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Analytic properties of smooth measures in the non-sectorial case

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Summary. We show that the smooth measures related to a generalized Dirichlet form have similar analytic properties than in the symmetric case (cf. [2])

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1 Foreword

In the following presentation we will derive analytic properties of so-called smooth measures and by this complete the theoretical results of [8], [9]. Smooth measures play a central role in the analytic potential theory as well as in the calculus of additive functionals related to (generalized) Dirichlet forms and associated Markov processes. As far as possible we provide direct and simple proofs, though some of the results (e.g. Lemma 3.3, Theorem 4.7) may previously have been shown by more sophisticated means. In particular Theorem 4.7 may also follow from results in [1].

2 Framework

Let E be a Hausdorff space such that its Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $\mathcal{C}(E)$ of all continuous functions on E. Let m be a σ-finite measure on $(E,\mathcal{B}(E))$ such that $\mathcal{H}=L^2(E,m)$ is a separable (real) Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$. Let $(\mathcal{A},\mathcal{V})$ be a real valued coercive closed form on $\mathcal{H}$, i.e. $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $\mathcal{A}:\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is a positive definite bilinear map, $\mathcal{V}$ is a Hilbert space with inner product $\widetilde{\mathcal{A}}_1(u,v):=\frac{1}{2}(\mathcal{A}(u,v)+\mathcal{A}(v,u))+(u,v)_{\mathcal{H}}$, and $\mathcal{A}$ satisfies the *weak sector condition*

$$|\mathcal{A}_1(u,v)|\le K\mathcal{A}_1(u,u)^{1/2}\mathcal{A}_1(v,v)^{1/2},$$

$u, v \in \mathcal{V}$, with *sector constant* K. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously. Since $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $(\mathcal{V}, \widetilde{\mathcal{A}}_1(\cdot,\cdot)^{1/2})$ is again a separable real Hilbert space. Let $\|\cdot\|_{\mathcal{V}}$ be the corresponding norm.
For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ or $\mathcal{V}'$ we will use from now on the notation (Λ, D). Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0-semigroup of contractions $(U_t)_{t\geq 0}$.
(ii) $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restricted semigroup. From [7, Lemma I.2.3,p.12] we have that if $(\Lambda, D(\Lambda, \mathcal{H}))$ satisfies **D1** then $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ is closable as an operator from $\mathcal{V}$ into $\mathcal{V}'$. Let $(\Lambda, \mathcal{F})$ denote its closure, then $\mathcal{F}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2.$$

By [7, Lemma I.2.4,p.13] the adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ can be extended to a C_0-semigroup on $\mathcal{V}'$ and the corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Let $\widehat{\mathcal{F}} := D(\widehat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$. Then $\widehat{\mathcal{F}}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\widehat{\mathcal{F}}}^2 := \|u\|_{\mathcal{V}}^2 + \|\widehat{\Lambda} u\|_{\mathcal{V}'}^2.$$

Let the form $\mathcal{E}$ be given by

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \widehat{\Lambda} v, u\rangle & \text{for } u \in \mathcal{V},\ v \in \widehat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha > 0$. $\mathcal{E}$ is called the *bilinear form associated with* $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$.
Here, $\langle\cdot,\cdot\rangle$ denotes the dualization between $\mathcal{V}'$ and $\mathcal{V}$. Note that $\langle\cdot,\cdot\rangle$ restricted to $\mathcal{H}\times\mathcal{V}$ coincides with $(\cdot,\cdot)_{\mathcal{H}}$ and that $\mathcal{E}$ is well-defined. It follows, from [7, Proposition I.3.4.,p.19], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \to \widehat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u\rangle = \mathcal{E}_\alpha(u, \widehat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. Furthermore $(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha) W_\alpha W_\beta \ \textit{and}\ \widehat{W}_\alpha - \widehat{W}_\beta = (\beta-\alpha)\widehat{W}_\alpha \widehat{W}_\beta.$$

Restricting W_α to $\mathcal{H}$ we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ in $\mathcal{V}$ for all $f \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha>0}$ is called the *resolvent associated with* $\mathcal{E}$. Let $(\widehat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in $\mathcal{H}$. $(\widehat{G}_\alpha)_{\alpha>0}$ is called the *coresolvent associated with* $\mathcal{E}$.
A bounded linear operator $G : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* if $0 \leq Gf \leq 1$ for all $f \in \mathcal{H}$ with $0 \leq f \leq 1$. By [7, Proposition I.4.6, p.24] we have that $(G_\alpha)_{\alpha>0}$ is sub-Markovian if and only if

$$\textbf{D2} \quad u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \ \textit{and}\ \mathcal{E}(u, u - u^+ \wedge 1) \geq 0$$

is satisfied.

Definition 2.1 *The bilinear form $\mathcal{E}$ associated with $(\mathcal{A},\mathcal{V})$ and $(\Lambda, D(\Lambda,\mathcal{H}))$ is called a* **generalized Dirichlet form** *if* **D2** *holds.*

Examples 2.2 *(i) Let $(\mathcal{A},\mathcal{V})$ be a Dirichlet form (cf. [2], [3]) and $\Lambda = 0$. Then $\mathcal{F} = \mathcal{V} = \widehat{\mathcal{F}}$. And $\mathcal{E} = \mathcal{A}$ is a generalized Dirichlet form since the resolvent of $\mathcal{A}$ is sub-Markovian and therefore* **D2** *is satisfied.*
(ii) Let $\mathcal{A} = 0$ on $\mathcal{V} := \mathcal{H}$ and $(\Lambda, D(\Lambda))$ be a Dirichlet operator (cf. e.g. [3]) generating a C_0-semigroup of contractions on $\mathcal{H}$. In this case $\mathcal{F} = D(\Lambda)$, $\widehat{\mathcal{F}} = D(\widehat{\Lambda})$ and the corresponding bilinear form $\mathcal{E}(u,v) = (-\Lambda u, v)_{\mathcal{H}}$ if $u \in D(\Lambda)$, $v \in \mathcal{H}$, and $\mathcal{E}(u,v) = (u, -\widehat{\Lambda} v)_{\mathcal{H}}$ if $u \in \mathcal{H}$, $v \in D(\widehat{\Lambda})$, is a generalized Dirichlet form.

An element u of $\mathcal{H}$ is called 1-excessive (resp. 1-coexcessive) if $\beta G_{\beta+1} u \leq u$ (resp. $\beta \widehat{G}_{\beta+1} u \leq u$) for all $\beta \geq 0$. Let $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of $\mathcal{V}$. Let $\mathcal{C},\mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} \mid \exists f \in \mathcal{C},\ u \leq f\}$. For an arbitrary Borel set $B \in \mathcal{B}(E)$ and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} \mid v \geq u \cdot 1_B\} \cap \mathcal{F} \neq \emptyset$ (resp. $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$) let $u_B := e_{u\cdot 1_B}$ be the 1-*reduced function* (resp. $\hat{u}_B := \hat{e}_{\hat{u}\cdot 1_B}$ be the 1-*coreduced function*) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [7, Definition III.1.8., p.65]. Here we use the notation 1_B for the characteristic function of B. Note that in general only if B is open our definition of reduced function coincides with the one of [2, p.92], [3, Exercise III.3.10(ii), p.84]. In particular, if $B \in \mathcal{B}(E)$ is such that $m(B) = 0$, then $u_B = 0$. We will use the following quite often in the sequel (cf. [7, Proposition III.1.6. and proof of Proposition III.1.7.]): for $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$, $B \in \mathcal{B}(E)$ there exists $\hat{u}_B^\alpha \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$ such that $\hat{u}_B^\alpha \leq \hat{u}_B^\beta$, $0 < \alpha \leq \beta$, $\hat{u}_B^\alpha \to \hat{u}$, $\alpha \to \infty$, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$, and

$$\mathcal{E}_1(v, \hat{u}_B^\alpha) = \alpha((\hat{u}_B^\alpha - \hat{u} \cdot 1_B)^-, v)_{\mathcal{H}} \text{ for any } v \in \mathcal{V} \tag{1}$$

where f^- denotes the negative part of f. Similarly for $u \in \mathcal{P}_{\mathcal{F}}$ there exists $u_B^\alpha \in \mathcal{F} \cap \mathcal{P}$ such that $u_B^\alpha \leq u_B^\beta$, $0 < \alpha \leq \beta$, $u_B^\alpha \to u_B$, $\alpha \to \infty$, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$ and

$$\mathcal{E}_1(u_B^\alpha, v) = \alpha((u_B^\alpha - u \cdot 1_B)^-, v)_{\mathcal{H}} \text{ for any } v \in \mathcal{V}.$$

Since by [7, Proposition III.1.7.(ii)] $\hat{u}_B \cdot 1_B = \hat{u} \cdot 1_B$, $u_B \cdot 1_B = u \cdot 1_B$ we then have for any $\alpha > 0$

$$\lim_{\alpha \to \infty} \mathcal{E}_1(u_B^\alpha, \hat{u}) = \lim_{\alpha \to \infty} \mathcal{E}_1(u, \hat{u}_B^\alpha).$$

Note that then (by our definition of reduced functions for not necessarily open sets) [7, Lemma III.2.9] extends to general Borel sets, i.e. $\mathcal{E}_1(f_B, \hat{f}) = \mathcal{E}_1(f, \hat{f}_B)$ for any $f \in \mathcal{F} \cap \mathcal{P}$, $\hat{f} \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$, $B \in \mathcal{B}(E)$.
Let $A \subset E$. We set $A^c := E \setminus A$, i.e. the complement of A in E. An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an $\mathcal{E}$-nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} E \setminus F_k$. A property of points in E holds $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set. A function f defined up to some $\mathcal{E}$-exceptional set $N \subset E$ is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.)(resp. $\mathcal{E}$-quasi-lower-semicontinuous ($\mathcal{E}$-q.l.s.c.)) if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $\bigcup_{k\geq 1} F_k \subset E \setminus N$ and $f_{|F_k}$ is continuous (resp. lower-semicontinuous) for all k.
We denote by $\widetilde{f}$ an $\mathcal{E}$-q.c. m-version of f, conversely f denotes the m-class represented by an $\mathcal{E}$-q.c. m-version $\widetilde{f}$ of f.

Definition 2.3 *The generalized Dirichlet form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called quasi-regular if:*
(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ consisting of compact sets.
(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.
(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$-q.c. m-versions $\widetilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\widetilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

3 Measures associated to coexcessive functions

Let us first make a remark about a notational convention: in the sequel before each statement we will name the assumptions on the generalized Dirichlet form which we need to show the statement. We do this in the following way: we define abbreviations for these assumptions and put the abbreviations in brackets just before the statement (cf e.g. Theorem 3.1 below).
From now on we assume that we are given a quasi-regular generalized Dirichlet form. We write **QR** as an abbreviation for this assumption.
By quasi-regularity every element in $\mathcal{F}$ admits an $\mathcal{E}$-q.c. m-version (cf. [7, Proposition IV.1.8.]). For a subset $\mathcal{G} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{G}}$ all the $\mathcal{E}$-q.c. m-versions of elements in $\mathcal{G}$. In particular $\widetilde{\mathcal{P}}_{\mathcal{F}}$ denotes the set of all $\mathcal{E}$-q.c. m-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements of $\mathcal{F}$. Note that $\widetilde{\mathcal{F} \cap \mathcal{P}} \subset \widetilde{\mathcal{P}}_{\mathcal{F}}$ and that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ is a linear lattice, that is $\widetilde{u} \wedge \alpha \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ for all $\alpha \geq 0$ and all $\widetilde{u} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. We emphasize that an element in $\mathcal{P}_{\mathcal{F}}$ not necessarily admits an $\mathcal{E}$-q.c. m-version.
We denote by $\mathcal{B}$ the $\mathcal{B}(E)$-measurable functions on E and by $\mathcal{B}_b$, $\mathcal{B}^+$ the bounded respectively positive elements in $\mathcal{B}$. We also set $\mathcal{B}_b^+ := \mathcal{B}_b \cap \mathcal{B}^+$. Let $\mathcal{D} \subset \mathcal{H}$. We denote by $\mathcal{D}_b$, $\mathcal{D}^+$ the bounded respectively positive elements of $\mathcal{D}$. As above we set $\mathcal{D}_b^+ := \mathcal{D}_b \cap \mathcal{D}^+$.
We are now in the situation to state an integral representation theorem for elements in $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$ whose proof can be found in [8].

Theorem 3.1 *(**QR**) Let $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$. Then there exists a unique σ-finite and positive measure $\mu_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set, such that*

$$\int \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} .$$

Let $\mathcal{D} \subset \mathcal{H}$. For a linear operator G on $\mathcal{H}$ with domain $D(G) \supset \mathcal{D}$ we set $G\mathcal{D} := \{Gh \mid h \in \mathcal{D}\}$.

Remark 3.2 *In some time dependent cases (cf. e.g. [6]), whereas in the case of classical Dirichlet forms we have $\mathcal{P}_{\mathcal{F}} = \mathcal{P}$ and $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} = \widehat{\mathcal{P}}$. More generally this holds for any generalized Dirichlet form with $\mathcal{F} = \widehat{\mathcal{F}}$ and $-\Lambda f = \widehat{\Lambda} f$ for any $f \in G_1\mathcal{H}_b \cup \widehat{G}_1\mathcal{H}_b$. Indeed, let us show in this case that $\mathcal{P}_{\mathcal{F}} = \mathcal{P}$. Let $u \in \mathcal{P}$, $h \in \mathcal{H}_b^+$. Since $(G_\alpha)_{\alpha>0}$ is positivity preserving by the assumption $\mathcal{F} = \widehat{\mathcal{F}}$ we have $f := \alpha G_{\alpha+1} h \in \widehat{\mathcal{F}}^+ \cap G_1\mathcal{H}_b$ hence $0 \leq \mathcal{E}_1(u, f)$ by [7,* *Proposition III.1.4.]. Now*

$$\begin{aligned} 0 \le \mathcal{E}_1(u,f) &= 2\widetilde{\mathcal{A}}_1(u,f) - \mathcal{E}_1(f,u) \\ &= {}_{\mathcal{V}'}\langle 2\widetilde{\mathcal{A}}_1(u,\cdot), f\rangle_{\mathcal{V}} - \mathcal{E}_1(f,u) \\ &= \mathcal{E}_1(f, \widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u) \\ &= (h, \{\widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u\} - \alpha\widehat{G}_{\alpha+1}\{\widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u\})_{\mathcal{H}}. \end{aligned}$$

implies that $\widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u$ is 1-coexcessive. In particular we have $u \le \widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) \in \mathcal{F}$ and therefore $u \in \mathcal{P}_{\mathcal{F}}$. The converse inclusion is trivial and $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} = \widehat{\mathcal{P}}$ can be shown similarly.

From now on we fix an m-tight special standard process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\ge 0}, (Y_t)_{t\ge 0}, (P_z)_{z\in E_\Delta})$ with lifetime ζ and shift operator $(\theta_t)_{t\ge 0}$ such that the resolvent $R_\alpha f$ of $\mathbb{M}$ is an $\mathcal{E}$-q.c. m-version of $G_\alpha f$ for all $\alpha > 0$, $f \in \mathcal{H} \cap \mathcal{B}_b$. $\mathbb{M}$ is then said to be properly associated in the resolvent sense with $\mathcal{E}$. The exact definition of such a process $\mathbb{M}$ can be found in [3]. We always assume that $(\mathcal{F}_t)_{t\ge 0}$ is the (universally completed) natural filtration of $(Y_t)_{t\ge 0}$ and that any real-valued function u on E is extended to E_Δ by setting $u(\Delta) = 0$. We use the abbreviation $\mathbf{M}^{\mathbf{ex}}$ for the assumption that such a process exists.
In addition to quasi-regularity a structural condition on the domain $\mathcal{F}$ of the generalized Dirichlet form is imposed in [7, IV.2,D3] in order to construct explicitly an associated m-tight special standard process. Since we make no use of this technical assumption and since it may be subject to some further progress we instead prefer to assume the existence of $\mathbb{M}$. We will use the resolvent of $\mathbb{M}$ in the proofs of Lemma 3.3, Lemma 3.4 and Theorem 3.5 below but we remark that the statement of the main result Theorem 3.5 is independent of $\mathbb{M}$ and only depends on the generalized Dirichlet form.
Let P be a probability measure on $(\Omega, \mathcal{F}_\infty)$. Let $A, B \in \mathcal{F}_\infty$ be two events. We say that A holds P-a.s. on B, if $P(A; B) := P(A \cap B) = P(B)$. An $(\mathcal{F}_t)$-stopping time τ is called a *terminal time* provided $t + \tau \circ \theta_t = \tau$ P_z-a.s. on $\{\tau > t\}$ for any $z \in E$. Define for $A \subset E_\Delta$

$$\sigma_A := \inf\{t > 0 \mid Y_t \in A\}, \qquad D_A := \inf\{t \ge 0 \mid Y_t \in A\}.$$

A terminal time τ is called *exact* provided $t_n \downarrow 0$ implies that $t_n + \tau \circ \theta_{t_n} \downarrow \tau$ P_z-a.s. for every $z \in E$. Note that if $A \subset E_\Delta$ is such that σ_A, D_A are $(\mathcal{F}_t)$-stopping times, then σ_A is an exact terminal time, whereas D_A is in general only a terminal time and may fail to be exact since $\lim_{t\downarrow 0} t + D_A \circ \theta_t \downarrow \sigma_A$ P_z-a.s. for every $z \in E$. For $(\mathcal{F}_t)$-stopping times σ, τ define

$$R_\alpha^{\sigma,\tau} f(z) := E_z[\int_\sigma^\tau e^{-\alpha s} f(Y_s)ds], \quad \alpha > 0, \quad z \in E, \quad f \in \mathcal{B}^+.$$

By $(p_t)_{t>0}$ we denote the transition semigroup of $\mathbb{M}$.
Let $B \in \mathcal{B}(E)$. Then $\{\sigma_B = 0\} \in \mathcal{F}_0$ and according to Blumenthal's 0-1 law we know that $P_z(\sigma_B = 0) = 0$ or 1. Let us denote the regular points for B by

$$B^{reg} := \{z \in E \mid P_z(\sigma_B = 0) = 1\}.$$

From its definition we see that B^{reg} is universally measurable. Also obviously by right-continuity of the associated process we have $B^{reg} \subset \overline{B}$ where $\overline{B}$ denotes the closure of B in E.

Lemma 3.3 *(**QR**, $\mathbf{M}^{ex}$) Let $B \in \mathcal{B}(E)$. Then $m(B \setminus B^{reg}) = 0$ and $P_m(D_B \neq \sigma_B) = 0$.*

Proof Let $\varphi \in L^2(E;m) \cap \mathcal{B}$, $0 < \varphi \leq 1$. Then $0 \leq R_1^{0,D_B}\varphi \leq R_1\varphi$ and therefore $R_1^{0,D_B}\varphi \in L^2(E;m) \cap \mathcal{B}_b^+$. By strong continuity of $(U_t)_{t>0}$ we may find a decreasing sequence $(t_n)_{n\in\mathbb{N}} \subset (0,\infty)$ converging to zero such that $\lim_{n\to\infty} U_{t_n} R_1^{0,D_B}\varphi(z) = R_1^{0,D_B}\varphi(z)$ for m-a.e. $z \in E$. Since $p_{t_n} R_1^{0,D_B}\varphi$ is an m-version of $U_{t_n} R_1^{0,D_B}\varphi$ for every n we have also $\lim_{n\to\infty} p_{t_n} R_1^{0,D_B}\varphi(z) = R_1^{0,D_B}\varphi(z)$ for m-a.e. $z \in E$. Note that $\lim_{t\downarrow 0} D_B \circ \theta_t + t = \sigma_B$. Now, using the strong Markov property and Lebesgue's Theorem we have for any $z \in E$

$$
\begin{aligned}
\lim_{n\to\infty} p_{t_n} E_\cdot\Big[\int_0^{D_B} e^{-s}\varphi(Y_s)\,ds\Big](z) &= \lim_{n\to\infty} E_z\Big[E_{Y_{t_n}}\Big[\int_0^{D_B} e^{-s}\varphi(Y_s)\,ds\Big]\Big] \\
&= \lim_{n\to\infty} e^{t_n} E_z\Big[\int_{t_n}^{D_B\circ\theta_{t_n}+t_n} e^{-s}\varphi(Y_s)\,ds\Big] \\
&= E_z\Big[\int_0^{\sigma_B} e^{-s}\varphi(Y_s)\,ds\Big].
\end{aligned}
$$

It follows that $E_z[\int_{D_B}^{\sigma_B} e^{-s}\varphi(Y_s)\,ds] = 0$ for m-a.e. $z \in E$. But

$$
E_z\Big[\int_{D_B}^{\sigma_B} e^{-s}\varphi(Y_s)\,ds\Big] \text{ is } \begin{cases} = 0 & \text{for } z \in B^{reg} \cup B^c \\ > 0 & \text{for } z \in B \setminus B^{reg} \end{cases}
$$

and therefore $m(B \setminus B^{reg}) = 0$. Clearly $P_z(D_B = \sigma_B) = 1$ for all $z \in B^{reg} \cup B^c$ hence $P_m(D_B \neq \sigma_B) = 0$.

□

Given a finite measure μ on measurable space $(G, \mathcal{G})$. The completion of $\mathcal{G}$ w.r.t. μ is denoted by $\mathcal{G}^\mu$. An element of $\mathcal{B}^*(E) := \bigcap_{P\in\mathcal{P}(E)} \mathcal{B}(E)^P$ where $\mathcal{P}(E)$ denotes the family of all probability measures on $(E, \mathcal{B}(E))$ is called a universally measurable set. Let $\mathcal{B}^*$ denote the $\mathcal{B}^*(E)$-measurable functions on E.
Let $\gamma \geq 0$. A function $f \in \mathcal{H} \cap \mathcal{B}^{*+}$ is called *γ-supermedian for $(R_\alpha)_{\alpha>0}$* if $\alpha R_{\alpha+\gamma} f \leq f$, $\alpha > 0$. In particular γ-supermedian functions $f \in \mathcal{H} \cap \mathcal{B}^{*+}$ are m-versions of γ-excessive elements in $\mathcal{H}$. $f \in \mathcal{H} \cap \mathcal{B}^{*+}$ is called *γ-excessive for $(R_\alpha)_{\alpha>0}$* if f is γ-supermedian for $(R_\alpha)_{\alpha>0}$ and if $\lim_{\alpha\to\infty} \alpha R_{\alpha+\gamma} f = f$.
We already remarked that $u \in \mathcal{P}$ not necessarily admits an $\mathcal{E}$-q.c. m-version. By quasi-regularity however we know that there exists an an $\mathcal{E}$-q.c. m-version $\widetilde{\alpha G_{\alpha+1} u}$ of $\alpha G_{\alpha+1} u$. Since $\alpha G_{\alpha+1} u$ increases m-a.s. if α increases we know from [7, Corollary III.3.3.] that $\widetilde{\alpha G_{\alpha+1} u}$ increases $\mathcal{E}$-q.e. if α increases. Hence we may define an $\mathcal{E}$-q.l.s.c. m-version of u by

$$
\overline{u} := \sup_{\alpha>0} \widetilde{\alpha G_{\alpha+1} u}
$$

$\overline{u}$ is called an $\mathcal{E}$-q.l.s.c. regularization of $u \in \mathcal{P}$. Surely any two $\mathcal{E}$-q.l.s.c. regularizations of $u \in \mathcal{P}$ coincide $\mathcal{E}$-q.e. hence any $\mathcal{E}$-q.l.s.c. regularization of $u \in \mathcal{P}$ coincides $\mathcal{E}$-q.e. with

the "canonical" regularization $\overline{u} = \sup_{\alpha>0} \alpha R_{\alpha+1} u$. If not otherwise stated we will always choose the canonical regularization for $u \in \mathcal{P}$.
Let μ be a positive measure on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set. Since by assumption there exists an $\mathcal{E}$-nest consisting of compact sets, the support of μ $supp(\mu)$ is defined.

Lemma 3.4 *(**QR**, $\mathbf{M}^{ex}$)*
(i) Let τ be a terminal time. Let $f \in L^2(E;m) \cap \mathcal{B}_b^{*+}$. Then $R_1^{\tau,\infty} f$ is 1-supermedian for $(R_\alpha)_{\alpha>0}$ and $R_1^{\tau,\infty} f \in \mathcal{P} \cap \mathcal{B}_b^*$. If in addition τ is exact then $R_1^{\tau,\infty} f$ is 1-excessive for $(R_\alpha)_{\alpha>0}$. In this case we have in particular that $\overline{R_1^{\tau,\infty} f}(z) = R_1^{\tau,\infty} f(z)$ for every $z \in E$.
(ii) Let $g \in L^2(E;m)^+$, $F \subset E$ be closed. Then $\mu_{(\widehat{G}_1 g)_F}(E \setminus F^{reg}) = 0$. In particular $supp(\mu_{(\widehat{G}_1 g)_F}) \subset F$.

Proof (i) Since τ is a terminal time we have $\tau \circ \theta_t + t \geq \tau$ P_z-a.s. for any $z \in E$. Hence the strong Markov property of $\mathbb{M}$ implies

$$\begin{aligned} e^{-t} p_t R_1^{\tau,\infty} f(z) &= E_z[e^{-t} E_{Y_t}[\int_\tau^\infty e^{-s} f(Y_s) ds]] \\ &= E_z[\int_{\tau\circ\theta_t + t}^\infty e^{-s} f(Y_s) ds] \leq R_1^{\tau,\infty} f(z). \end{aligned}$$

It follows that $R_1^{\tau,\infty} f$ is 1-supermedian for $(R_\alpha)_{\alpha>0}$ because

$$\alpha R_{\alpha+1} R_1^{\tau,\infty} f(z) = \int_0^\infty \alpha e^{-\alpha t} E_z[\int_{\tau\circ\theta_t+t}^\infty e^{-s} f(Y_s) ds] dt \leq R_1^{\tau,\infty} f(z).$$

Furthermore $R_1^{\tau,\infty} f \leq R_1 f$ implies $R_1^{\tau,\infty} f \in \mathcal{V} \cap \mathcal{B}^*$ by [7, Lemma III.2.1.(i)]. Note that $R_1^{\tau,\infty} f$ is finite $\mathcal{E}$-q.e. Then, using the exactness of τ and Lebesgue's Theorem we have

$$\begin{aligned} \lim_{\alpha\to\infty} \alpha R_{\alpha+1} R_1^{\tau,\infty} f(z) &= \lim_{\alpha\to\infty} \int_0^\infty \alpha e^{-(\alpha+1)t} p_t R_1^{\tau,\infty} f(z) dt \\ &= \lim_{\alpha\to\infty} \{ \int_0^\infty e^{-t} p_{\frac{t}{\alpha+1}} R_1^{\tau,\infty} f(z) dt - \int_0^\infty e^{-(\alpha+1)t} p_t R_1^{\tau,\infty} f(z) dt \} \\ &= R_1^{\tau,\infty} f(z) \quad \text{for every} \quad z \in E. \end{aligned}$$

Clearly $\lim_{\alpha\to\infty} \alpha R_{\alpha+1} R_1^{\tau,\infty} f(z) = \sup_{\alpha>0} \alpha R_{\alpha+1} R_1^{\tau,\infty} f(z)$ for every $z \in E$ hence $\overline{R_1^{\tau,\infty} f} = R_1^{\tau,\infty} f$.
(ii) Fix $\varphi \in L^1(E;m) \cap \mathcal{B}$ such that $0 < \varphi \leq 1$. Since σ_F is an exact terminal time we know from (i) that $R_1^{\sigma_F,\infty} \varphi$ is 1-excessive for $(R_\alpha)_{\alpha>0}$ and $R_1^{\sigma_F,\infty} \varphi \in \mathcal{V} \cap \mathcal{B}^*$. Furthermore by Lemma 3.3 $R_1^{\sigma_F,\infty} \varphi = R_1 \varphi$ m-a.s. on F and therefore by [7, Proposition III.1.7.(ii)] $R_1^{\sigma_F,\infty} \varphi \geq (G_1 \varphi)_F$ m-a.e. Hence

$$\overline{(G_1\varphi)}_F = \sup_{n\geq 1} n R_{n+1} (G_1 \varphi)_F \leq \sup_{n\geq 1} n R_{n+1} R_1^{\sigma_F,\infty} \varphi = R_1^{\sigma_F,\infty} \varphi \quad \mathcal{E}\text{-q.e.} \tag{2}$$

Furthermore

$$R_1 \varphi - R_1^{\sigma_F,\infty} \varphi \text{ is } \begin{cases} > 0 & \mathcal{E} - q.e. \text{ on } E \setminus F^{reg} \\ = 0 & \text{on } F^{reg}. \end{cases}$$

Using (2), and (1) the rest of the proof follows is exactly as in [8, Lemma 2.4].

□

In order to show (see [8]) the equivalence of (i) and (ii) in Theorem 3.5 below one uses another equivalent description for $\mathcal{E}$-exceptional sets via a finite Choquet capacity called the φ-capacity. To explain this let $\varphi \in L^2(E;m) \cap \mathcal{B}$, $0 < \varphi \leq 1$. For $U \subset E$, U open let $\text{cap}_\varphi(U) := ((G_1\varphi)_U, \varphi)_\mathcal{H}$ and for arbitrary $A \subset E$ let $\text{cap}_\varphi(A) := \inf\{((G_1\varphi)_U, \varphi)_\mathcal{H} \mid U \supset A,\ U \text{ open}\}$. It is shown in [7, Proposition III.2.10.] that an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is an $\mathcal{E}$-nest if and only if $\lim_{k\to\infty} \text{cap}_\varphi(F_k^c) = 0$. Hence the $\mathcal{E}$-exceptional sets are exactly the zero sets of the set function cap_φ restricted to $\mathcal{B}(E)$.
As a generalization of [2, p.78] we introduce the following class of measures

$$\widehat{S}_{00} := \{\mu_{\hat{u}} \mid \ \hat{u} \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+} \text{ and } \mu_{\hat{u}}(E) < \infty\}$$

where $\widehat{G}_1\mathcal{H}_b^+ := \{\widehat{G}_1 h \mid h \in \mathcal{H}_b^+\}$.

Theorem 3.5 *(**QR**, **M**$^{\text{ex}}$) For $B \in B(E)$ the following conditions are equivalent:*

(i) B is $\mathcal{E}$-exceptional

(ii) $\mu(B) = 0\ \forall \mu \in \hat{S}_{00}$

Remark 3.6 *(i) $A \subset E$ is called nearly Borel if there exists $B_1, B_2 \in \mathcal{B}(E)$ such that $B_1 \subset A \subset B_2$ and $B_2 \setminus B_1$ is $\mathcal{E}$-exceptional. Then Theorem 3.5 extends to nearly Borel sets. Indeed, we have $A \subset B_1 \cup (B_2 \setminus B_1)$ and*

$$cap_\varphi(A) = cap_\varphi(B_1) = cap_\varphi(B_2)$$

hence if $cap_\varphi(A) > 0$ by Theorem 3.5 there exists $\mu \in \widehat{S}_{00}$ with $\mu(B_1) > 0$ but then $\mu(A) = \mu(B_1) > 0$. The fact that A is in general not $\mathcal{B}(E)$-measurable doesn't matter since for convenience only we restricted ourselves to $(E, \mathcal{B}(E))$ in Theorem 3.1. Actually $\mu \in \widehat{S}_{00}$ is defined on $\bigcap_{\mu\in\widehat{S}_{00}} \sigma(\widetilde{\mathcal{P}}_\mathcal{F} - \widetilde{\mathcal{P}}_\mathcal{F})^\mu$ (cf. the paragraph before Lemma 3.4 for the meaning of this) which contains any nearly Borel measurable set. Finally, we can call the nearly Borel set A $\mathcal{E}$-exceptional if $cap_\varphi(B_1) = 0$.
(ii) Since we may divide each $\mu \in \widehat{S}_{00} \setminus \{0\}$ by its total mass the assertion of Theorem 3.5 remains true if we replace $\widehat{S}_{00}$ by $\{\mu \in \widehat{S}_{00} \mid \mu(E) = 1\}$. Note also that if $(\widehat{G}_\alpha)_{\alpha>0}$ is sub-Markovian we may replace $\widehat{S}_{00}$ by the larger class $\{\mu_{\hat{u}} \mid \ \|\hat{u}\|_\infty < \infty \text{ and } \mu_{\hat{u}}(E) < \infty\}$ and then our definition coincides with the one of [2, p.78].

4 Smooth measures

In this section similar to [2], [5] we will define smooth measures and measures of finite (co-)energy integral and show that these measures have properties similar to those in [2],

[5]. Throughout the whole section we assume that we are given a quasi-regular generalized Dirichlet form (**QR**) and an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$ ($\mathbf{M}^{\mathbf{ex}}$).

Definition 4.1 *A positive measure μ on $(E, \mathcal{B}(E))$ is said to be of finite 1-order co-energy integral if there exists $\widehat{U}_1\mu \in \mathcal{V}$, such that*

$$\int_E \widetilde{G_1 h}\, d\mu = \mathcal{E}_1(G_1 h, \widehat{U}_1\mu) \tag{3}$$

for all $h \in \mathcal{H}$ and for all $\mathcal{E}$-q.c. m-versions $\widetilde{G_1 h}$ of $G_1 h$. The measures of finite 1-order co-energy integral are denoted by $\widehat{S}_0$.

Let $\hat{u} \in \widehat{\mathcal{P}}_{\widetilde{\mathcal{F}}}$ and $\mu_{\hat{u}}$ be the associated measure of Theorem 3.1. Then $\widehat{U}_1\mu_{\hat{u}} = \hat{u}$. Hence obviously $\widehat{S}_{00} \subset \widehat{S}_0$. Clearly $\mu \in \widehat{S}_0$ does not charge $\mathcal{E}$-exceptional sets. Furthermore $\widehat{U}_1\mu \in \widehat{\mathcal{P}}$ because for any $f \in \mathcal{H}^+$ we have

$$\begin{aligned}(f, \widehat{U}_1\mu - \alpha\widehat{G}_{\alpha+1}\widehat{U}_1\mu)_{\mathcal{H}} &= \int_E \widetilde{G_1 f}\, d\mu - (\alpha G_{\alpha+1} f, \widehat{U}_1\mu)_{\mathcal{H}} \\ &= \int_E \widetilde{G_1 f} - \alpha\widetilde{G_{\alpha+1} G_1 f}\, d\mu \geq 0\end{aligned}$$

since $\widetilde{G_1 f} - \alpha\widetilde{G_{\alpha+1}G_1 f} \geq 0$ $\mathcal{E}$-q.e. hence μ-a.e.
Let $\overline{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ denote the totality of $\mathcal{E}$-q.l.s.c. regularizations of the elements in $\mathcal{P}_{G_1\mathcal{H}_b^+}$. Let $\overline{u} \in \overline{\mathcal{P}}_{G_1\mathcal{H}_b^+}$. Then $\int_E \overline{u}\, d\mu = \sup_{\alpha>0}\int_E \alpha R_{\alpha+1} u\, d\mu = \lim_{\alpha\to\infty}\mathcal{E}_1(u, \alpha\widehat{G}_{\alpha+1}\widehat{U}_1\mu)$ exists as a bounded and increasing limit for all $\overline{u} \in \overline{\mathcal{P}}_{G_1\mathcal{H}_b^+}$. Now let $\widetilde{w} \in \widetilde{\mathcal{P}}$. Since $\lim_{\alpha\to\infty}\alpha R_{\alpha+1}\widetilde{v} = \widetilde{v}$ $\mathcal{E}$-q.e. for any $\mathcal{E}$-q.c. function $\widetilde{v} \in B_b$ we have $\mathcal{E}$-q.e.

$$\begin{aligned}\widetilde{w} = \sup_{n\geq 1}\widetilde{w}\wedge n &= \sup_{n\geq 1}\lim_{\alpha\to\infty}\alpha R_{\alpha+1}(\widetilde{w}\wedge n) \\ &= \sup_{n\geq 1}\sup_{\alpha>0}\alpha R_{\alpha+1}(\widetilde{w}\wedge n) \\ &= \sup_{\alpha>0}\alpha R_{\alpha+1}\widetilde{w}.\end{aligned}$$

Hence if a function $u \in \mathcal{P}_{G_1\mathcal{H}_b^+}$ admits an $\mathcal{E}$-q.c. m-version then this m-version coincides $\mathcal{E}$-q.e. with its canonical regularization. Thus $\int_E \widetilde{u}\, d\mu = \int_E \overline{u}\, d\mu$ and therefore (3) extends to all $\widetilde{f} \in \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ in the sense of Theorem 3.1. Note that $\widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ is also a vector lattice which separates the points of $E \setminus N$ and hence could have also been used as a space of test functions in Theorem 3.1.
On the other hand only if $\widehat{U}_1\mu \in \widehat{\mathcal{P}}_{\widetilde{\mathcal{F}}}$ similarly to **4.Step** of the proof of Theorem 3.5 in [8] we can show that (3) extends to all $\widetilde{f} \in \widetilde{\mathcal{F}}$. Also only if $\hat{u} \in \widehat{\mathcal{P}}_{\widetilde{\mathcal{F}}}$ (and not for all $\hat{u} \in \widehat{\mathcal{P}}$!) by Theorem 3.1 we can show the existence of $\mu_{\hat{u}} \in \widehat{S}_0$.
In the following proof $(ii) \Rightarrow (i)$ of Lemma 4.2 we shall see that $\mu \in \widehat{S}_0$ can be identified with some $\overline{L}_\mu \in (\mathcal{V}')'$, i.e. the bidual of $\mathcal{V}$.

Lemma 4.2 *(QR) The following statements are equivalent for a positive measure μ on $(E, \mathcal{B}(E))$:*
(i) μ is of finite 1-order co-energy integral.
(ii) There exists $C > 0$, such that

$$|\int_E \widetilde{G_1 h}\, d\mu| \leq C\|G_1 h\|_{\mathcal{F}}$$

for all $h \in \mathcal{H}$ and for all $\mathcal{E}$-q.c. m-versions $\widetilde{G_1 h}$ of $G_1 h$.

Proof (cf. [5]) Let us assume that (ii) holds. Clearly μ then does not charge $\mathcal{E}$-exceptional sets. Define $L_\mu(h) = \int_E \widetilde{G_1 h}\, d\mu$, $h \in \mathcal{H}$. Since $|L_\mu(h)| \leq C\|G_1 h\|_{\mathcal{F}} \leq C\|W_1\|_{L(\mathcal{V}')}\|h\|_{\mathcal{V}'}$ where $\|W_1\|_{L(\mathcal{V}')}$ denotes the operator norm of $W_1 : \mathcal{V}' \to \mathcal{F}$. Since $\mathcal{H} \subset \mathcal{V}'$ dense we may extend L_μ to a continuous linear functional $\overline{L}_\mu$ on $\mathcal{V}'$. But then by [10, IV.8.Theorem 1] there exists a unique $\widehat{U}_1\mu \in \mathcal{V}$, such that $\overline{L}_\mu(f) =_{\mathcal{V}'} \langle f, \widehat{U}_1\mu\rangle_{\mathcal{V}}$ for all f in $\mathcal{V}'$ and (i) holds.
(i) $\Rightarrow$ (ii) is clear.

□

Definition 4.3 *A positive measure μ on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set is called smooth if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact subsets of E, such that*

$$\mu(F_k) < \infty \quad \textit{for all} \quad k \in \mathbb{N}.$$

The smooth measures are denoted by S.

From now on we assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ is sub-Markovian. We abbreviate this assumption by $\widehat{\textbf{SUB}}$. The following lemma will be needed as a preparation for Lemma 4.5 below.

Lemma 4.4 *(QR, $\widehat{\textbf{SUB}}$) Let $\widetilde{u} \in \widetilde{\mathcal{F}}$, $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. Then $cap_\varphi(|\widetilde{u}| > \lambda) \leq 2\frac{(K+1)^2}{\lambda^2}\|u\|^2_{\mathcal{F}}$.*

Proof Let $U := \{\widetilde{u} > \lambda\}$, $V := \{-\widetilde{u} > \lambda\}$. Then, since $G_1\varphi \leq \frac{u}{\lambda}$ m-a.e. on U, $G_1\varphi \leq -\frac{u}{\lambda}$ m-a.e. on V

$$\begin{aligned}
\text{cap}_\varphi(\{|\,\widetilde{u}\,|> \lambda\}) &\leq \text{cap}_\varphi(\{\widetilde{u} > \lambda\}) + \text{cap}_\varphi(\{-\widetilde{u} > \lambda\}) \\
&= \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_U) + \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_V) \\
&\leq \mathcal{E}_1(\frac{u}{\lambda}, (\widehat{G}_1\varphi)_U) + \mathcal{E}_1(-\frac{u}{\lambda}, (\widehat{G}_1\varphi)_V) \\
&\leq \frac{(K+1)}{\lambda}\|u\|_{\mathcal{F}}(\|(\hat{G}_1\varphi)_U\|_{\mathcal{V}} + \|(\widehat{G}_1\varphi)_V\|_{\mathcal{V}}).
\end{aligned}$$

By sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ we have in particular that $(\widehat{G}_1\varphi)^\alpha_U \leq 1$ m-a.e. on U, hence

$$\begin{aligned}
\|(\widehat{G}_1\varphi)_U\|^2_{\mathcal{V}} &\leq \overline{\lim_{\alpha\to\infty}}\ \mathcal{E}_1((\widehat{G}_1\varphi)^\alpha_U, (\widehat{G}_1\varphi)^\alpha_U) \\
&\leq \overline{\lim_{\alpha\to\infty}}\ \mathcal{E}_1(\frac{u}{\lambda}, (\widehat{G}_1\varphi)^\alpha_U) \\
&\leq \frac{(K+1)}{\lambda}\|u\|_{\mathcal{F}}\|(\widehat{G}_1\varphi)_U\|_{\mathcal{V}}.
\end{aligned}$$

Therefore $\|(\widehat{G}_1\varphi)_U\|_\mathcal{V} \leq \frac{(K+1)}{\lambda}\|u\|_\mathcal{F}$. Similarly we get $\|(\widehat{G}_1\varphi)_V\|_\mathcal{V} \leq \frac{(K+1)}{\lambda}\|u\|_\mathcal{F}$ and the assertion follows.

□

Using the preceding Lemma 4.4 and Lemma 4.2(ii) the following lemma can be shown exactly as in [2, Lemma 2.2.8.,p.81].

Lemma 4.5 *(**QR**, $\widehat{\textbf{SUB}}$) Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. Let ν be a finite positive measure on $(E, \mathcal{B}(E))$ such that there exists $C > 0$ with*

$$\nu(B) \leq C\, cap_\varphi(B) \text{ for all } B \in \mathcal{B}(E).$$

Then $\nu \in \widehat{S}_0$.

Since cap_φ, $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$ is a Choquet capacity and the proof of [2, Lemma 2.2.9.,p.81] only uses general properties of Choquet capacities the following lemma can be shown exactly as [2, Lemma 2.2.9.,p.81].

Lemma 4.6 *(**QR**) Let ν be a finite positive measure on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set (i.e. a finite smooth measure). Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. Then there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that*

$$\nu(A) \leq 2^k cap_\varphi(A) \quad \text{for any Borel set } A \subset F_k.$$

Theorem 4.7 *(**QR**, $\widehat{\textbf{SUB}}$) Let μ be a positive measure on $(E, \mathcal{B}(E))$. Then the following statements are equivalent:*
(i) $\mu \in S$.
(ii) There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E, such that

$$1_{F_k} \cdot \mu \in \widehat{S}_0 \text{ for each } k$$

where $1_A \cdot \mu(B) := \mu(A \cap B)$ for $A \subset E$, $B \in \mathcal{B}(E)$.

Proof Let us assume (i). Then there exists an $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E, such that $1_{E_k} \cdot \mu$ is a finite positive measure charging no $\mathcal{E}$-exceptional set for any k. Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. By Lemma 4.6 we can find an $\mathcal{E}$-nest $(\tilde{E}_k)_{k\in\mathbb{N}}$ such that $1_{E_k \cap \tilde{E}_k} \cdot \mu(A) \leq 2^k \text{cap}_\varphi(A)$ for any k and for any Borel set $A \subset \tilde{E}_k$ but then also for for any k and for any $A \in \mathcal{B}(E)$. Therefore (ii) follows by Lemma 4.5 with $F_k := E_k \cap \tilde{E}_k$. Let us assume (ii). Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. There exits an $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E and an $\mathcal{E}$-q.c. m-version of $\widetilde{G_1\varphi}$ of $G_1\varphi$, such that $1_{E_k} \cdot \mu \in \hat{S}_0$ and such that $k\widetilde{G_1\varphi} \geq 1$ on E_k for each k. Therefore $\mu(E_k) \leq k \int_E \widetilde{G_1\varphi}\, 1_{E_k}\, d\mu = \mathcal{E}_1(G_1\varphi, \widehat{U}_1(1_{E_k} \cdot \mu)) < \infty$.

□

In the following we will need some preparations in order establish a relation between the classes $\widehat{S}_{00}$ (which we defined in section 3) and S. The methods in [2], [4] to develop such a relation rely heavily on the symmetry of the domain of the form, the sector condition and the invariance of the Dirichlet space under truncation. Since in general none of the above mentioned properties are available for generalized Dirichlet forms we have to develop a different procedure. We remark that this procedure takes advantage of the behaviour of the associated process in an essential way.

For $B \in \mathcal{B}(E)$ let

$$B^0 := \{z \in E \mid P_z(\sigma_{B^c} > 0) = 1\}.$$

If $F \subset E$ is closed then F^0 is called the fine interior of F.

In the following Lemma 4.8 we shall not make use of the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$.

Lemma 4.8 *(QR, $\mathbf{M^{ex}}$) Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest. Then*

$$\bigcap_{k\geq 1}(E\setminus F_k^0) \quad \text{and} \quad \bigcap_{k\geq 1}(E\setminus F_k^{reg}) \quad \text{is } \mathcal{E}\text{-exceptional.}$$

Proof Let $B \in \mathcal{B}(E)$. We first remark that B^0, B^{reg} is nearly Borel. To show this let $\varphi \in L^2(E;m)\cap\mathcal{B}$, $0<\varphi\leq 1$. Then

$$\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi > 0\} = \bigcup_{n\in\mathbb{N}}\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi \geq \frac{1}{n}\}.$$

Thus, since by Lemma 3.4(i) $R_1^{\sigma_{B^c},\infty}\varphi$ is $\mathcal{E}$-q.l.s.c. it is easy to see that $\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi > 0\}$ is nearly Borel. Since $\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi > 0\} = B^0$ up to an $\mathcal{E}$-exceptional set B^0 is nearly Borel too. The same is also true for B^{reg} since $B^{reg} = ((B^c)^0)^c$. Now let $\mu \in \widehat{S}_{00}$. Then

$$\begin{aligned}\int_E R_1^{\sigma_{F_k^c},\infty}\varphi\, d\mu &\geq \int_{E\setminus F_k^0} E_z\Big[\int_{\sigma_{F_k^c}}^\infty e^{-s}\varphi(Y_s)\,ds\Big]\,\mu(dz)\\ &= \int_{E\setminus F_k^0} R_1\varphi\, d\mu.\end{aligned}$$

By [7, Lemma IV.3.9.] $R_1^{\sigma_{F_k^c},\infty}\varphi$ is an $\mathcal{E}$-q.l.s.c. m-version of $(G_1\varphi)_{F_k^c}$ and since $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest we have $\lim_{k\to\infty}(G_1\varphi)_{F_k^c} = 0$ weakly in $\mathcal{V}$. Therefore

$$0 = \lim_{k\to\infty}\int R_1^{\sigma_{F_k^c},\infty}\varphi\, d\mu \geq \int_{\cap_{k\geq 1}(E\setminus F_k^0)} R_1\varphi\, d\mu$$

which implies $\mu(\cap_{k\geq 1}(E\setminus F_k^0)) = 0$. By Remark 3.6(i) we then have that $\cap_{k\geq 1}(E\setminus F_k^0)$ is $\mathcal{E}$-exceptional. Since $B^0 \subset B^{reg}$ for any $B \subset E$ we have $\cap_{k\geq 1}(E\setminus F_k^{reg}) \subset \cap_{k\geq 1}(E\setminus F_k^0)$ and then $\cap_{k\geq 1}(E\setminus F_k^{reg})$ is $\mathcal{E}$-exceptional too.

□

Remark 4.9 *In contrast to the case of regular symmetric Dirichlet forms (cf. [2, Theorem 4.1.3., p.139]) $F_k \setminus F_k^{reg}$ is not polar. In our framework, as it is well known from the parabolic case, semi-polar sets are not polar in general. This is from the potential theoretic point of view an important difference to the case of classical Dirichlet forms in the sense* *of [2], [4], [3]. As an example consider the uniform motion to the right on the real line, i.e.* $\mathcal{H} = \mathcal{V} = L^2(\mathbb{R}, dx)$, $\mathcal{F} = \widehat{\mathcal{F}} = H^{1,2}(\mathbb{R})$, $p_t f(x) = f(x+t)$, $x \in \mathbb{R}, t \geq 0$. *Let* $[a,b]$ *be the closed interval from a to b. Then* $[a,b] \setminus [a,b]^{reg} = \{b\}$ *is semi-polar but surely hit if we start at* $c < b$. *Thus* $[a,b] \setminus [a,b]^{reg}$ *is not polar. Furthermore, since the Dirac measure* δ_x *is in* S_0 *for any* $x \in \mathbb{R}$ *we have also that* $[a,b] \setminus [a,b]^{reg}$ *is not* $\mathcal{E}$*-exceptional.*

For the rest of the section let us assume that in **D1** (ii) the adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ can also be restricted to a C_0-semigroup on $\mathcal{V}$. Let $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{H}))$ denote the generator of $(\widehat{U}_t)_{t\geq 0}$ on $\mathcal{H}$, $\widehat{\mathcal{A}}(u,v) := \mathcal{A}(v,u)$, $u,v \in \mathcal{V}$ and let the coform $\widehat{\mathcal{E}}$ be defined as the bilinear form associated with $(\widehat{\mathcal{A}}, \mathcal{V})$ and $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{H}))$. Note that since $(\widehat{G}_\alpha)_{\alpha>0}$ was assumed to be sub-Markovian the corresponding statement of **D2** holds for the coform. The coform is hence a generalized Dirichlet form too. Let us further assume up to the end of this section that the coform $\widehat{\mathcal{E}}$ is quasi-regular too. We will abbreviate the assumption that $\widehat{\mathcal{E}}$ is a quasi-regular generalized Dirichlet form by $\widehat{\mathbf{QR}}$.
We fix an m-tight special standard process $\widehat{\mathbb{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}}_t)_{t\geq 0}, (\widehat{Y}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E_\Delta})$ with lifetime $\widehat{\zeta}$ and shift operator $(\widehat{\theta}_t)_{t\geq 0}$ such that the resolvent $\widehat{R}_\alpha f = \widehat{E}.[\int_0^\infty e^{-\alpha t} f(\widehat{Y}_t)dt]$ is an $\widehat{\mathcal{E}}$-q.c. m-version of $\widehat{G}_\alpha f$ for all $f \in \mathcal{H} \cap \mathcal{B}_b$. $\widehat{\mathbb{M}}$ is then said to be properly coassociated in the resolvent sense with $\widehat{\mathcal{E}}$. As before we assume that $(\widehat{\mathcal{F}}_t)_{t\geq 0}$ denotes the (universally completed) natural filtration. Necessary and sufficient conditions for the existence of such a process are given in [7]. $\widehat{\mathbb{M}}$ is in duality to $\mathbb{M}$ w.r.t. m. We will use the abbreviation $\widehat{\mathbf{M}}^{\mathbf{ex}}$ to express our assumption that such a process exists. Symbols with a superposed hat as

$$\widehat{E}.[...], \widehat{\sigma}_B, \widehat{D}_B, B^{\widehat{0}}, B^{\widehat{reg}}, \widehat{\mathcal{E}}\text{-nest}, \widehat{\mathcal{E}}\text{-exceptional}, \widehat{\mathcal{E}}\text{-}q.c., ... \ etc.$$

correspond to the coassociated process or the coform and are defined analoguous to the corresponding objects in terms of the associated process $\mathbb{M}$.
We remark that by the discussion right below (1) we have for any open set U that

$$\text{cap}_\varphi(U) = \mathcal{E}_1((G_1\varphi)_U, \widehat{G}_1\varphi) = \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_U) =: \widehat{\text{cap}}_\varphi(U).$$

But since analoguously to the corresponding statement for $\mathcal{E}$ (cf. paragraph before Theorem 3.5) we have that an increasing sequence of closed sets $(F_k)_{k\in\mathbb{N}}$ is an $\widehat{\mathcal{E}}$-nest if and only if $\lim_{k\to\infty} \widehat{\text{cap}}_\varphi(F_k^c) = 0$ we can see that $\widehat{\mathcal{E}}$-nests and $\mathcal{E}$-nests coincide hence $\widehat{\mathcal{E}}$-exceptional sets and $\mathcal{E}$-exceptional sets coincide.

Lemma 4.10 *(i)* (**QR**, $\mathbf{M}^{\mathbf{ex}}$) *Let* $g \in L^2(E;m) \cap \mathcal{B}_b^+$. *Let* $F \subset E$, F *closed. Then there exists relatively compact subsets* $(B_{n,k})_{n,k\geq 1}$ *of* E *(resp. compact subsets* $(\overline{B}_{n,k})_{n,k\geq 1}$ *of* E*) such that* $B_{n+1,k} \subset \overline{B}_{n+1,k} \subset B_{n,k} \subset B_{n,k+1}$, $P_\mu(\bigcup_{k\geq 1}\bigcap_{n\geq 1} B_{n,k}) = P_\mu(F)$ *for any* $\mu \in \widehat{S}_{00}$ *and*

$$R_1^{D_F,\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}},\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} R_1^{\sigma_{B_{n,k}},\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1 g)_{\overline{B}_{n,k}}}(z)$$

for $\mathcal{E}$*-q.e.* $z \in E$. *In particular there exists open subsets* $(U_{n,k})_{n,k\geq 1}$ *of* E *such that*

$$R_1^{D_F,\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} R_1^{\sigma_{U_{n,k}},\infty} g(z)$$

for $\mathcal{E}$-q.e. $z \in E$.

(ii) ($\mathbf{QR}$, $\mathbf{M}^{\mathrm{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathrm{ex}}$) Let $F \subset E$, F closed. Then

$$\mathrm{supp}(\mu_{\widehat{E}\cdot(\int_{D_F}^{\infty} e^{-s}g(Y_s)ds)}) \subset F$$

for any $g \in L^2(E;m) \cap \mathcal{B}^+$.

Proof (i) By quasi-regularity there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact sets. Let ρ_k, $k \geq 1$ be a metric on F_k compatible with the relative topology on F_k inherited from E (ρ_k can be constructed analogous to [3, Remark IV.3.2, p.101]). Define for $n, k \geq 1$

$$B_{n,k} := \{z \in F_k \mid \rho_k(F \cap F_k, z) < \frac{1}{n}\}, \quad \overline{B}_{n,k} := \{z \in F_k \mid \rho_k(F \cap F_k, z) \leq \frac{1}{n}\}.$$

Obviously $\lim_{n\to\infty} D_{B_{n,k}} \leq D_{F\cap F_k}$. Also note that since $D_{B_{n,k}}$ is increasing in n and $B_{n,k} \supset F \cap F_k$ for all n we have $\{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\} = \bigcap_{n\geq 1}\{D_{B_{n,k}} < \zeta\} \supset \{D_{F\cap F_k} < \zeta\}$. Fix $z \in E$. Since $\mathbb{M}$ is special standard by quasi-left continuity up to ζ we have

$$\lim_{n\to\infty} Y_{D_{B_{n,k}}} = Y_{\lim_{n\to\infty} D_{B_{n,k}}} \quad P_z\text{-a.s. on } \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}.$$

But on $\{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}$ we have P_z-a.s. $Y_{D_{B_{n,k}}} \in \overline{B}_{n,k}$ and hence $\lim_{n\to\infty} Y_{D_{B_{n,k}}} = Y_{\lim_{n\to\infty} D_{B_{n,k}}} \in \bigcap_{n\geq 1} \overline{B}_{n,k} = F \cap F_k$. It follows that

$$\lim_{n\to\infty} D_{B_{n,k}} = D_{F\cap F_k} \quad P_z\text{-a.s. on } \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}.$$

Since $z \in E$ was arbitrary this holds for every $z \in E$. For $A \in \mathcal{F}_\infty$, f $\mathcal{F}_\infty$-measurable, let $E_z[f;A] := E_z[f1_A]$. Now using that $\lim_{k\to\infty} R_1^{\sigma_{F_k^c},\infty} g = 0$ $\mathcal{E}$-q.e. and $\{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\} \supset \{D_{F\cap F_k} < \zeta\}$ we obtain for $\mathcal{E}$-q.e. $z \in E$

$$\begin{aligned}
R_1^{D_F,\infty} g(z) &= \lim_{k\to\infty} R_1^{D_{F\cap F_k},\infty} g(z) \\
&= \lim_{k\to\infty} E_z[\int_{D_{F\cap F_k}}^{\infty} e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}] \\
&= \lim_{k\to\infty} E_z[\int_{D_{F\cap F_k}\wedge\sigma_{F_k^c}}^{\infty} e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}] \\
&= \lim_{k\to\infty}\lim_{n\to\infty} E_z[\int_{D_{B_{n,k}}\wedge\sigma_{F_k^c}}^{\infty} e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}] \\
&= \lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}}\wedge\sigma_{F_k^c},\infty} g(z)
\end{aligned}$$

where the last identity followed since by Lebesgue's Theorem for $\mathcal{E}$-q.e. $z \in E$

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} E_z[\int_{D_{B_{n,k}}\wedge\sigma_{F_k^c}}^{\infty} e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} \geq \zeta\}] \\
= \; &\lim_{k\to\infty} E_z[\int_{\lim_{n\to\infty} D_{B_{n,k}}\wedge\sigma_{F_k^c}}^{\infty} e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} \geq \zeta\}] \\
\leq \; &\lim_{k\to\infty} E_z[\int_{\sigma_{F_k^c}}^{\infty} e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} \geq \zeta\}] = 0.
\end{aligned}$$

Observe that $U_{n,k} := B_{n,k} \cup F_k^c$ is open in E hence $D_{B_{n,k}} \wedge \sigma_{F_k^c} = \sigma_{B_{n,k}} \wedge \sigma_{F_k^c} = \sigma_{U_{n,k}}$. Therefore

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}}\wedge\sigma_{F_k^c},\infty} g(z) &= \lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}},\infty} g(z)\\ &= \lim_{k\to\infty}\lim_{n\to\infty} R_1^{\sigma_{B_{n,k}},\infty} g(z)\end{aligned}$$

for $\mathcal{E}$-q.e. $z \in E$. Finally, we also have (recall that the bar over an 1-excessive function denotes when not otherwise stated the "canonical" $\mathcal{E}$-q.l.s.c. regularization) $\overline{(G_1g)_{\overline{B}_{n+1,k}}}(z) \le \overline{(G_1g)_{B_{n,k}}}(z) \le \overline{(G_1g)_{\overline{B}_{n,k}}}(z)$ and $\overline{(G_1g)_{B_{n,k}\cup F_k^c}}(z) \le \overline{(G_1g)_{B_{n,k}}}(z) + \overline{(G_1g)_{F_k^c}}(z)$ for $\mathcal{E}$-q.e. $z \in E$. Hence

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}}\wedge\sigma_{F_k^c},\infty} g(z) &= \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1g)_{B_{n,k}\cup F_k^c}}(z)\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1g)_{B_{n,k}}}(z)\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1g)_{\overline{B}_{n,k}}}(z)\end{aligned}$$

for $\mathcal{E}$-q.e. $z \in E$ and (i) follows.

(ii) Let $(B_{n,k})_{n,k\ge1}$, be as in (i). Let $\varphi \in L^2(E;m) \cap \mathcal{B}$, $0 < \varphi \le 1$. By (i) but in terms of the coassociated process we have

$$\widehat{E}.[\int_{\widehat{D}_F}^{\infty} e^{-s} g(\widehat{Y}_s)ds] = \lim_{k\to\infty}\lim_{n\to\infty} (\widehat{G}_1 g)_{\overline{B}_{n,k}} \quad \text{m-a.s.}$$

Now similar to the proof of Theorem 3.5 $(\widehat{G}_1g)_{\overline{B}_{n,k}}$ converges weakly in $\mathcal{V}$ (as $n \to \infty$) to some $(\widehat{G}_1g)_{\overline{B}_{\infty,k}}$ such that $supp(\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}) \subset F \cap F_k$. Hence

$$\begin{aligned}\int R_1\varphi \, d\mu_{\widehat{E}.(\int_{\widehat{D}_F}^{\infty} e^{-s}g(\widehat{Y}_s)ds)} &= \lim_{k\to\infty} \int R_1\varphi \, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}\\ &= \lim_{k\to\infty} \int R_1^{D_{F\cap F_k},\infty} \varphi \, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}.\end{aligned}$$

Define for $k \ge 1$

$$B'_{l,m} := \{z \in F_m \mid \rho_m(F \cap F_k, z) < \frac{1}{l}\}.$$

Then by (i) and since $\sigma_{B'_{l,m}}$, l, $m \ge 1$, is exact we have

$$\begin{aligned}&\lim_{k\to\infty} \int R_1^{D_{F\cap F_k},\infty} \varphi \, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}\\ =\ &\lim_{k\to\infty}\lim_{m\to\infty}\lim_{l\to\infty}\lim_{\alpha\to\infty} \int \alpha R_{\alpha+1} R_1^{\sigma_{B'_{l,m}},\infty} \varphi \, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}\end{aligned}$$

$$\leq \lim_{k\to\infty}\lim_{m\to\infty}\lim_{l\to\infty}\lim_{\alpha\to\infty}\int \alpha R_{\alpha+1}R_1^{\sigma_{B'_{l,m}},\infty}\varphi\, d\mu_{\widehat{E}\cdot(\int_{\widehat{D}_F}^{\infty} e^{-s}g(\widehat{Y}_s)ds)}$$
$$= \int R_1^{D_F,\infty}\varphi\, d\mu_{\widehat{E}\cdot(\int_{\widehat{D}_F}^{\infty} e^{-s}g(\widehat{Y}_s)ds)}$$

and the assertion follows.

□

We are now in the situation to formulate the main structural theorem.

Theorem 4.11 *(*$\mathbf{QR}$*,* $\mathbf{M}^{\mathbf{ex}}$*,* $\widehat{\mathbf{QR}}$*,* $\widehat{\mathbf{M}^{\mathbf{ex}}}$*) Let $\mu \in S$. Then there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E such that*

$$1_{F_k^{reg}}\cdot\mu \in \widehat{S}_{00} \quad \textit{for each} \quad k \geq 1.$$

Proof By Theorem 4.7 we know that there exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ consisting of compact subsets of E, such that $1_{E_k}\cdot\mu \in \widehat{S}_0$ for each k. Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. Let $\widetilde{\widehat{G}_1\varphi}$ be an $\mathcal{E}$-q.c. m-version of $\widehat{G}_1\varphi$, let $\overline{\widehat{U}_1(1_{E_k}\cdot\mu)}$ be an $\mathcal{E}$-q.l.s.c. regularization of $\widehat{U}_1(1_{E_k}\cdot\mu)$. By making E_k smaller if necessary, we may assume that both m-versions are chosen w.r.t. $(E_k)_{k\geq 1}$ and that $\widetilde{\widehat{G}_1\varphi} \geq \frac{1}{k}$ $\mathcal{E}$-q.e. on E_k for each k. Observe that $\overline{\widehat{U}_1(1_{E_k}\cdot\mu)}$, $\widetilde{\widehat{G}_1\varphi}$ are finite $\mathcal{E}$-q.e. and that $\widetilde{\widehat{G}_1\varphi} > 0$ $\mathcal{E}$-q.e. Define

$$C_k := \left\{ z \in E_k \mid \overline{\widehat{U}_1(1_{E_k}\cdot\mu)} \leq a_k\widetilde{\widehat{G}_1\varphi} \right\}; \quad a_k := \mu(E_k)k^2,\ k \geq 1.$$

Obviously C_k is a family of compact subsets of E. Note that we don't claim that C_k is increasing. Furthermore

$$\begin{aligned}
\overline{\lim_{k\to\infty}}\operatorname{cap}_\varphi(C_k^c) &= \overline{\lim_{k\to\infty}}\operatorname{cap}_\varphi\left(E_k^c \cup \left\{ z \in E_k \mid \overline{\widehat{U}_1(1_{E_k}\cdot\mu)} > a_k\widetilde{\widehat{G}_1\varphi} \right\}\right) \\
&\leq \overline{\lim_{k\to\infty}}\operatorname{cap}_\varphi(E_k^c) + \lim_{k\to\infty}\operatorname{cap}_\varphi\left(\left\{ z \in E_k \mid \overline{\widehat{U}_1(1_{E_k}\cdot\mu)} > \mu(E_k)k \right\}\right) \\
&\leq \overline{\lim_{k\to\infty}}\operatorname{cap}_\varphi\left(\left\{\overline{\widehat{U}_1(1_{E_k}\cdot\mu)} > \mu(E_k)k\right\}\right) \\
&\leq \overline{\lim_{k\to\infty}}\lim_{\alpha\to\infty}\frac{1}{k\mu(E_k)}\mathcal{E}_1\left((G_1\varphi)^{\alpha}_{\left\{\overline{\widehat{U}_1(1_{E_k}\cdot\mu)} > \mu(E_k)k\right\}}, \widehat{U}_1(1_{E_k}\cdot\mu)\right) \\
&\leq \overline{\lim_{k\to\infty}}\frac{1}{k\mu(E_k)}\int_{E_k} R_1\varphi\, d\mu \leq \overline{\lim_{k\to\infty}}\frac{1}{k} = 0
\end{aligned}$$

Since $C_k^{reg} \subset C_k \subset E_k$ implies $1_{C_k^{reg}}\cdot\mu(B) \leq 1_{E_k}\cdot\mu(B)$ for any $B \in \mathcal{B}(E)$ we know further from Lemma 4.6, Lemma 4.5 that $1_{C_k^{reg}}\cdot\mu \in \widehat{S}_0$ for each k. Lemma 3.3 implies that $\underline{R_1^{\sigma_{C_k},\infty}\varphi(z) = R_1^{D_{C_k},\infty}\varphi(z)}$ for m-a.e. $z \in E$. Since $C_k^{reg} \subset E_k$ we obtain that $\overline{\widehat{U}_1(1_{C_k^{reg}}\cdot\mu)} \leq \overline{\widehat{U}_1(1_{E_k}\cdot\mu)} \leq a_k\widetilde{\widehat{G}_1\varphi}$ $\mathcal{E}$-quasi-everywhere on C_k. Applying Lemma 4.10(ii) we obtain

$$\begin{aligned}
(\varphi, \widehat{U}_1(1_{C_k^{reg}} \cdot \mu))_{\mathcal{H}} &= \int R_1\varphi \, 1_{C_k^{reg}} \, d\mu \\
&= \int E_z[\int_{\sigma_{C_k}}^{\infty} e^{-s}\varphi(Y_s)\, ds] \, 1_{C_k^{reg}}(z)\, \mu(dz) \\
&= \lim_{\alpha\to\infty} \int \alpha R_{\alpha+1} E_{\cdot}[\int_{\sigma_{C_k}}^{\infty} e^{-s}\varphi(Y_s)\, ds](z)\, 1_{C_k^{reg}}(z)\, \mu(dz)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\alpha\to\infty} \int \alpha \widehat{R}_{\alpha+1}\widehat{U}_1(1_{C_k^{reg}} \cdot \mu))\, d\mu_{\widehat{E}_{\cdot}\int_{D_{C_k}}^{\infty} e^{-s}\varphi(Y_s)\,ds} \\
&= \int \overline{\widehat{U}_1(1_{C_k^{reg}} \cdot \mu) \wedge a_k \widetilde{\widehat{G}_1\varphi}}\, d\mu_{\widehat{E}_{\cdot}\int_{D_{C_k}}^{\infty} e^{-s}\varphi(Y_s)\,ds} \\
&\leq (\varphi, \widehat{U}_1(1_{C_k^{reg}} \cdot \mu) \wedge a_k\widehat{G}_1\varphi)_{\mathcal{H}}.
\end{aligned}$$

Therefore $\widehat{U}_1(1_{C_k^{reg}} \cdot \mu) \leq a_k\widehat{G}_1\varphi$ for any $k \geq 1$. Define $F_k := \bigcup_{l=1}^k C_l$, $k \geq 1$. Since by the above $\lim_{k\to\infty} \mathrm{cap}_\varphi(C_k^c) = 0$, and $F_k^c \subset C_k^c$, we obtain that $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest consisting of compact sets. Since $F_k^{reg} = \bigcup_{l=1}^k C_l^{reg}$ we obtain

$$\widehat{U}_1(1_{F_k^{reg}} \cdot \mu) = \widehat{U}_1(1_{\bigcup_{l=1}^k C_l^{reg}} \cdot \mu) \leq \sum_{l=1}^{k} \widehat{U}_1(1_{C_l^{reg}} \cdot \mu) \leq \sum_{l=1}^{k} a_l\widehat{G}_1\varphi.$$

This implies the assertion.

□

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