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A universal framework of the homogenization problem of infinite dimensional diffusions

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Abstract

By generalizing the concrete formulations in [ABRY1,2], A universal frame work of the homogenization problem of infinite dimensional diffusions is proposed. The corresponding general structure is considered.

1 Probability space $(\Theta, \overline{B}, \mu)$ on which the random coefficients are defined

Suppose that we are given the following:

$\{(\Theta_k, B_k, \lambda_k)\}_{k \in \mathbb{Z}^d}$: a system of complete probability spaces, where $d$ is a given natural number.

$(\Theta, \overline{B}, \overline{\lambda})$: the probability space that is the completion of $\prod_k \Theta_k$, $\otimes_k B_k$, $\prod_k \lambda_k$, i.e., the completion of the direct product probability space.

$(\Theta, \overline{B}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows: for $\forall D \subset \subset \mathbb{Z}^d$ and for any bounded measurable function $\varphi$ defined on $\prod_{k \in D^c} \Theta_k$ with some $\forall D' \subset \subset \mathbb{Z}^d$, $\mu$ satisfies

$$(E^D \varphi, \mu) = (\varphi, \mu),$$

where

$$(E^D \varphi)(\theta) = \int_{\Theta} E^D(d\theta' | \theta)$$

$$\equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \overline{\lambda}(d\theta'),$$

with

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{k \in D} U_k,$$

$$\Theta \ni \theta \mapsto \theta_D \equiv \prod_{k \in D} \Theta_k.$$

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is the natural projection, $\theta'_D \cdot \theta_D^r$ is the element $\theta'' \in \Theta$ such that

$$\theta'' = \theta'_D, \quad \theta''_D^r = \theta_D^r,$$

also, for each $k \in \mathbb{Z}^d$, $U_k$ is a given bounded measurable function of which support is in $\prod_{|k' - k| \leq L} \Theta_{k'}$, where the number $L$ (the range of interactions) does not depend on $k$, and $Z_D(\theta_D^r)$ is the normalizing constant.

2 The ergodic flow

On $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ we are given an ergodic flow $T_x$ (which is also a map on $(\Theta, \overline{\mathcal{B}}, \mu)$, but is not a measure preserving map on it) as follows:\n
Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{Z}^d \quad \text{there exists a } d_k \text{ such that } d_k < M_1. \quad (2.1)$$

For each $x \in \prod_k \mathbb{R}^{d_k}$ such that $x = (x^k)_{k \in \mathbb{Z}^d}$ with $x^k = (x^k_1, \ldots, x^k_{d_k})$ the map $T_x$ on $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ is defined by

i) $T_x : \Theta \rightarrow \Theta$

that is a measure preserving transformation with respect to the measure $\overline{\lambda}$; ii)

$$T_0 = \text{the identity,}$$

for $x, x' \in \prod_k \mathbb{R}^{d_k}$ \quad $T_{x + x'} = T_x \circ T_{x'}$,

where

$$x + x' \equiv (x^k + x'^k)_{k \in \mathbb{Z}^d},$$

with

$$x^k + x'^k = (x^k_1 + x'^k_1, \ldots, x^k_{d_k} + x'^k_{d_k}),$$

for

$x = (x^k)_{k \in \mathbb{Z}^d}$, \quad $x^k = (x^k_1, \ldots, x^k_{d_k}),$

$x' = (x'^k)_{k \in \mathbb{Z}^d}$, \quad $x'^k = (x'^k_1, \ldots, x'^k_{d_k}),$

and

$0 \equiv (0^k)_{k \in \mathbb{Z}^d}$, \quad $0^k = (0, \ldots, 0) \in \mathbb{R}^{d_k};$

iii) $(x, \theta) \in (\prod_k \mathbb{R}^{d_k}) \times \theta \rightarrow T_x(\theta) \in \Theta$

is $\mathcal{B}(\prod_k \mathbb{R}^{d_k}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}}$-measurable, where, $\prod_k \mathbb{R}^{d_k}$ is assumed to be the topological space with the direct product topology;
iv) A function which is \(T_x\) invariant for all \(x \in \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}\) is a constant function;

v) For \(D \subset \mathbb{Z}^d\), let

\[
\prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k} \ni x \mapsto x_D \in \prod_{k \in D} \mathbb{R}^{d_k}
\]

be the natural projection. If \(x_{D'} = 0_{D'}\), then

\[
(T_x(\theta))_{D'} = \theta_{D'}, \quad \forall \theta \in \Theta, \quad \forall D \subset \subset \mathbb{Z}^d.
\]

3 The core

We assume that an existence of a core \(D\). Namely, there exists \(D\) which is a dense subset of both \(L^2(P)\) and \(L^1(P)\), and \(\forall \varphi \in D\) satisfies

(D-1) \(\varphi\) is a bounded measurable function having only a finite number of variables \(\theta_D\) for some \(D \subset \subset \mathbb{Z}^d\),

(D-2)

\[
\varphi(T_x(\theta)) \in C^\infty(\prod_{k \in D} \mathbb{R}^{d_k} \to \mathbb{R}), \quad \forall \theta \in \Theta,
\]

(cf. v) in the previous section) where we identify \(x_D \in \prod_{k \in D} \mathbb{R}^{d_k}\) with an \(x \in (\prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k})\) of which projection to \(\prod_{k \in D} \mathbb{R}^{d_k}\) is \(x_D\),

(D-3) in (D-2) for each \(\theta \in \Theta\), all the partial derivatives of all orders of the function \(\varphi(T_x(\theta))\) (with the variables \(x_D\)) are bounded and

\[
\forall \varphi \in D, \exists M < \infty; \quad |\nabla_k \varphi(T_x(\theta))| < M, \quad \forall \theta \in \Theta, \forall x, \forall k \in \mathbb{Z}^d, \quad (3.1)
\]

where

\[
\nabla_k = \left(\frac{\partial}{x_1^k}, \ldots, \frac{\partial}{x_{d_k}^k}\right).
\]

4 Probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\) on which the infinite dimensional diffusions are defined

Suppose that we are given a system of family of functions \(a_{ij}^k, \ k \in \mathbb{Z}^d, \ 1 \leq i, j \leq d_k\) on \((\Theta, \overline{\mathcal{B}}, \overline{\mu})\) such that for each \(k \in \mathbb{Z}^d\) and each \(1 \leq i, j \leq d_k\) \(a_{ij}^k\) is a measurable function on \(\Theta_k\) and there exists \(M_2 \in (0, \infty)\) and

\[
M_2^{-1} \leq \sum_{1 \leq i, j \leq d_k} a_{ij}^k(\theta_k)x_i x_j \leq M_2, \quad \forall k \in \mathbb{Z}^d, \forall \theta_k \in \Theta_k, \forall (x_1, \ldots, x_{d_k}) \in \mathbb{R}^{d_k}. \quad (4.1)
\]

We assume that

\[
U_k, \ a_{ij}^k \in D, \quad k \in \mathbb{Z}^d, \ 1 \leq i, j \leq d_k.
\]
Also, we assume that there exists a common $M < \infty$ by which the evaluation (3.1) holds for all $a_{ij}^k$ and $U_k$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{k,i}(t))_{t \geq 0}\}_{k \in \mathbb{Z}, 1 \leq i \leq d_k}.$$

Now, for each $\theta \in \Theta$, let

$$X^{\theta} \equiv \{(X^{\theta,k,i}(t))_{t \geq 0}\}_{k \in \mathbb{Z}, 1 \leq i \leq d_k}.$$

be the unique solution of

$$X^{\theta,k,i}(t) = X^{\theta,k,i}(0) + \int_0^t \sum_{1 \leq j \leq d_k} \left\{ \frac{\partial}{\partial x_j^k} a_{ij}^k(T_{X^{\theta,k}(s)}(\theta)) ight. \\
- a_{ij}^k(T_{X^{\theta,k}(s)}(\theta)) \left( \frac{\partial}{\partial x_j^k} \left( \sum_{|k-k'| \leq L} U_{k'}(T_{X^{\theta}(s)}(\theta)) \right) \right) \right\} ds \\
+ \int_0^t \sum_{1 \leq j \leq d_k} \sigma_{ij}^k(T_{X^{\theta,k}(s)}(\theta)) dB^{k,j}(s), \quad t \geq 0,$$

where

$$\sigma_{ij}^k = (2a_{ij}^k)^{\frac{1}{2}},$$

and

$$X^{\theta,k}(t) = (X^{\theta,k,1}(t), \ldots, X^{\theta,k,d_k}(t)),$$

also, by $X^{\theta}(t)$ we denote the vector

$$(X^{\theta,k}(t))_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} \mathbb{R}^{d_k}.$$

Then, the random variable on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in

$$C([0, \infty) \rightarrow \prod_{k \in \mathbb{Z}} \mathbb{R}^{d_k}).$$

Through $(X^{\theta}(t))_{t \geq 0}$ we define a $\Theta$ valued process such that

$$\{T_{X^{\theta}(t)}(\theta)\}_{t \geq 0}.$$

**Proposition 4.1** The following hold:

i) If $T_{X^{\theta}(0)}(\theta) = T_{X^{\theta'}(0)}(\theta')$, $P$–a.s. $\omega \in \Omega$,

then

$$(X^{\theta}(t) - X^{\theta}(0))_{t \geq 0} = (X^{\theta'}(t) - X^{\theta'}(0))_{t \geq 0}, \quad P$–a.s. $\omega \in \Omega,$

ii) For

$$\theta' = T_{X^{\theta}(0)}(\theta),$$

(4.3)
\[
(T_{X^{	heta}(t)}(\theta))_{t \geq 0} = (T_{X^{	heta'}(t)}(\theta'))_{t \geq 0}, \quad P-a.s. \quad \omega \in \Omega,
\]
(4.4)

where \(X^\theta_0(t)(\theta')\) is the diffusion defined by (4.2) with \(X^\theta_0(0) = 0\) and replacing \(\theta\) by \(\theta'\) in it.

By (4.4) such \((T_{X^\theta(t)}(\theta))_{t \geq 0}\) are represented by

\[
(Y_{\theta'}(t))_{t \geq 0} \equiv (T_{X^\theta_0(t)}(\theta'))_{t \geq 0}
\]

iii) The process \((Y_{\theta'}(t))_{t \geq 0}\) satisfies \(Y_{\theta'}(0) = \theta'\) and is a Markov process.

\[\square\]

**Definition 4.1** By Proposition 4.1, we define Markovian semi-groups corresponding to \((X^\theta(t))_{t \geq 0}\) and \((Y(t))_{t \geq 0}\):

For bounded measurable \(f \in C(\prod_{k \in D} \mathbb{R}^{d_k} \rightarrow \mathbb{R})\) with some bounded \(D \subset \subset \mathbb{Z}^d\),

\[
(p^X_{t} \varphi)(x) = E[f(X^\theta(t)) | X^\theta(0) = x], \quad x \in \prod_{k} \mathbb{R}^{d_k};
\]

For \(\varphi \in \mathcal{D}\)

\[
(p^Y_t \varphi)(y) = E[\varphi(T_{X^\theta(t)}(\theta)) | T_{X^\theta}(0)(\theta) = y], \quad y \in \Theta.
\]

\[\square\]

5 Key assumption and the result

As was done in [ABRY1,2], we assume the following:

There exist \(K < \infty\) and \(\gamma > 0\) such that

\[
\sup_{y \in \Theta} |(p^Y_t \varphi)(y) - \langle \varphi, \mu \rangle| \leq K e^{-\gamma t} (\|\nabla \varphi\|_{L^\infty} + \|\varphi\|_{L^\infty}), \quad \forall \theta \in \Theta, \quad \forall \varphi \in \mathcal{D},
\]

(5.1)

where

\[
\|\nabla \varphi\|_{L^\infty} = \sup_{x, \theta} |\nabla \varphi(T_x(\theta))|, \quad \|\varphi\|_{L^\infty} = \sup_{\theta} |\varphi(\theta)|.
\]

\[\square\]

**Definition 5.1** For each \(k \in \mathbb{Z}^d\) and \(i = 1, \ldots, d_k\), define an operator \(D^{k,i} : \mathcal{D} \rightarrow \mathcal{D}\) such that

\[
(D^{k,i} \varphi)(y) = \frac{\partial}{\partial x^k_i} \varphi(T_x(y))|_{x=0}, \quad \varphi \in \mathcal{D}, \quad y \in \Theta.
\]

\[\square\]

**Proposition 5.1** For each \(k \in \mathbb{Z}^d, \ i = 1, \ldots, d_k\) let

\[
b^k_i(y) \equiv \sum_{1 \leq j \leq d_k} \{(D^{k,j} a^{k}_{ij}(y) - a^{k}_{ij}(y)(D^{k,j}( \sum_{|k-k'| \leq L} U_{k'})(y))\}, \quad y \in \Theta,
\]

then

\[
\chi^k_i(y) = \int_0^\infty (p^Y_t \varphi)(y) dt \in L^2(\mu).
\]
Through the same discussions performed in [ABRY1,2], for the present general framework we are also able to show the following (cf. [ABRY1,2] for the exact statement and the terminologies)

**Theorem 5.1** By taking a subsequence of the scaling process with the scaling parameter $\epsilon > 0$ such that $\{\epsilon X^\theta \left( \frac{t}{\epsilon^2} \right) \}_{t \geq 0}$, for $\mu - a.e.$ $\theta \in \Theta$, it converges weakly to a Gaussian process with a constant covariance matrix, characterized by $\sigma_{ij}^k, \chi_i^k, k \in \mathbb{Z}^d, 1 \leq i, j \leq d_k$, and $\theta \in \Theta$, as $\epsilon_n \downarrow 0$ where $\{\epsilon_n\}_{n \in \mathbb{N}}$ is the sequence of the parameter corresponding to the subsequence of $\{\epsilon X^\theta \left( \frac{t}{\epsilon^2} \right) \}_{t \geq 0}$.

**References**
