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Branching Brownian motions in random environment

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Abstract

This article is a survey on branching Brownian motions in time-space random environment associated with the Poisson random measure. We show the existence of the phase transitions in terms of the population growth rate and of the diffusivity as follows: if the effect from the randomness of the environment is weak enough, the population growth rate is the same with its expectation almost surely and the population density satisfies the central limit theorem. In contrast with this, if the effect is strong enough, the population growth rate is strictly less that its expectation almost surely and particles concentrate on small sets infinitely often.

1 Introduction

We consider a model of branching Brownian motions in time-space random environment associated with the Poisson random measure. In particular, we are concerned with the population growth rate and the diffusivity of the population density. For the non-random environment model, it is well known that the growth rate is the same with that of the expected total population size and the population density satisfies the central limit theorem. However, our model may have quite different properties because of the correlation among Brownian particles caused by the randomness of the environment. In the present article, we give a survey on this problem and related topics by following [32] and [33].

Smith-Wilkinson [35] and Athreya-Karlin [1], [2] formulated models of discrete time branching processes in random environment as a generalization of the classical Galton-Watson process (see also [3]). There the offspring distribution forms a stochastic process indexed by generation. A notable feature of these models is the correlation among particles caused by the randomness of the offspring distribution. On account of this, these models are different from the classical Galton-Watson process in many properties such as the extinction condition and the population growth rate. For instance, see [34], [35] for the Smith-Wilkinson model and [1], [2] for the Athreya-Karlin model. Kaplan [25] also introduced a continuous time model and obtained the counterparts of the results as we mentioned above. In this model, the offspring and the splitting time distributions form stochastic processes and are independent to each other.

The case of interest here is a model of branching processes with spatial motions; particles reproduce according to a Galton-Watson process and move in space according to

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a stochastic process. Then the peculiar property to this model is the diffusivity of particles. For the non-random environment model, we may guess that the diffusivity of this model is the same with that of the underlying process. In fact, this is true for branching Brownian motions and branching random walks as proved by S. Watanabe (see [3, p.243, Theorem 1]) and Biggins [6], respectively (see also [11], [18], [37] and references therein for related results on asymptotic properties of branching Markov processes). However, if we cope with models of branching processes with spatial motions in random environment, both the population growth rate and the diffusivity are affected by the correlation among particles. In particular, N. Yoshida [38] and Hu-N. Yoshida [20] proved the existence of the phase transitions of these properties for a model of branching random walks in discrete time-space random environment. There the offspring distributions attached to time-space points are independent and identically distributed (see also [7], [17] and [29] for more results on this model).

The goal of the present article is to study the population growth rate and the diffusivity for a continuous time-space model, that is, a model of branching Brownian motions in time-space random environment. To consider this subject, we formulate the model so that the splitting time distributions of particles are correlated to each other by the time-space Poisson random measure. Roughly speaking, each Brownian particle splits early in proportion to the number of Poisson points over the trajectory of the particle. We can then prove the following: if the spatial dimension is high and the correlation is weak enough, the growth rate of the population size is the same with that of the expectation almost surely and the population density satisfies the central limit theorem. Therefore, the situation of our model is the same with that of the non-random environment model. In contrast with this, if the spatial dimension is low or the correlation is strong enough, the growth rate of the population size is strictly less than that of the expectation almost surely and particles concentrate on small sets infinitely often.

The results we stated above are continuous counterparts of those established by N. Yoshida [38] and Hu-N. Yoshida [20], and we take an approach similar to theirs. Here we explain our motivation for introducing and studying the continuous time model: we can often investigate the continuous time model more in detail than the discrete time one by stochastic analysis. For directed polymers in random environment, Comets and N. Yoshida [15, 16] introduced a continuous time-space model which is called Brownian directed polymers in random environment, and obtained several detailed results by applying stochastic analysis. Concerning our model, we do not satisfy the motivation yet, but we hope that our model is applicable for the detailed study.

Here it should be mentioned that the phase transitions of the population growth rate (or the growth rate of the partition function) and of the diffusivity appear in many models such as directed polymers in random environment ([8], [9], [12], [13], [14], [15], [16]), branching random walks in random environment ([17], [20], [29], [38]), linear stochastic evolutions ([28], [40], [41]) and linear systems ([26], [27]). Furthermore, similar techniques can be applied for the study of these models and our model. Among them, Brownian directed polymers in random environment ([15], [16]) are closely related to our model. In fact, if we fix an environment, the expected population size of the branching model coincides with the so called partition function of the directed polymer model. This relation will be explained more precisely in Section 4 below.

2 Model

2.1 Construction

A branching process we consider in this article is defined by the Brownian motion on $\mathbb{R}^d$ and the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ for $\mathbb{R}_+ := [0, \infty)$. Following [39], we first give some notations of them and then construct branching Brownian motions in time-space random environment. We remark that Savits [31] also constructed branching Markov processes in time-space random environment by applying the results by N. Ikeda, M. Nagasawa and S. Watanabe ([21], [22], [23]), but our construction is more direct and self-contained.

Let η denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with unit intensity on a probability space $(\mathcal{M}, \mathcal{G}, Q)$. Namely, η is a non-negative integer valued random measure such that, $\eta(A_1), \ldots, \eta(A_n)$ are mutually independent for disjoint and bounded sets $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ and

$$Q(\eta(A) = k) = e^{-|A|}\frac{|A|^k}{k!} \quad \text{for } A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d).$$

Here $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ is the family of all Borel measurable sets on $\mathbb{R}_+ \times \mathbb{R}^d$ and $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^{1+d}$. Let $\{\theta_t\}_{t\geq 0}$ be the time shift operator of the Poisson random measure, that is, $\theta_t\eta = \theta_t\eta(\mathrm{d}s, \mathrm{d}x) = \eta(\{t\} + \mathrm{d}s, \mathrm{d}x)$ identically for any $s, t \geq 0$. The notation $\theta_t\eta$ is often abbreviated to η_t. We denote by $\{\mathcal{G}_t\}_{t\geq 0}$ the family of the sub-σ-field of $\mathcal{G}$ defined by

$$\mathcal{G}_t = \sigma\left(\eta(A \cap ((0, t] \times \mathbb{R}^d)), A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)\right).$$

Let $\mathbf{M} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}, \{\theta_t\}_{t\geq 0})$ be the Brownian motion on $\mathbb{R}^d$, where $\{\theta_t\}_{t\geq 0}$ is the time shift operator of paths, that is, for each path $\omega \in \Omega$, $B_t(\theta_s\omega) = B_{t+s}(\omega)$ identically for any $s, t \geq 0$. Note that we use the same notation $\{\theta_t\}_{t\geq 0}$ as the time shift operators of paths and of the Poisson random measure, respectively. Denote by V_t the tube around the graph $\{(s, B_s)\}_{0<s\leq t}$ defined by

$$V_t = V_t(\omega) = \{(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \mid s \in (0, t],\ x \in U(B_s(\omega))\} \quad \text{for } \omega \in \Omega,$$

where $U(x)$ is a closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with unit volume. Here we recall that a Brownian particle can not hit any point for $d \geq 2$, but we can recognize $\eta(V_t)$ as the number of Poisson points hit by the particle. We learn this idea from Comets and N. Yoshida [16].

Let τ be a non-negative random variable on $(\Omega, \mathcal{F}, P_x)$, independently of the Brownian motion, of exponential distribution with the mean 1; $P_x(\tau > a) = e^{-a}$ for any $a \geq 0$. Fix a parameter $\alpha > 0$ and set

$$S = S(\omega, \eta) = \inf\{t > 0 \mid \alpha\eta(V_t(\omega)) > \tau(\omega)\} \quad \text{for } (\omega, \eta) \in \Omega \times \mathcal{M}.$$

We then have

$$P_x(S(\cdot,\eta) > t) = E_x\left[e^{-\alpha\eta(V_t)}\right].$$

Here we note that $\{\eta(V_t(\omega))\}_{t\geq 0}$ is a standard Poisson process on the half line for each $\omega \in \Omega$. In particular, the jump size of this process is equal to one Q-a.s. (for instance, see [30, p.472, Proposition 1.4]). Let $\{p_n\}_{n=0}^{\infty}$ be a probability function, that is, $p_n \geq 0$ for any $n \geq 0$ and $\sum_{n=0}^{\infty} p_n = 1$. In the sequel, we assume $p_0 + p_1 < 1$ to avoid the case where the numbers of particles do not increase for branching Brownian motions which will be introduced below. We define

$$m^{(q)} = \sum_{n=0}^{\infty} n^q p_n \quad \text{for } q \geq 0.$$

We also let I be an $\mathbb{N} \cup \{0\}$-valued random variable on $(\Omega, \mathcal{F}, P_x)$, independently of the Brownian motion and τ, associated with $\{p_n\}_{n=0}^{\infty}$ so that $P_x(I = n) = p_n$.

We now introduce the index sets. Define

$$K^0 = \{(0)\}, \quad K^1 = \{(1)\}, \quad K^n = \{(1, k_2, \ldots, k_n) \mid k_2, \ldots, k_n \in \mathbb{N}\} \quad \text{for } n \geq 2$$

and

$$\mathbf{K} = \sum_{n=0}^{\infty} K^n.$$

In addition, it is useful to set

$$\overline{K^0} = \{(0, 1)\}, \quad \overline{K^n} = K^{n+1} \quad \text{for } n \geq 1$$

and

$$\overline{\mathbf{K}} = \sum_{n=0}^{\infty} \overline{K^n}.$$

If $\mathbf{k} = (1, k_2, \ldots, k_n) \in K^n$ for some $n \geq 1$ and $k \in \mathbb{N}$, then we define $\mathbf{k} \cdot k = (1, k_2, \ldots, k_n, k) \in \overline{K^n}$. By the same way, we define $(0) \cdot 1 = (0, 1) \in \overline{K^0}$.

Let $\{B_t^{\mathbf{k}}\}_{t\geq 0}$ and $\tau^{\mathbf{k}}$, $\mathbf{k} \in \mathbf{K}$, be independent copies of $\{B_t\}_{t\geq 0}$ and τ, respectively. Denote by $V_t^{\mathbf{k}}$ the tube V_t associated with the Brownian motion $\{B_t^{\mathbf{k}}\}_{t\geq 0}$, and by $S^{\mathbf{k}}$ the random variable S with τ and V_t replaced by $\tau^{\mathbf{k}}$ and $V_t^{\mathbf{k}}$, respectively. In addition, we set $I^{(0)} = 1$ and let $I^{\mathbf{k}}$, $\mathbf{k} \in \mathbf{K} \setminus K^0$, be independent copies of I, respectively.

We consider the family of random variables $T^{\mathbf{k}}$ and $\{\mathbf{B}_t^{\mathbf{k}}\}_{t\geq 0}$ indexed by $\mathbf{k} \in \mathbf{K}$ on the measurable space $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G})$ as follows; for each fixed $(\omega, \eta) \in \Omega \times \mathcal{M}$, let $T^{(0)}(\omega, \eta) = 0$ and $\mathbf{B}_t^{(0)}(\omega, \eta) = B_t^{(0)}(\omega)$ identically for any $t \geq 0$. We then define inductively for $\mathbf{k} \cdot k \in \overline{\mathbf{K}}$,

$$T^{\mathbf{k}\cdot k} = T^{\mathbf{k}\cdot k}(\omega, \eta) = \begin{cases} T^{\mathbf{k}}(\omega, \eta) + S^{\mathbf{k}\cdot k}(\theta_{T^{\mathbf{k}}(\omega,\eta)}\omega, \theta_{T^{\mathbf{k}}(\omega,\eta)}\eta) & \text{if } k \leq I^{\mathbf{k}}(\omega) \\ \infty & \text{if } k \geq I^{\mathbf{k}}(\omega) + 1, \end{cases}$$

and

$$\begin{aligned} \mathbf{B}_t^{\mathbf{k}\cdot k} &= \mathbf{B}_t^{\mathbf{k}\cdot k}(\omega, \eta) \\ &= \begin{cases} \mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}(\omega,\eta)}(\omega, \eta) + B_t^{\mathbf{k}\cdot k}(\omega) - B^{\mathbf{k}\cdot k}_{T^{\mathbf{k}}(\omega,\eta)}(\omega) & \text{for } T^{\mathbf{k}}(\omega, \eta) \leq t < T^{\mathbf{k}\cdot k}(\omega, \eta) \quad \text{if } k \leq I^{\mathbf{k}}(\omega) \\ \Delta & \text{otherwise,} \end{cases} \end{aligned}$$

where Δ is a cemetery point, $T^{(1)} := T^{(0,1)}$ and $\mathbf{B}_t^{(1)} := \mathbf{B}_t^{(0,1)}$. We use the notations $\mathbf{B}_t^{\mathbf{k}}$ and $T^{\mathbf{k}}$ to denote, respectively, the position and the splitting time of the particle with index $\mathbf{k}$ of a branching Brownian motion. More precisely, we can describe our branching Brownian motion as follows:

- At time 0, the Brownian particle with index 1 starts from $\mathbf{B}_0^{(0)}$.
- The Brownian particle with index $\mathbf{k} \in \mathbf{K} \setminus K^0$ splits into n Brownian particles with probability p_n at site $\mathbf{B}_{T^{\mathbf{k}}}^{\mathbf{k}}$ at time $T^{\mathbf{k}}$.
- These Brownian particles, indexed by $\mathbf{k}\cdot 1, \mathbf{k}\cdot 2, \dots, \mathbf{k}\cdot n$, respectively, start from $\mathbf{B}_{T^{\mathbf{k}}}^{\mathbf{k}}$ independently.

The definition of the splitting time says that each Brownian particle is apt to split if the associated ball with unit volume catches many Poisson points.

Let us introduce the notion of branching Brownian motions in random environment. We define the probability measures $\{\mathbb{P}_x^\eta\}_{x\in\mathbb{R}^d}$ and $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ on $(\Omega \times \mathcal{M}, \mathcal{F}\otimes\mathcal{G})$, respectively, by

$$\mathbb{P}_x^\eta = P_x \otimes \delta_\eta \quad \text{and} \quad \mathbb{P}_x = \int_{\mathcal{M}} Q(\mathrm{d}\eta)\mathbb{P}_x^\eta,$$

where δ_η is the Dirac measure at $\eta \in \mathcal{M}$. We call

$$\overline{\mathbf{M}}^\eta = (\Omega\times\mathcal{M}, \mathcal{F}\otimes\mathcal{G}, \{\mathcal{F}_t\otimes\mathcal{G}_t\}_{t\geq 0}, \{\{\mathbf{B}_t^{\mathbf{k}}\}_{t\geq 0}\}_{\mathbf{k}\in\mathbf{K}}, \{T^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{K}}, \{\mathbb{P}_x^\eta\}_{x\in\mathbb{R}^d})$$

a branching Brownian motion in environment η with offspring distribution $\{p_n\}_{n=0}^\infty$, and

$$\overline{\mathbf{M}} = (\Omega\times\mathcal{M}, \mathcal{F}\otimes\mathcal{G}, \{\mathcal{F}_t\otimes\mathcal{G}_t\}_{t\geq 0}, \{\{\mathbf{B}_t^{\mathbf{k}}\}_{t\geq 0}\}_{\mathbf{k}\in\mathbf{K}}, \{T^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{K}}, \{\mathbb{P}_x\}_{x\in\mathbb{R}^d})$$

a branching Brownian motion in random environment with offspring distribution $\{p_n\}_{n=0}^\infty$. Here it should be emphasized that, for each fixed $\mathbf{k} \in \mathbf{K}$, $T^{\mathbf{k}\cdot 1} - T^{\mathbf{k}}, T^{\mathbf{k}\cdot 2} - T^{\mathbf{k}}, \dots$ are independent to each other under the law $\mathbb{P}^\eta(\cdot\,|\,T^{\mathbf{k}})$, but this is not true under the law $\mathbb{P}(\cdot\,|\,T^{\mathbf{k}})$.

Let $N_t(A)$ be the number of particles on the set $A \in \mathcal{B}(\mathbb{R}^d)$ at time t, that is,

$$N_t(A) = \sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}}\leq t<T^{\mathbf{k}\cdot k},\, \mathbf{B}_t^{\mathbf{k}\cdot k}\in A\}}.$$

We can then regard $N_t(\cdot)$ as a configuration measure of particles at time t. We denote by $\overline{N}_t$ the total number of particles at time t, that is, $\overline{N}_t = N_t(\mathbb{R}^d)$. We also use the notation

$$N_t(f) = \sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}} f\left(\mathbf{B}_t^{\mathbf{k}\cdot k}\right)\mathbf{1}_{\{T^{\mathbf{k}}\leq t<T^{\mathbf{k}\cdot k}\}} \quad \text{for } f\in\mathcal{B}_b(\mathbb{R}^d),$$

where $\mathcal{B}_b(\mathbb{R}^d)$ stands for the set of all bounded Borel measurable functions on $\mathbb{R}^d$.

Remark 2.1. (Extinction. [32]) Since the branching mechanism $\{p_n\}_{n=0}^\infty$ is deterministic in our model, the extinction condition is similar to the Galton-Watson process. In fact, we can prove

$$m^{(1)} \leq 1 \Longrightarrow \mathbb{P}\left(\lim_{t\to\infty}\overline{N}_t = 0\right) = 1$$

by comparing our model with the continuous time Galton-Watson process with branching rate $1-e^{-\alpha}$ and branching mechanism $\{p_n\}_{n=0}^\infty$.

2.2 Moments

Here we give some results on the moments of N_t. In the sequel, we assume that $m^{(1)}$ is finite. Let us define

$$\beta = \log\left\{m^{(1)} - e^{-\alpha}(m^{(1)} - 1)\right\} \quad \text{and} \quad \lambda = \lambda(\beta) := e^{\beta} - 1. \tag{2.1}$$

Lemma 2.2. ([32]) *For any $s, t \geq 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, we have*

$$\mathbb{E}_x^{\eta}\left[N_{t+s}(f) \mid \mathcal{F}_t \otimes \mathcal{G}_t\right] = \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[e^{\beta \eta_t(V_s)} f(B_s)\right] \quad Q\text{-a.s.}$$

and

$$\mathbb{E}_x\left[N_{t+s}(f) \mid \mathcal{F}_t \otimes \mathcal{G}_t\right] = e^{\lambda s} \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[f(B_s)\right].$$

In particular, we obtain

$$\mathbb{E}_x^{\eta}\left[N_t(f)\right] = E_x\left[e^{\beta\eta(V_t)} f(B_t)\right] \quad Q\text{-a.s.} \tag{2.2}$$

and

$$\mathbb{E}_x\left[N_t(f)\right] = e^{\lambda t} E_x\left[f(B_t)\right]. \tag{2.3}$$

By (2.1), (2.2) and (2.3), we have

$$\mathbb{E}_x^{\eta}\left[\overline{N}_t\right] = E_x\left[\left\{m^{(1)} - e^{-\alpha}(m^{(1)} - 1)\right\}^{\eta(V_t)}\right] \quad Q\text{-a.s.}$$

and

$$\mathbb{E}_x\left[\overline{N}_t\right] = e^{\lambda t} = e^{(m^{(1)}-1)(1-e^{-\alpha})t}.$$

Therefore, if we fix an environment, the expected population size of our model is similar to that of discrete time branching processes. On the other hand, if we randomize the environment, the situation is similar to continuous time branching processes.

Let $\overline{M}_t$ be a normalization of the total population size defined by

$$\overline{M}_t = e^{-\lambda t}\overline{N}_t \quad \text{for } t \geq 0. \tag{2.4}$$

Lemma 2.2 then implies that $\overline{M}_t$ is a non-negative martingale on $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G}, \{\mathcal{F}_t \otimes \mathcal{G}_t\}_{t\geq 0}, \mathbb{P}_x)$, whence there exists a limit $\lim_{t\to\infty} \overline{M}_t =: \overline{M}_\infty$ $\mathbb{P}$-a.s. Here we note that the martingale $\overline{M}_t$ includes the information on asymptotic properties similar to branching processes in non-random environment (for instance, see [3]). We can then derive this information by the moment calculation and by Ito's formula.

In what follows, we further assume that $m^{(2)}$ is finite. Let us define

$$c = m^{(2)} - m^{(1)} = \sum_{n=0}^{\infty} n(n-1)p_n \quad \text{and} \quad \mu = 1 - e^{-\alpha}.$$

We denote by $(\{B_t^1\}_{t\geq 0}, \{P_x^1\}_{x\in\mathbb{R}^d})$ and $(\{B_t^2\}_{t\geq 0}, \{P_x^2\}_{x\in\mathbb{R}^d})$ the independent Brownian motions on $\mathbb{R}^d$. We let $P_{x,y} = P_x^1 \otimes P_y^2$ and abbreviate $P_{x,x}$ to P_x.

Lemma 2.3. ([33]) *For any $s, t \geq 0$ and $f, g \in \mathcal{B}_b(\mathbb{R}^d)$, we have*

$$\mathbb{E}_x\left[N_{t+s}(f)N_{t+s}(g)|\mathcal{F}_t \otimes \mathcal{G}_t\right] = \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}\}} \Big(e^{\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[f(B_s)g(B_s)\right]$$

$$+ c\mu e^{2\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[\int_0^s e^{-\lambda u} E_{B_u}\left[\exp\left(\lambda^2 \int_0^{s-u} |U(B_v^1) \cap U(B_v^2)|\, \mathrm{d}v\right) f(B_{s-u}^1)g(B_{s-u}^2)\right] \mathrm{d}u\right]\Big)$$

$$+ \sum_{\substack{\mathbf{k}\cdot k, \bar{\mathbf{k}}\cdot\bar{k} \in \overline{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \bar{\mathbf{k}}\cdot\bar{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k} \\ T^{\bar{\mathbf{k}}} \leq t < T^{\bar{\mathbf{k}}\cdot\bar{k}}}\right\}} e^{2\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\bar{\mathbf{k}}\cdot\bar{k}}}\left[\exp\left(\lambda^2 \int_0^s |U(B_u^1) \cap U(B_u^2)|\, \mathrm{d}u\right) f(B_s^1)g(B_s^2)\right].$$

In particular, we have

$$\mathbb{E}_x\left[\overline{N}_{t+s}^2|\mathcal{F}_t \otimes \mathcal{G}_t\right]$$

$$= \overline{N}_t\left(e^{\lambda s} + c\mu e^{2\lambda s}\int_0^s e^{-\lambda u} E\left[\exp\left(\lambda^2 \int_0^{s-u} |U(B_v^1) \cap U(B_v^2)|\, \mathrm{d}v\right)\right] \mathrm{d}u\right)$$

$$+ \sum_{\substack{\mathbf{k}\cdot k, \bar{\mathbf{k}}\cdot\bar{k} \in \overline{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \bar{\mathbf{k}}\cdot\bar{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k} \\ T^{\bar{\mathbf{k}}} \leq t < T^{\bar{\mathbf{k}}\cdot\bar{k}}}\right\}} e^{2\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\bar{\mathbf{k}}\cdot\bar{k}}}\left[\exp\left(\lambda^2 \int_0^s |U(B_u^1) \cap U(B_u^2)|\, \mathrm{d}u\right)\right].$$

Related to Lemma 2.3, we should keep in mind that the value

$$\exp\left(\lambda^2 \int_0^t |U(B_s^1) \cap U(B_s^2)|\, \mathrm{d}s\right)$$

expresses how often two independent Brownian particles "meet" together. This value comes from the fact that some Brownian balls can catch a Poisson point at the same time. In other words, this value measures the magnitude of the correlation among particles caused by the Poisson random measure.

Let $\langle\overline{M}\rangle_t$ be a predictable quadratic variation of the martingale $\overline{M}_t$, that is, $\langle\overline{M}\rangle_t$ is a unique predictable and locally integrable increasing process such that $\overline{M}_t^2 - \langle\overline{M}\rangle_t$ is a locally square integrable martingale (see [19, p.199, 7.28 Lemma]).

Proposition 2.4. ([33]) *We get the following equality.*

$$\langle\overline{M}\rangle_t = \left\{\sum_{n=1}^{\infty}(n-1)^2 p_n\right\}\mu\int_0^t e^{-\lambda s}\overline{M}_s\,\mathrm{d}s + \lambda^2\int_0^t\left(\int_{\mathbb{R}^d} M_s(U(x))^2\,\mathrm{d}x - e^{-\lambda s}\overline{M}_s\right)\mathrm{d}s \tag{2.5}$$

for $t \geq 0$.

Here we give a remark on the predictable quadratic variation $\langle\overline{M}\rangle_t$. The equality

$$\int_{\mathbb{R}^d} M_s(U(x))^2\,\mathrm{d}x - e^{-\lambda s}\overline{M}_s = e^{-2\lambda s}\int_{\mathbb{R}^d}\sum_{\substack{\mathbf{k}\cdot k, \bar{\mathbf{k}}\cdot\bar{k} \in \overline{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \bar{\mathbf{k}}\cdot\bar{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \leq s < T^{\mathbf{k}\cdot k} \\ \mathbf{B}_s^{\mathbf{k}\cdot k} \in U(x)}\right\}}\mathbf{1}_{\left\{\substack{T^{\bar{\mathbf{k}}} \leq s < T^{\bar{\mathbf{k}}\cdot\bar{k}} \\ \mathbf{B}_s^{\bar{\mathbf{k}}\cdot\bar{k}} \in U(x)}\right\}}\,\mathrm{d}x$$

implies that the second term of the right hand side of (2.5) is closely related to the correlation among particles because the magnitude of the correlation is proportional to the degree to which pairs of particles meet together

3 Results

In this section, we state the results in this article. These results are the continuous model version of those obtained by N. Yoshida [38] and Hu-N. Yoshida [20] for branching random walks in random environment. In the sequel, we denote by P, $\mathbb{P}^\eta$, $\mathbb{P}$, etc. the quantities P_x, $\mathbb{P}^\eta_x$, $\mathbb{P}_x$, etc. for $x = 0$, respectively.

3.1 Regular growth and diffusivity

In this subsection, we show that, if the correlation among particles is weak enough, then the properties of our model are similar to branching Brownian motions in non-random environment. Here the non-random environment means that the splitting times of particles are independent and identically distributed with the given exponential distribution.

Define

$$M_t(\mathrm{d}x) = e^{-\lambda t} N_t(\mathrm{d}x) \quad \text{and} \quad \rho(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right).$$

Let $C_b(\mathbb{R}^d)$ stand for the set of all bounded and continuous functions on $\mathbb{R}^d$.

Theorem 3.1. ([32]) *Assume*

$$d \geq 3, \quad m^{(1)} > 1 \quad \text{and} \quad m^{(2)} < \infty.$$

Then the following conditions are equivalent to each other:

(i) $E\left[\exp\left(\lambda^2 \int_0^\infty |U(B_t^1) \cap U(B_t^2)| \,\mathrm{d}t\right)\right] < \infty$;

(ii) $\lim_{t\to\infty} \overline{M}_t = \overline{M}_\infty$ *in* $L^2(\mathbb{P})$;

(iii) $\lim_{t\to\infty} \int_{\mathbb{R}^d} f\left(\frac{x}{\sqrt{t}}\right) M_t(\mathrm{d}x) = \overline{M}_\infty \int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x$ *in* $L^2(\mathbb{P})$ *for any* $f \in C_b(\mathbb{R}^d)$.

Remark 3.2. Related to the comment after Lemma 2.3, Condition (i) means that the correlation among particles is weak enough. Furthermore, since Lemma 2.3 implies

$$\mathbb{E}_x\left[\overline{M}_t^2\right] = e^{-\lambda t} + c\mu \int_0^t e^{-\lambda s} E\left[\exp\left(\lambda^2 \int_0^{t-s} |U(B_u^1) \cap U(B_u^2)| \,\mathrm{d}u\right)\right] \mathrm{d}s, \tag{3.1}$$

Conditions (i) and (ii) are equivalent to each other. From another point of view, Condition (i) says that the randomness of the Brownian motion moderates that of the environment. In fact, if we formally replace both B_t^1 and B_t^2 in Condition (i) with the origin, that is, we assume that particles stay at the origin forever, then the expectation diverges to infinity.

Here we give another remark on Condition (i). Recall first the relation

$$(\{B_t^2 - B_t^1\}_{t\geq 0}, P_x) \stackrel{\mathrm{d}}{=} (\{B_{2t}\}_{t\geq 0}, P), \tag{3.2}$$

where $\stackrel{\mathrm{d}}{=}$ means that the both hand sides have the same law. Since this implies

$$E\left[\exp\left(\lambda^2 \int_0^\infty |U(B_t^1) \cap U(B_t^2)| \,\mathrm{d}t\right)\right] = E\left[\exp\left(\frac{\lambda^2}{2} \int_0^\infty |U(0) \cap U(B_t)| \,\mathrm{d}t\right)\right],$$

we see from [10, Theorem 5.1] and [36, Theorem 2.4] that Condition (i) is equivalent to say

$$\inf\left\{\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u(x)|^2\,\mathrm{d}x\,\middle|\,u\in C_0^\infty(\mathbb{R}^d),\ \frac{\lambda^2}{2}\int_{\mathbb{R}^d}u(x)^2\,|U(0)\cap U(x)|\,\mathrm{d}x=1\right\}>1,$$

where $C_0^\infty(\mathbb{R}^d)$ denotes the totality of infinitely differentiable functions with compact support in $\mathbb{R}^d$. [16, Proposition 4.2.1] also yields that Condition (i) holds if

$$\beta\in\left(0,\log\left(1+\frac{\gamma_d}{2r_d}\right)\right),$$

where $r_d=\beta((d+2)/2)^{1/d}/\sqrt{\pi}$ is the radius of $U(0)$ and γ_d is the smallest positive zero of the Bessel function $J_{(d-4)/2}$ defined by

$$J_\nu(\gamma)=\left(\frac{\gamma}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-\gamma^2/4)^k}{k!\gamma(\nu+k+1)}\quad\text{for }\gamma\geq 0\text{ and }\nu>-1.$$

In contrast with $d\geq 3$, when $d=1$ or 2, the Brownian motion is recurrent and a pair of particles is apt to meet together as we can see from (3.2). Hence the correlation among particles is so strong that Condition (i) does not hold.

Let $\rho_t(\mathrm{d}x)$ be the population density at time t defined by

$$\rho_t(\mathrm{d}x)=\frac{N_t(\mathrm{d}x)}{\overline{N}_t}.$$

We then get

Corollary 3.3. (Central limit theorem. [32]) *Assume*

$$d\geq 3,\quad m^{(1)}>1\quad\textit{and}\quad m^{(2)}<\infty.$$

If one of the conditions in Theorem 3.1 *holds, then*

$$\lim_{t\to\infty}\int_{\mathbb{R}^d}f\left(\frac{x}{\sqrt{t}}\right)\rho_t(\mathrm{d}x)=\int_{\mathbb{R}^d}f(x)\rho(x)\,\mathrm{d}x\text{ in }\mathbb{P}(\cdot\,|\,\overline{M}_\infty>0)\text{-probability}$$

for any $f\in C_b(\mathbb{R}^d)$.

Corollary 3.3 says that the population density $\rho_t(\mathrm{d}x)$ converges weakly to the standard normal distribution under the Brownian scale. We note that S. Watanabe and Nakashima proved respectively almost sure central limit theorems of this type for branching Brownian motions in non-random environment (see [3, p.245]) and for branching random walks in random environment ([29]).

Related to the population density $\rho_t(\mathrm{d}x)$, we let

$$\overline{\rho}_t=\sup_{x\in\mathbb{R}^d}\rho_t(U(x))\quad\text{and}\quad R_t=\int_{\mathbb{R}^d}\rho_t(U(x))^2\,\mathrm{d}x.\tag{3.3}$$

We can then regard $\overline{\rho}_t$ as the density at the most populated site and R_t as the replica overlap by analogy with the spin glass theory. Furthermore, by the same way as that in [16, Theorem 2.3.2], there exists a constant $c = c(d) \in (0,1)$ such that

$$c\overline{\rho}_t^2 \le R_t \le \overline{\rho}_t \quad \text{for any } t \ge 0. \tag{3.4}$$

We now characterize the diffusive behavior of our model in terms of the decay rate of the replica overlap:

Proposition 3.4. ([32]) *Assume*

$$d \ge 3, \quad m^{(1)} > 1 \quad \textit{and} \quad m^{(2)} < \infty.$$

If one of the conditions in Theorem 3.1 *holds, then*

$$R_t = O(t^{-d/2}) \text{ in } \mathbb{P}(\cdot \mid \overline{M}_\infty > 0)\text{-probability}.$$

3.2 Slow growth and localization

In this subsection, we assume that the spatial dimension d is one or two, or the parameter λ is large enough. For $d = 1$ or 2, the correlation among particles becomes strong enough as we mentioned above. Even for $d \ge 3$, the situation is similar to the former case for large λ. Therefore, under such situations, the population growth rate and the diffusivity of our model change dramatically.

We first consider the population growth rate. Since the exponential growth rate of $\mathbb{E}^\eta[\overline{M}_t]$ is strictly negative Q-a.s. as we will see in Section 4, we have the following:

Theorem 3.5. (Slow growth. [32]) *For $d = 1$ or 2, $\mathbb{P}(\overline{M}_\infty = 0) = 1$ holds for any $\beta > 0$. On the other hand, for $d \ge 3$, there exists a positive constant $\beta_0(d) > 0$ such that $\mathbb{P}(\overline{M}_\infty = 0) = 1$ holds for any $\beta > \beta_0(d)$. Moreover, for any dimension d, there exists a non-negative constant $\beta_1(d) \ge 0$ such that, for each $\beta > \beta_1(d)$,*

$$\limsup_{t\to\infty} \frac{\log \overline{M}_t}{t} < -c(\beta) \quad \mathbb{P}\text{-a.s.}$$

holds with some non-random constant $c(\beta) > 0$. In particular, we have $\beta_1(1) = \beta_1(2) = 0$ and $\beta_1(d) > 0$ for $d \ge 3$.

Theorem 3.5 says that, if the randomness of the environment is strong enough, the growth rate of the population size is strictly less than its expectation almost surely. This result contrasts with the non-random environment case and the weak random environment case as we discussed before.

We next consider the diffusivity. Here we recall that each particle splits early in proportion to the number to Poisson points over the passage area of the associated ball. Since Brownian balls can catch common Poisson points at the same time, the splitting places of some particles may be close to each other. Moreover, if the correlation is strong enough, such a tendency increases so that particles may concentrate on small sets. To confirm this property, we establish the following relations between the slow population growth and the localization property in terms of the replica overlap:

Theorem 3.6. ([33]) (i) *Assume*

$$p_0 = 0, \quad m^{(1)} > 1 \quad \text{and} \quad m^{(2)} < \infty. \tag{3.5}$$

Then we have the relation

$$\{\overline{M}_\infty = 0\} \subset \left\{ \int_0^\infty R_t \, \mathrm{d}t = \infty \right\} \quad \mathbb{P}\text{-a.s.}$$

Furthermore, if $\mathbb{P}(\overline{M}_\infty = 0) = 1$ *holds, then there exists a non-random positive constant* $c > 0$ *such that*

$$\int_0^t R_s \, \mathrm{d}s \geq -c \log \overline{M}_t \quad \text{for any } t \geq T$$

for some random positive constant $T > 0$.
(ii) *Assume*

$$p_0 = 0 \quad \text{and there exists } L \geq 2 \text{ such that } p_n = 0 \text{ for any } n \geq L + 1. \tag{3.6}$$

Then we also have the relation

$$\{\overline{M}_\infty = 0\} = \left\{ \int_0^\infty R_t \, \mathrm{d}t = \infty \right\} \quad \mathbb{P}\text{-a.s.}$$

If $\mathbb{P}(\overline{M}_\infty = 0) = 1$, *then there exist non-random positive constants* $c_1, c_2 > 0$ *such that*

$$-c_1 \log \overline{M}_t \leq \int_0^t R_s \, \mathrm{d}s \leq -c_2 \log \overline{M}_t \quad \text{for any } t \geq T$$

for some random positive constant $T > 0$.

We now give a sketch of the proof of Theorem 3.6 (ii). In the sequel, we use the following notations: for functions f and g defined on a set $A \subset \mathbb{R}^d$, we write $f \asymp g$ on a set A if there exist two positive constants $c_1, c_2 > 0$ such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ holds for any $x \in A$. For functions f and g defined on $\mathbb{R}_+$, we write $f \sim g$ as $t \to \infty$ if $\lim_{t \to \infty} f(t)/g(t) = 1$ holds.

We first note that $\overline{M}_t$ is a purely discontinuous martingale because $\overline{M}_t$ is of finite variation on each finite time interval (see [24, p.41, 4.14 Lemma (b)]). Therefore, if $[\overline{M}]_t$ denotes the quadratic variation of $\overline{M}_t$, then we get

$$[\overline{M}]_t = \overline{M}_0^2 + \sum_{\substack{0 < s \leq t \\ \Delta \overline{M}_s \neq 0}} (\Delta \overline{M}_s)^2$$

for

$$\overline{M}_{t-} := \lim_{s \uparrow t} \overline{M}_s \quad \text{and} \quad \Delta \overline{M}_t := \overline{M}_t - \overline{M}_{t-}.$$

Moreover, by Ito's formula ([24, p.57, Theorem 4.57]) applied to $-\log \overline{M}_t$ and (3.6), we have

$$-\log \overline{M}_t \asymp -\int_0^t \frac{1}{\overline{M}_{s-}} \, \mathrm{d}\overline{M}_s + \int_0^t \frac{1}{\overline{M}_{s-}^2} \, \mathrm{d}[\overline{M}]_s \quad \text{for any } t > 0.$$

By [19, p.291, 10.7] and Proposition 2.4, we know

$$\int_0^t \frac{1}{\overline{M}_{s-}^2} \mathrm{d}[\overline{M}]_s \sim \int_0^t \frac{1}{\overline{M}_s^2} \mathrm{d}\langle \overline{M} \rangle_s \asymp \int_0^t R_s \,\mathrm{d}s \quad \text{as } t \to \infty.$$

In addition, the finite variation part $\int_0^t 1/\overline{M}_{s-}^2 \,\mathrm{d}[\overline{M}]_s$ dominates the martingale part $-\int_0^t 1/\overline{M}_{s-} \,\mathrm{d}\overline{M}_s$ by the law of large numbers ([19, p.247, 9.38 Corollary]). Hence $-\log \overline{M}_t$ is comparable to $\int_0^t R_s \,\mathrm{d}s$, which completes the proof.

Using Theorems 3.5 and 3.6 with (3.4), we can derive the strong localization property in terms of the population density.

Corollary 3.7. (Localization. [33]) *Assume the condition* (3.5). *Then, for any* $\beta > \beta_1(d)$, *we have*

$$\limsup_{t \to \infty} \overline{\rho}_t \geq \limsup_{t \to \infty} R_t \geq c'(\beta) \qquad \mathbb{P}\text{-a.s.}$$

with some non-random positive constant $c'(\beta) \in (0, 1)$.

4 Connection with Brownian directed polymers in random environment

In this section, we confirm a connection between the model of branching Brownian motions in random environment and the model of Brownian directed polymers in random environment introduced by Comets and N. Yoshida [16]. Let μ_t^x be a probability measure on $(\Omega, \mathcal{F})$, the so called polymer measure, defined by

$$\mu_t^x(\mathrm{d}\omega) = \frac{e^{\beta\eta(V_t)}}{Z_t^x} P_x(\mathrm{d}\omega) \quad \eta \in \mathcal{M},$$

where $\beta \in \mathbb{R}$ is a parameter and Z_t^x is the partition function defined by

$$Z_t^x = E_x\left[e^{\beta\eta(V_t)}\right].$$

The size of $\eta(V_t)$ is then considered as the total number of impurities governed by η in the tube V_t, and thus the polymer measure is nothing but the law of the Brownian motion in environment η.

Let

$$W_t = e^{-\lambda t} Z_t$$

for $\lambda = \lambda(\beta) = e^\beta - 1$ as we defined in (2.1). Then W_t is called the normalized partition function because $Q[W_t] = 1$ holds. In addition, since the process $\{\eta(V_t(\omega))\}_{t \geq 0}$ has independent Poisson increments for each $\omega \in \Omega$, W_t is a mean-one, right continuous and left limited, positive martingale on $(\mathcal{M}, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, Q)$, whence the limit $W_\infty := \lim_{t\to\infty} W_t$ exists Q-a.s. By noting that $e^{\beta\eta(V_t)} > 0$ holds for all $t \geq 0$, the event $\{W_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{t \geq 1} \sigma(\eta|_{[t,\infty) \times \mathbb{R}^d}).$$

Furthermore, Kolmogorov's 0–1 law implies $Q(W_\infty > 0) = 1$ or $Q(W_\infty = 0) = 1$. The situation $Q(W_\infty > 0) = 1$ is called the weak disorder and another situation $Q(W_\infty = 0) = 1$ the strong disorder.

In the sequel, let β and $\lambda = \lambda(\beta)$ be the same as we defined in (2.1). Since (2.3) yields

$$\mathbb{E}^\eta [N_t(A)] = E\left[e^{\beta\eta(V_t)}; B_t \in A\right] \quad \text{and} \quad \mathbb{E}^\eta \left[\overline{N_t}\right] = Z_t \tag{4.1}$$

for any $\eta \in \mathcal{M}$, we obtain

$$\mathbb{E}^\eta [M_t(A)] = e^{-\lambda t} E\left[e^{\beta\eta(V_t)}; B_t \in A\right] \quad \text{and} \quad \mathbb{E}^\eta \left[\overline{M_t}\right] = W_t, \tag{4.2}$$

and thus

$$\mu_t(B_t \in A) = \frac{\mathbb{E}^\eta [N_t(A)]}{\mathbb{E}^\eta \left[\overline{N_t}\right]} = \frac{\mathbb{E}^\eta [M_t(A)]}{\mathbb{E}^\eta \left[\overline{M_t}\right]}.$$

Moreover, (4.1) says that the model of branching Brownian motions in random environment is more random than that of Brownian directed polymers in random environment. However, as we already saw before, we can study the properties of the population growth rate and of the diffusivity behavior of the former model in a similar way to the latter model (see [16]).

We finally explain how Theorem 3.5 follows from the relation (4.2). Comets and N. Yoshida [16, Theorem 2.1.1] showed the existence of the phase transition for Brownian directed polymers in random environment in terms of the so called free energy defined by

$$\psi(\beta) := \lim_{t\to\infty} -\frac{1}{t} \log W_t \quad Q\text{-a.s.}$$

(the existence of the limit follows from the subadditive argument and $\psi(\beta) \geq 0$ holds for any $\beta > 0$). More precisely, they proved that there exists a critical value $\beta_c = \beta_c(d) \geq 0$ such that

$$\psi(\beta) = 0 \iff 0 < \beta \leq \beta_c$$

and

$$\beta_c(d) > 0 \quad \text{for } d \geq 3, \quad \lim_{d\to\infty} \beta_c(d) = \infty.$$

Furthermore, Bertin ([4], [5]) recently proved

$$\beta_c(1) = \beta_c(2) = 0.$$

Hence, combining these results with (4.2), we get Theorem 3.5.

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