is a function $f\in\mathcal{F}$ so that $u=f$

m-a.e. on D_1. To show these two definitions are equivalent, the crux of the difficulty is to

(i) appropriately extend the definition of $\mathcal{E}(u,v)$ to functions u in $\mathcal{F}_{D,\mathrm{loc}}$ that satisfy some minimal integrability condition when X is discontinuous so that $\mathcal{E}(u,v)$ is well defined for every $v \in \mathcal{F} \cap C_c(D)$;

(ii) show that if u is subharmonic in D in the probabilistic sense, then $u \in \mathcal{F}_{D,\mathrm{loc}}$ and $\mathcal{E}(u,v) \le 0$ for every non-negative $v \in \mathcal{F} \cap C_c(D)$.

The question (i) is solved in the previous work [3]. The main focus of this paper is to address the second question (ii). For (ii), we establish a Riesz type decomposition theorem (Lemma 3.7 in [6]) for $\mathcal{E}$-subharmonic functions, which is a crucial step in proving our main result.

If one assumes a priori that $u \in \mathcal{F}$, then the equivalence of (a) and (b) is easy to establish. In next section, we give precise definitions, statements of the main results and their proofs. Four examples are given to illustrate the main results of this paper. We use ":="as a way of definition. For two real numbers a and b, $a \wedge b := \min\{a,b\}$.

2 Main result

Let $X = (\Omega, \mathfrak{F}_\infty, \mathfrak{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ be an m-symmetric right Markov process on a space E, where m is a positive σ-finite measure with full topological support on E. A cemetery state ∂ is added to E to form $E_\partial := E \cup \{\partial\}$, and Ω is the totality of right-continuous, left-limited sample paths from $[0,\infty)$ to E_∂ that hold the value ∂ once attaining it. Throughout this paper, every function f on E is automatically extended to be a function on E_∂ by setting $f(\partial) = 0$. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \ge 0 \mid X_t(\omega) = \partial\}$ be the life time of X. Throughout this paper, we use the convention that $X_\infty(\omega) := \partial$. As usual, $\mathfrak{F}_\infty$ and $\mathfrak{F}_t$ are the minimal augmented σ-algebras obtained from $\mathfrak{F}^0_\infty := \sigma\{X_s \mid 0 \le s < \infty\}$ and $\mathfrak{F}^0_t := \sigma\{X_s \mid 0 \le s \le t\}$ under $\{\mathbf{P}_x : x \in E\}$. For a Borel subset B of E, $\tau_B := \inf\{t \ge 0 \mid X_t \notin B\}$ (the *exit time* of B) is an $(\mathfrak{F}_t)$-stopping time.

The transition semigroup $\{P_t : t \ge 0\}$ of X is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta], \qquad t \ge 0.$$

Each P_t may be viewed as an operator on $L^2(E;m)$, and taken as a whole these operators form a strongly continuous semigroup of self-adjoint contractions. The Dirichlet form associated with X is the bilinear form

$$\mathcal{E}(u,v) := \lim_{t \downarrow 0} t^{-1}(u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \left\{ u \in L^2(E;m) \,\middle|\, \sup_{t>0} t^{-1}(u - P_t u, u)_m < \infty \right\}.$$

Here we use the notation $(f,g)_m := \int_E f(x)g(x)\,m(dx)$ and we shall use $|f|_2 := \sqrt{(f,f)_m}$ for $f, g \in L^2(E;m)$. P_t is extended to be a strongly continuous semigroup $\{T_t; t \geq 0\}$ on $L^2(E;m)$. Without loss of generality, we may assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E;m)$ and the X is an m-symmetric Hunt process, where E is a locally compact separable metric space having a one point compactification $E_\partial := E \cup \{\partial\}$ and m is a positive Radon measure with full topological support (see [7]).

A set $B \subset E_\partial$ is called *nearly Borel* if for each probability measure μ on E_∂, there exist Borel sets $B_1, B_2 \subset E_\partial$ such that $B_1 \subset B \subset B_2$ and $\mathbf{P}_\mu(X_t \in B_2 \setminus B_1$ for some $t \geq 0) = 0$. Any hitting time $\sigma_B := \inf\{t > 0 \mid X_t \in B\}$ is an $(\mathfrak{F}_t)$-stopping time for nearly Borel subset of E_∂ (see Theorem 10.7 and the remark after Definition 10.21 in [1]). A subset B of E_∂ is said to be X-invariant if B is nearly Borel and

$$\mathbf{P}_x(X_t \in B_\partial, X_{t-} \in B_\partial \text{ for all } t \geq 0) = 1 \qquad \text{for any } x \in B.$$

A set A is *finely open* if for each $x \in A$ there exists a nearly Borel subset $B = B(x)$ of E such that $B \supset E \setminus A$ and $\mathbf{P}_x(\sigma_B > 0) = 1$. A set N is called *properly exceptional* if $E \setminus N$ is X-invariant and $m(N) = 0$. A nearly Borel set N is called *m-polar* if $\mathbf{P}_m(\sigma_N < \infty) = 0$ and any subset N of E is called *exceptional* if there exists an m-polar set $\tilde{N}$ containing N. Clearly any properly exceptional set N is exceptional. A function defined q.e. on an open subset D of E is said to be *q.e. finely continuous* on D if there exists a properly exceptional Borel set N such that u is Borel measurable and finely continuous on $D \setminus N$. It is known (cf. [12]) a quasi-continuous function on D is q.e. finely continuous on D.

Let $\mathcal{F}_e$ be the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\{u_n\}$ as above an approximating sequence for $u \in \mathcal{F}_e$. For any $u, v \in \mathcal{F}_e$ and its approximating sequences $\{u_n\}$, $\{v_n\}$ the limit $\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}(u_n, v_n)$ exists and does not depend on the choices of the approximating sequences for u, v. It is known that $\mathcal{E}^{1/2}$ on $\mathcal{F}_e$ is a semi-norm and $\mathcal{F} = \mathcal{F}_e \cap L^2(E;m)$. We call $(\mathcal{E}, \mathcal{F}_e)$ *the extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$. Any $u \in \mathcal{F}_e$ admits a quasi-continuous m-version $\tilde{u}$. Throughout this paper, we always take quasi-continuous m-version of the element of $\mathcal{F}_e$, that is, we omit *tilde* from $\tilde{u}$ for $u \in \mathcal{F}_e$

Let D be an open subset of E. We define

$$\begin{cases} \mathcal{F}_D := \{u \in \mathcal{F} \mid u = 0 \ \ \mathcal{E}\text{-q.e. on } E \setminus D\}, \\ \mathcal{E}^D(u,v) := \mathcal{E}(u,v) \qquad \text{for } u, v \in \mathcal{F}_D. \end{cases}$$

Then $(\mathcal{E}^D, \mathcal{F}_D)$ is again a regular Dirichlet form on $L^2(D;m)$, which is called the *part space* in D. Denote by $\mathcal{F}_{D,\mathrm{loc}}$ (resp. $(\mathcal{F}_D)_{\mathrm{loc}}$) *the space of functions locally in* $\mathcal{F}$ on D (resp. *the space of functions locally in* $\mathcal{F}_D$); that is, $u \in \mathcal{F}_{D,\mathrm{loc}}$ (resp. $u \in (\mathcal{F}_D)_{\mathrm{loc}}$) if and only if for any relatively compact open set U with $\overline{U} \subset D$ there exists $u_U \in \mathcal{F}$ (resp. $u_U \in \mathcal{F}_D$) such that $u = u_U$ m-a.e. on U. Note that $(\mathcal{F}_D)_{\mathrm{loc}} \subset \mathcal{F}_{D,\mathrm{loc}}$ and $\mathbf{1}_D \in (\mathcal{F}_D)_{\mathrm{loc}}$. Any $u \in \mathcal{F}_{D,\mathrm{loc}}$ admits an m-version $\tilde{u}$ of u which is quasi-continuous on D. As remarked above, we always take such m-version and omit *tilde* from $\tilde{u}$ for $u \in \mathcal{F}_{D,\mathrm{loc}}$. We can see that $\mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m) \subset (\mathcal{F}_D)_{\mathrm{loc}}$. Indeed, for $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$, we can take $u_U \in \mathcal{F}_b$ such that $u = u_U$ m-a.e. on U, because $u_U = (-\|u\|_{U,\infty}) \vee u_U \wedge \|u\|_{U,\infty}$ m-a.e. on U, where $\|u\|_{U,\infty} := m\text{-ess-}\sup_U |u|$. Taking $\phi \in \mathcal{F} \cap C_c(E)$ with $\phi = 1$ on U and $\phi = 0$ on D^c, we see $u_U\phi \in \mathcal{F}_D$ and $u = u_U\phi$ m-a.e. on U.

Definition 2.1 (Sub/Super-harmonicity) Let D be an open set in E. We say that a nearly Borel measurable function u defined on E is *subharmonic (resp. superharmonic) in D* if for any relatively compact open subset U of D with $\overline{U} \subsetneq D$, $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable right continuous $\mathbf{P}_x$-submartingale (resp. $\mathbf{P}_x$-supermartingale) for q.e. $x \in E$. A nearly Borel function u on E is said to be ***harmonic in D u*** is both superharmonic and subharmonic in D.

Definition 2.2 (Sub/Super-harmonicity in the weak sense) Let D be an open set in E. We say that a nearly Borel function u defined on E is *subharmonic (resp. superharmonic) in D in the weak sense* if u is q.e. finely continuous in D and for any relatively compact open subset U with $\overline{U} \subsetneq D$, $\mathbf{E}_x[|u|(X_{\tau_U})] < \infty$ for q.e. $x \in E$ and for q.e. $x \in E$, $u(x) \le \mathbf{E}_x[u(X_{\tau_U})]$ (resp. $u(x) \ge \mathbf{E}_x[u(X_{\tau_U})]$) holds if $\mathbf{P}_x(\tau_U < \infty) > 0$. A nearly Borel measurable function u on E is said to be *harmonic in D in the weak sense* if u is both superharmonic and subharmonic in D in the weak sense.

Clearly $\mathbf{1}_D$ is superharmonic in D in the weak sense.

Remark 2.3 Our definition on the subharmonicity or superharmonicity in the weak sense is different from what is defined in the Dynkin's textbook [11] and is weaker than it when X is an m-irreducible diffusion process satisfying (2.1) below. Actually, superharmonicity of u in [11] requires u be locally bounded from below instead of the $\mathbf{P}_x$-integrability of $u(X_{\tau_U})$ for any relatively compact open U with $\overline{U} \subset D$. Indeed, suppose that X is a diffusion process and u is a superharmonic function in D in the sense of [11]. Then for U as above, we have

$$\mathbf{E}_x[|u(X_{\tau_U})|] \le \mathbf{E}_x[u(X_{\tau_U})] + 2\mathbf{E}_x[(-u)^+(X_{\tau_U})] \le u(x) + 2(-\inf_{\partial U} u)^+ < \infty$$

for q.e. $x \in E$. □

We introduce the following condition:

$$\text{For any relatively compact open set } U \text{ with } \overline{U} \subsetneq D,\ \mathbf{P}_x(\tau_U < \infty) > 0 \ \text{ for q.e. } x \in U. \tag{2.1}$$

Condition (2.1) is satisfied if $(\mathcal{E}, \mathcal{F})$ is *m-irreducible*, that is, any (T_t)-invariant set B is trivial in the sense that $m(B) = 0$ or $m(B^c) = 0$.

It will be shown that under condition (2.1), every subharmonic function in D is a subharmonic function in D in the weak sense.

In what follows, all functions denoted by u or u_i, $(i = 1, 2)$ are defined on E and are (nearly) Borel measurable and finite quasi everywhere.

For an open set $D \subset E$, we consider the following conditions for a (nearly) Borel function u on E that are introduced in [3]. For any relatively compact open sets U, V with $\overline{U} \subset V \subset \overline{V} \subset D$,

$$\int_{U\times(E\setminus V)} |u(y)| J(dxdy) < \infty \tag{2.2}$$

and

$$\mathbf{1}_U \mathbf{E}_\cdot[(1-\phi_V)|u|(X_{\tau_U})] \in (\mathcal{F}_U)_e, \tag{2.3}$$

where $\phi_V \in \mathcal{F} \cap C_c(E)$ with $0 \le \phi_V \le 1$ and $\phi_V = 1$ on V.

As is noted in [3], in many concrete cases such as in Examples 2.12-2.14 in [3] (see also Examples 3.1-3.2 below), one can show that condition (2.2) implies condition (2.3).

Remark 2.4 (i) In view of [3, Lemma 2.3], every nearly Borel bounded function u on E satisfies both (2.2) and (2.3).

(ii) If $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D; m)$, then u is bounded q.e. on any relatively compact open U with $\overline{U} \subset D$, so for any U, V as above, (2.2) is equivalent to

$$\int_{U\times(E\setminus V)} |u(y) - u(x)| J(dxdy) < \infty \tag{2.4}$$

for such u. Clearly, any $u \in \mathcal{F}_e$ satisfies

$$\int_{U\times(E\setminus V)} |u(y) - u(x)| J(dxdy) \le J(U \times V^c)^{1/2} \left(\int_{E\times E} |u(y) - u(x)|^2 J(dxdy) \right)^{1/2} < \infty;$$

that is, (2.4) is satisfied by $u \in \mathcal{F}_e$. Furthermore, by Lemma 2.5 of [3], both (2.2) and (2.3) hold for every $u \in \mathcal{F}_e \cap L^\infty_{\mathrm{loc}}(D; m)$. □

The following is proved in [3].

Lemma 2.5 (cf. Lemma 2.6 in [3]) *Let D be an open set of E. Suppose that u is a locally bounded function on D such that u belongs to $\mathcal{F}_{D,\mathrm{loc}}$ and it satisfies condition* (2.2). *Then for every $v \in \mathcal{F} \cap C_c(D)$, the expression*

$$\frac{1}{2}\mu^c_{\langle u,v\rangle}(D) + \frac{1}{2}\int_{E\times E} (u(x) - u(y))(v(x) - v(y)) J(dxdy) + \int_D u(x)v(x)\kappa(dx)$$

is well-defined and finite; it will still be denoted as $\mathcal{E}(u, v)$.

Definition 2.6 ($\mathcal{E}$-sub/super-harmonicity) *Let $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$ be a function satisfying the condition (2.2). We say that u is $\mathcal{E}$-subharmonic (resp. $\mathcal{E}$-superharmonic) in D if and only if $\mathcal{E}(u,v) \le 0$ (resp. $\mathcal{E}(u,v) \ge 0$) for every non-negative $v \in \mathcal{F} \cap C_c(D)$. A function $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$ satisfying condition (2.2) is said to be $\mathcal{E}$-harmonic in D if u is both $\mathcal{E}$-superharmonic and $\mathcal{E}$-subharmonic in D. When $D = E$, we omit the phrase 'in D'.*

Note that $\mathbf{1}_D \in \mathcal{F}_{D,\mathrm{loc}}$ satisfies (2.2) and is $\mathcal{E}$-superharmonic in D. It is $\mathcal{E}$-harmonic in D provided $\kappa(D) = 0$ and $J(D, D^c) = 0$.

Our main theorem below is an analogy of Theorem 2.11 in [3] for subharmonic functions.

Theorem 2.7 *Let D be an open subset of E. Suppose that a nearly Borel $u \in L^\infty_{\mathrm{loc}}(D;m)$ satisfies conditions (2.2) and (2.3). Then*

(i) *u is subharmonic in D if and only if $u \in (\mathcal{F}_D)_{\mathrm{loc}}$ and it is $\mathcal{E}$-subharmonic in D.*

(ii) *Assume that (2.1) holds. Then u is subharmonic in D if and only if u is subharmonic in D in the weak sense, that is, for any relatively compact open set U with $\overline{U} \subsetneq D$, $u(X_{\tau_U})$ is $\mathbf{P}_x$-integrable and $u(x) \le \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x \in E$.*

Theorem 2.7 will be established through Lemma 3.7 and Theorems 3.8–3.10 in [6]. As an application of Theorem 2.7, we have the following.

Corollary 2.8 (i) *Let $\eta \in C^1(\mathbb{R})$ be a convex function and $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$ be an $\mathcal{E}$-harmonic function in D satisfying conditions (2.2)–(2.3). Suppose that η has bounded first derivative or u is bounded on E. Then $\eta(u) \in \mathcal{F}_{D,\mathrm{loc}}$ and is $\mathcal{E}$-subharmonic in D satisfying conditions (2.2)–(2.3).*

(ii) *The conclusion of (i) remains to true if $\eta \in C^1(\mathbb{R})$ is an increasing convex function and $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$ is an $\mathcal{E}$-subharmonic function in D satisfying conditions (2.2)–(2.3).*

(iii) *Let $p \ge 1$ and $u \in \mathcal{F}_{D,\mathrm{loc}}$ be an $\mathcal{E}$-harmonic function in D that is locally bounded in D and satisfies conditions (2.2)–(2.3). Suppose that $|u|^p$ satisfies conditions (2.2) and (2.3), and that (2.1) holds. Then $|u|^p \in \mathcal{F}_{D,\mathrm{loc}}$ and is $\mathcal{E}$-subharmonic in D.*

(iv) *Let $u_1, u_2 \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$ be $\mathcal{E}$-subharmonic functions in D satisfying conditions (2.2)–(2.3). Then $u_1 \vee u_2 \in \mathcal{F}_{D,\mathrm{loc}}$ satisfies (2.2)–(2.3) and is $\mathcal{E}$-subharmonic in D.*

We say that X satisfies *the absolute continuity condition with respect to m* if the transition kernel $P_t(x, dy)$ of X is absolutely continuous with respect to $m(dy)$ for any $t > 0$ and $x \in E$.

As a consequence of Corollary 2.8(iv), we have the following strong maximum principle.

Theorem 2.9 (Strong maximum principle) *Assume that D is an open subset of E, X satisfies the absolute continuity condition with respect to m and $(\mathcal{E}^D, \mathcal{F}_D)$ is m-irreducible. Suppose that $u \in \mathcal{F}_{D,\mathrm{loc}}$ satisfying conditions (2.2)-(2.3) is a locally bounded finely continuous $\mathcal{E}$-subharmonic function in D. If u attains a maximum at a point $x_0 \in D$. Then $u^+ \equiv u^+(x_0)$ on D. If in addition $\kappa(D) = 0$, then $u \equiv u(x_0)$ on D.*

3 Examples

Example 3.1 (Stable-like process on $\mathbb{R}^d$) Consider the following Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$, where

$$\begin{cases} \mathcal{F} = W^{\alpha/2,}(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 |x-y|^{d+\alpha} dxdy < \infty \right\}, \\ \mathcal{E}(u,v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)|x-y|^{d+\alpha} c(x,y) dxdy \text{ for } u, v \in \mathcal{F}. \end{cases}$$

Here $d \geq 1$, $\alpha \in]0,2[$, and $c(x,y)$ is a symmetric function in (x,y) that is bounded between two positive constants. In literature, $W^{\alpha/2,2}(\mathbb{R}^d)$ is called the Sobolev space on $\mathbb{R}^d$ of fractional order $(\alpha/2, 2)$. For an open set $D \subset \mathbb{R}^d$, $W^{\alpha/2,2}(D)$ is similarly defined as above but with D in place of $\mathbb{R}^d$. It is easy to check that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ and its associated symmetric Hunt process X is called symmetric α-stable-like process on $\mathbb{R}^d$, which is studied in [4]. When $c(x,y) \equiv A(d,-\alpha) := \frac{\alpha 2^{d+\alpha}\Gamma(\frac{d+\alpha}{2})}{2^{d+1}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$, the process X is nothing but the rotationally symmetric α-stable process on $\mathbb{R}^d$. It is shown in [4] that the symmetric α-stable-like process X has strictly positive jointly continuous transition density function $p_t(x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and hence is irreducible. Moreover, there is constant $c > 0$ such that

$$p_t(x,y) \leq ct^{-d/\alpha} \qquad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{3.1}$$

Consequently, by [10, Theorem],

$$\sup_{x \in U} \mathbf{E}_x[\tau_U] < \infty. \tag{3.2}$$

for any open set U having finite Lebesgue measure. Note that in this example, the jumping measure

$$J(dxdy) = \frac{c(x,y)}{|x-y|^{d+\alpha}} dxdy$$

Hence for any non-empty open set $D \subset \mathbb{R}^d$, condition (2.2) is satisfied if and only if $(1 \wedge |x|^{-d-\alpha})u(x) \in L^1(\mathbb{R}^d)$ (or equivalently, $u(x)/(1+|x|)^{d+\alpha} \in L^1(\mathbb{R}^d)$). As is shown in [3, Example 2.12], condition (2.3) is automatically satisfied for such u. When $\alpha \in]1,2[$, every (globally) Lipschitz function u on $\mathbb{R}^d$ satisfies the condition (2.2), that is, $(1 \wedge |x|^{-d-\alpha})u(x) \in L^1(\mathbb{R}^d)$ holds. Consequently (2.3) holds for any Lipschitz function u

provided $\alpha \in]1,2[$. Indeed, for any relatively compact open sets U, V with $\overline{U} \subset V \subset \overline{V} \subset D$,

$$\begin{aligned}\int_{U\times V^c} \frac{|u(y)-u(x)|}{|x-y|^{d+\alpha}}dxdy &\le \|u\|_{\mathrm{Lip}} \int_{U\times V^c} \frac{|x-y|}{|x-y|^{d+\alpha}}dxdy \\ &\le \|u\|_{\mathrm{Lip}}\sigma(\mathbb{S}^{d-1}) \int_U \int_{d(x,V^c)}^{\infty} r^{-\alpha} dr dx \\ &\le \|u\|_{\mathrm{Lip}}|U|\sigma(\mathbb{S}^{d-1})\frac{d(U,V^c)^{1-\alpha}}{\alpha-1} < \infty,\end{aligned}$$

and so by Remark 2.3, (2.2) holds. Here $\|u\|_{\mathrm{Lip}} := \sup_{x,y\in\mathbb{R}^d} \frac{|u(x)-u(y)|}{|x-y|}$, $|U|$ denotes the volume of U and $\sigma(\mathbb{S}^{d-1})$ is the $(d-1)$-dimensional volume of the unit sphere $\mathbb{S}^{d-1}$.

Theorem 2.7 says that for an open set D and a nearly Borel function u on $\mathbb{R}^d$ that is locally bounded on D with $(1\wedge |x|^{-d-\alpha})u(x) \in L^1(\mathbb{R}^d)$, the following are equivalent.

(i) u is subharmonic in D;

(ii) For every relatively compact open subset U of D, $u(X_{\tau_U}) \in L^1(\mathbf{P}_x)$ and $u(x) \le \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x \in U$;

(iii) $u \in \mathcal{F}_{D,\mathrm{loc}} = W^{\alpha/2,2}_{\mathrm{loc}}(D)$ and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy \le 0 \text{ for every } v \in W^{\alpha/2,2}(D)\cap C_c^+(D).$$

Example 3.2 (Symmetric Relativistic α-stable Process) Take $\alpha \in]0,2[$ and $m \ge 0$. Let $\mathrm{X}^{\mathrm{R},\alpha} = (\Omega, X_t, \mathbf{P}_x)_{x\in\mathbb{R}^d}$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}_0\left[e^{i\langle\xi,X_t\rangle}\right] = e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)}.$$

If $m>0$, it is called the *relativistic α-stable process with mass m* (see [20]). In particular, if $\alpha = 1$ and $m > 0$, it is called the *relativistic free Hamiltonian process* (see [13]). When $m = 0$, $\mathrm{X}^{\mathrm{R},\alpha}$ is nothing but the usual *symmetric α-stable process*. Let $(\mathcal{E}^{\mathrm{R},\alpha}, \mathcal{F}^{\mathrm{R},\alpha})$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $\mathbf{X}^{\mathrm{R},\alpha}$. Using Fourier transform $\widehat{f}(x) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\langle x,y\rangle}f(y)dy$, it follows from Example 1.4.1 of [12] that

$$\begin{cases} \mathcal{F}^{\mathrm{R},\alpha} &:= \left\{f\in L^2(\mathbb{R}^d)\;\middle|\; \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)d\xi < \infty\right\}, \\ \mathcal{E}^{\mathrm{R},\alpha}(f,g) &:= \int_{\mathbb{R}^d} \widehat{f}(\xi)\bar{\widehat{g}}(\xi)\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)d\xi \quad \text{for } f,g\in\mathcal{F}^{\mathrm{R},\alpha}. \end{cases}$$

It is shown by Ryznar [20] that the semigroup kernel $p_t(x,y)$ of $\mathbf{X}^{\mathrm{R},\alpha}$ is given by

$$p_t(x,y) = e^{mt}\int_0^\infty \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{|x-y|^2}{4s}}e^{-sm^{\frac{2}{\alpha}}}\theta_{\frac{\alpha}{2}}(t,s)ds,$$

where $\theta_\delta(t,s)$ is the nonnegative function called the *subordinator* whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda s}\theta_\delta(t,s)ds = e^{-t\lambda^\delta}.$$

Then we see the conservativeness of $\mathbf{X}^{\mathrm{R},\alpha}$ and the irreducibility of $(\mathcal{E}^{\mathrm{R},\alpha},\mathcal{F}^{\mathrm{R},\alpha})$. From Lemma 3 in [20], there exists $C(d,m)>0$ depending only on m and d such that

$$\sup_{x,y\in\mathbb{R}^d} p_t(x,y)\le C(d,m)\left(m^{d/\alpha-d/2}t^{-d/2}+t^{-d/\alpha}\right)\qquad \text{for any } t>0.$$

This yields by [10, Theorem 1] that (3.2) holds for any open set U having finite Lebesgue measure. It is shown in [8] that the corresponding jumping measure satisfies

$$J(dxdy)=\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy \quad \text{with} \quad c(x,y):=\frac{A(d,-\alpha)}{2}\Psi(m^{1/\alpha}|x-y|),$$

where $A(d,-\alpha)=\frac{\alpha 2^{d+\alpha}\Gamma(\frac{d+\alpha}{2})}{2^{d+1}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$, and the function Ψ on $[0,\infty[$ is given by $\Psi(r):=I(r)/I(0)$ with $I(r):=\int_0^\infty s^{\frac{d+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}ds$. Note that Ψ is decreasing and satisfies $\Psi(r)\asymp e^{-r}(1+r^{(d+\alpha-1)/2})$ near $r=\infty$, and $\Psi(r)=1+\Psi''(0)r^2/2+o(r^4)$ near $r=0$. In particular, for $b>0$ we have

$$0<\inf_{r\ge 0}\frac{\Psi(m^{1/\alpha}(r+b))}{\Psi(m^{1/\alpha}r)}\le\sup_{r\ge 0}\frac{\Psi(m^{1/\alpha}(r+b))}{\Psi(m^{1/\alpha}r)}<\infty \tag{3.3}$$

and

$$\begin{cases} \mathcal{F}^{\mathrm{R},\alpha} &= \left\{f\in L^2(\mathbb{R}^d)\;\middle|\; \int_{\mathbb{R}^d\times\mathbb{R}^d}|f(x)-f(y)|^2\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy<\infty\right\},\\ \mathcal{E}^{\mathrm{R},\alpha}(f,g) &= \int_{\mathbb{R}^d\times\mathbb{R}^d}(f(x)-f(y))(g(x)-g(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy \quad \text{for } f,g\in\mathcal{F}^{\mathrm{R},\alpha}. \end{cases}$$

Applying (3.3), we can obtain that for any relatively compact open sets U,V with $0\in U$ and $\overline{U}\subset V\subset\overline{V}\subset D$, condition (2.2) is satisfied if and only if $\Psi(m^{1/\alpha}|x|)(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$ (equivalently $\Psi(m^{1/\alpha}|x|)u(x)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$). Similarly, any function u with $\Psi(m^{1/\alpha}|x|)(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$ also satisfies the condition (2.3) in the same way as in Example 3.1. For $u\in L^\infty_{\mathrm{loc}}(D;m)\cap\mathcal{F}^{\mathrm{R},\alpha}_{D,\mathrm{loc}}$, we can deduce (2.2) and (2.3) if $\Psi(m^{1/\alpha}|x|)\left(1\wedge|x|^{-d-\alpha}\right)u(x)\in L^1(\mathbb{R}^d)$ without assuming $0\in U$. In this case, (2.2) for any relatively compact open U,V with $\overline{U}\subset V\subset\overline{V}\subset D$ is equivalent to $\Psi(m^{1/\alpha}|x|)\left(1\wedge|x|^{-d-\alpha}\right)u(x)\in L^1(\mathbb{R}^d)$. Moreover, any (globally) Lipschitz function u satisfies (2.2), consequently (2.3) holds for such u. Indeed, for any relatively compact open sets U, V with $\overline{U}\subset V$,

$$\begin{aligned}\int_{U\times V^c}\frac{|u(y)-u(x)|}{|x-y|^{d+\alpha}}c(x,y)dxdy &\le \frac{A(d,-\alpha)}{2}\|u\|_{\mathrm{Lip}}\int_{U\times V^c}\frac{|x-y|\Psi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}dxdy\\ &\le \frac{A(d,-\alpha)}{2}\|u\|_{\mathrm{Lip}}\sigma(\mathbb{S}^{d-1})\int_U\int_{d(x,V^c)}^\infty \Psi(m^{1/\alpha}r)r^{-\alpha}drdx\\ &\le C\int_{d(U,V^c)}^\infty e^{-m^{1/\alpha}r}(1+m^{\frac{d+\alpha-1}{2\alpha}}r^{\frac{d+\alpha-1}{2}})r^{-\alpha}dr<\infty,\end{aligned}$$

and so (2.2) holds by Remark 2.3. Here C is a positive constant.

By Theorem 2.7, for an open set D and a nearly Borel function u on $\mathbb{R}^d$ that is locally bounded on D with $\Psi(m^{1/\alpha}|x|)(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$, the following are equivalent.

(i) u is subharmonic in D;

(ii) For every relatively compact open subset U of D, $u(X_{\tau_U})\in L^1(\mathbf{P}_x)$ and $u(x)\le \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x\in U$;

(iii) $u\in\mathcal{F}^{\mathrm{R},\alpha}_{D,\mathrm{loc}}$ and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))\frac{\Psi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}dxdy\le 0 \qquad \text{for every } v\in\mathcal{F}^{\mathrm{R},\alpha}_D\cap C_c^+(D).$$

One may ask concrete examples of $\mathcal{E}$-(sub/super)-harmonicity on D. To answer this question, in what follows, we assume $d>2$ ($d>\alpha$ if $m=0$). Applying Theorems 3.1 and 3.3 in [19] to $\phi(\lambda):=(\lambda+m^{2/\alpha})^{\alpha/2}-m$, $\lambda>0$, we can obtain that the Green kernel $r(x,y):=\int_0^\infty p_t(x,y)dt$ of X satisfies $r(x,y)\asymp(K_\alpha(x,y)+K_2(x,y))$, $x,y\in\mathbb{R}^d$, where $K_\beta(x,y):=A(d,\beta)/|x-y|^{d-\beta}$ for $\beta\in]0,2]$. In particular, X is transient and $r(x,x)=\infty$ for $x\in\mathbb{R}^d$. Note that $r(x,y)=K_\alpha(x,y)$ provided $m=0$. Let u be a Borel function satisfying $u(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$. For $\varepsilon>0$ and $x\in\mathbb{R}^d$, we define the modified fractional Laplacian by

$$\Delta^{\alpha/2,m}_\varepsilon u(x):=A(d,-\alpha)\int_{|x-y|>\varepsilon}\frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\Psi(m^{1/\alpha}|x-y|)dy,$$

and put $\Delta^{\alpha/2,m}u(x):=\lim_{\varepsilon\to0}\Delta^{\alpha/2,m}_\varepsilon u(x)$ whenever the limit exists. It is essentially shown in Lemma 3.5 in [2] (resp. the remark after Definition 3.7 in [2]) that for any $u\in C_c^2(D)$ (resp. $u\in C^2(D)$ satisfying $u(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$), $\Delta^{\alpha/2,m}u$ always exists in $C(\mathbb{R}^d)$ (resp. in $C(D)$). Recall that for $u\in C^2(\mathbb{R}^d)$ with $u(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$, u satisfies (2.2) and (2.3). Hence, for such u and $\varphi\in C_c^2(D)$, $\mathcal{E}(u,\varphi)$ is well-defined and the proof of Lemma 2.6 in [3] shows

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|u(x)-u(y)||\varphi(x)-\varphi(y)|\frac{\Psi(m^{1/\alpha}|x-y|)dxdy}{|x-y|^{d+\alpha}}<\infty,$$

which implies $\mathcal{E}(u,\varphi)=(-\Delta^{\alpha/2,m}u,\varphi)$ and the $\mathcal{E}$-subharmonicity in D of u is equivalent to $\Delta^{\alpha/2,m}u\le0$ on D.

For $\varphi\in C_c(\mathbb{R}^d)$, we set

$$R^{(\alpha)}\varphi(x):=\int_{\mathbb{R}^d}r(x,y)\varphi(y)dy \quad x\in\mathbb{R}^d.$$

Then, we see $R^{(\alpha)}\varphi$ is locally bounded on $\mathbb{R}^d$ and $(R^{(\alpha)}\varphi)(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$ for such φ, because of $r(x,y)\asymp(K_\alpha(x,y)+K_2(x,y))$. Moreover, we see $R^{(\alpha)}\varphi\in$

$\mathcal{F}_{\mathrm{loc}}$ for such φ. Indeed, for any relatively compact open set D with $\overline{D} \subset \mathbb{R}^d$, $R^{(\alpha)}\varphi$ is a difference of excessive functions with respect to X^D and bounded on D, so $R^{(\alpha)}\varphi \in \mathcal{F}_{D,\mathrm{loc}}$ by Theorem 3.9 in [6]. Since D is arbitrary, $R^{(\alpha)}\varphi \in \mathcal{F}_{\mathrm{loc}}$. Thus $R^{(\alpha)}\varphi$ satisfies (2.2) and (2.3) for U, V with $\overline{U} \subset V \subset \overline{V} \subset \mathbb{R}^d$. Similarly, $r(a,\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{a\})$ satisfies $\int_{\mathbb{R}^d} \frac{r(a,x)\Psi(m^{1/\alpha}|x|)}{(1+|x|)^{d+\alpha}}dx < \infty$. We can obtain $r(a,\cdot) \in \mathcal{F}_{\mathbb{R}^d\setminus\{a\},\mathrm{loc}}$ in a similar way as above. Hence $r(a,\cdot)$ satisfies (2.2) and (2.3) for U, V with $\overline{U} \subset V \subset \overline{V} \subset \mathbb{R}^d \setminus \{a\}$. Note that for $\varphi \in C_c^\infty(D)$, $\Delta^{\alpha/2,m}\varphi = L^{\alpha,m}\varphi$ a.e. on $\mathbb{R}^d$ and $R^{(\alpha)}\Delta^{\alpha/2,m}\varphi = -\varphi$ on $\mathbb{R}^d$. Here $L^{\alpha,m}$ is the L^2-generator of $(\mathcal{E}^{\mathrm{R},\alpha}, \mathcal{F}^{\mathrm{R},\alpha})$.

For $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{a\})$, we then have

$$\begin{aligned}\mathcal{E}(r(a,\cdot),\varphi) &= -\int_{\mathbb{R}^d} r(a,x)\Delta^{\alpha/2,m}\varphi(x)dx \\ &= -(R^{(\alpha)}\Delta^{\alpha/2,m}\varphi)(a) = \varphi(a) = 0.\end{aligned}$$

This means the $\mathcal{E}$-harmonicity in $\mathbb{R}^d \setminus \{a\}$ of $r(a,\cdot)$. Similarly, for non-negative $\psi, \varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\mathcal{E}(R^{(\alpha)}\psi, \varphi) = (\psi, -R^{(\alpha)}\Delta^{\alpha/2,m}\varphi) = (\psi,\varphi) \geq 0,$$

which implies the $\mathcal{E}$-superharmonicity of $R^{(\alpha)}\psi$ for non-negative $\psi \in C_c^\infty(\mathbb{R}^d)$.

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