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A universal framework of the homogenization problem of infinite dimensional diffusions

Sergio ALBEVERIO 1, Michael RÖCKNER 2, and Minoru W. YOSHIDA 3

Abstract

By generalizing the concrete formulations in [ABRY1,2], a universal framework of the homogenization problem of infinite dimensional diffusions is proposed. The corresponding general structure is considered.

1 Probability space \((\Theta, \overline{\mathcal{B}}, \overline{\mu})\) on which the random coefficients are defined

Suppose that we are given the following:

\[ \{(\Theta_k, \mathcal{B}_k, \lambda_k)\}_{k \in \mathbb{Z}^d} : \text{a system of complete probability spaces, where } d \text{ is a given natural number.} \]

\[(\Theta, \overline{\mathcal{B}}, \overline{\lambda}) : \text{the probability space that is the completion of } (\prod_k \Theta_k, \otimes_k \mathcal{B}_k, \prod_k \lambda_k), \text{i.e., the completion of the direct product probability space.} \]

\[(\Theta, \overline{\mathcal{B}}, \mu) : \text{a complete probability space (corresponding to a Gibbs state) defined as follows: for } \forall D \subset \subset \mathbb{Z}^d \text{ and for any bounded measurable function } \varphi \text{ defined on } \prod_{k \in D'} \Theta_k \text{ with some } \forall D' \subset \subset \mathbb{Z}^d, \mu \text{ satisfies} \]

\[ (E^D \varphi, \mu) = (\varphi, \mu), \]

where

\[ (E^D \varphi)(\theta) = \int_{\Theta} E^D(d\theta' | \theta) \]

\[ = \int_{\Theta} \varphi(\theta' \cdot \theta_{D'}) m_D(\theta' \cdot \theta_{D'}) \overline{\lambda}(d\theta'), \]

with

\[ m_D(\theta' \cdot \theta_{D'}) \equiv \frac{1}{Z_D(\theta_{D'})} e^{-U_D(\theta' \cdot \theta_{D'})}, \quad U_D \equiv \sum_{k \in D} U_k, \]

\[ \Theta \ni \theta \mapsto \theta_D \in \prod_{k \in D} \Theta_k \]

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is the natural projection, \( \theta_D' \cdot \theta_{D^r} \) is the element \( \theta'' \in \Theta \) such that

\[
\theta'' = \theta_D', \quad \theta_{D^r}' = \theta_{D^r},
\]

also, for each \( k \in \mathbb{Z}^d \), \( U_k \) is a given bounded measurable function of which support is in \( \prod_{|k' - k| \leq L} \Theta_k' \), where the number \( L \) (the range of interactions) does not depend on \( k \), and \( Z_D(\theta_{D^r}) \) is the normalizing constant.

## 2 The ergodic flow

On \( (\Theta, \overline{\mathcal{B}}, \overline{\lambda}) \) we are given an ergodic flow \( T_x \) (which is also a map on \( (\Theta, \overline{\mathcal{B}}, \mu) \), but is not a measure preserving map on it) as follows:

Suppose that

\[
\exists M_1 < \infty \quad \text{and} \quad \forall k \in \mathbb{Z}^d \quad \text{there exists a} \quad d_k < M_1. \quad (2.1)
\]

For each \( x \in \prod_{k} \mathbb{R}^{d_k} \) such that \( x = (x^k)_{k \in \mathbb{Z}^d} \) with \( x^k = (x_1^k, \ldots, x_{d_k}^k) \)

the map \( T_x \) on \( (\Theta, \overline{\mathcal{B}}, \overline{\lambda}) \) is defined by

1) \( T_x : \Theta \rightarrow \Theta \)

that is a measure preserving transformation with respect to the measure \( \overline{\lambda} \);

2) \( T_0 = \) the identity,

for \( x, x' \in x \in \prod_{k} \mathbb{R}^{d_k} \) \( \quad T_{x+x'} = T_x \circ T_{x'} \),

where

\[
x + x' \equiv (x^k + x'^k)_{k \in \mathbb{Z}^d},
\]

with

\[
x^k + x'^k = (x_1^k + x'_1^k, \ldots, x_{d_k}^k + x'_{d_k}^k),
\]

for

\[
x = (x^k)_{k \in \mathbb{Z}^d}, \quad x^k = (x_1^k, \ldots, x_{d_k}^k),
\]

\[
x' = (x'^k)_{k \in \mathbb{Z}^d}, \quad x'^k = (x'_1^k, \ldots, x'_{d_k}^k),
\]

and

\[
0 \equiv (0^k)_{k \in \mathbb{Z}^d}, \quad 0^k = (0, \ldots, 0) \in \mathbb{R}^{d_k};
\]

3) \( (x, \theta) \in (\prod_{k} \mathbb{R}^{d_k}) \times \theta \rightarrow T_x(\theta) \in \Theta \)

is \( \mathcal{B}(\prod_{k} \mathbb{R}^{d_k}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}} \)-measurable, where, \( \prod_{k} \mathbb{R}^{d_k} \) is assumed to be the topological space with the direct product topology;
iv) A function which is $T_x$ invariant for all $x \in \prod_{k \in Z^d} \mathbb{R}^{d_k}$ is a constant function;

v) For $D \subset Z^d$, let

$$\prod_{k \in Z^d} \mathbb{R}^{d_k} \ni x \mapsto x_D \in \prod_{k \in D} \mathbb{R}^{d_k}$$

be the natural projection. If $x_{D^c} = 0_{D^c}$, then

$$(T_x(\theta)|_{D^c} = \theta_{D^c}, \forall \theta \in \Theta, \forall D \subset \subset Z^d.$$ 

\[\square\]

3 The core

We assume that an existence of a core $D$. Namely, there exists $D$ which is a dense subset of both $L^2(P)$ and $L^1(P)$, and $\forall \varphi \in D$ satisfies

(D-1) $\varphi$ is a bounded measurable function having only a finite number of variables $\theta_D$ for some $D \subset \subset Z^d,$

(D-2)

$$\varphi(T_x(\theta)) \in C^\infty(\prod_{k \in D} \mathbb{R}^{d_k} \rightarrow \mathbb{R}), \forall \theta \in \Theta,$$

(cf. v) in the previous section) where we identify $x_D \in \prod_{k \in D} \mathbb{R}^{d_k}$ with an $x \in (\prod_{k \in Z^d} \mathbb{R}^{d_k})$ of which projection to $\prod_{k \in D} \mathbb{R}^{d_k}$ is $x_D$,

(D-3) in (D-2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T(\theta))$ (with the variables $x_D$) are bounded and

$$\forall \varphi \in D, \exists M < \infty; \ |\nabla_k \varphi(T_x(\theta))| < M, \forall \theta \in \Theta, \forall x, \forall k \in Z^d, \quad (3.1)$$

where

$$\nabla_k = \left(\frac{\partial}{x_1^k}, \ldots, \frac{\partial}{x_{d_k}^k}\right).$$

\[\square\]

4 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ on which the infinite dimensional diffusions are defined

Suppose that we are given a system of family of functions $a_{ij}^k$, $k \in Z^d$, $1 \leq i, j \leq d_k$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $k \in Z^d$ and each $1 \leq i, j \leq d_k$, $a_{ij}^k$ is a measurable function on $\Theta_k$ and there exists $M_2 \in (0, \infty)$ and

$$M_2^{-1} \leq \sum_{1 \leq i, j \leq d_k} a_{ij}^k(\theta_k)x_i x_j \leq M_2, \forall k \in Z^d, \forall \theta_k \in \Theta_k, \forall (x_1, \ldots, x_{d_k}) \in \mathbb{R}^{d_k}. \quad (4.1)$$

We assume that

$$U_k, a_{ij}^k \in D, \quad k \in Z^d, \quad 1 \leq i, j \leq d_k.$$
Also, we assume that there exists a common $M < \infty$ by which the evaluation (3.1) holds for all $a_{ij}^k$ and $U_k$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{k,i}(t))_{t \geq 0}\}_{k \in \mathbb{Z}^d, 1 \leq i \leq d_k}.$$ 

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^{\theta,k,i}(t))_{t \geq 0}\}_{k \in \mathbb{Z}^d, 1 \leq i \leq d_k}.$$ 

be the unique solution of

$$X^{\theta,k,i}(t) = X^{\theta,k,i}(0) + \int_0^t \sum_{1 \leq j \leq d_k} \left\{ \frac{\partial}{\partial x_j^k} a_{ij}^k(T_{X^{\theta_k}(s)}(\theta)) - a_{ij}^k(T_{X^{\theta_k}(s)}(\theta))(\frac{\partial}{\partial x_j^k}(\sum_{|k-k'| \leq L} U_{k'}(T_{X^{\theta}(s)}(\theta)))) \right\} ds$$

$$+ \int_0^t \sum_{1 \leq j \leq d_k} \sigma_{ij}^k(T_{X^{\theta_k}(s)}(\theta)) dB^{k,j}(s), \quad t \geq 0,$$

where

$$(\sigma_{ij}^k) = (2a_{ij}^k)^{1/2},$$

and

$$X^{\theta,k}(t) = (X^{\theta,k,1}(t), \ldots, X^{\theta,k,d_k}(t)),$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,k}(t))_{k \in \mathbb{Z}^d} \in \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}.$$ 

Then, the random variable on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in

$$C([0, \infty) \rightarrow \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}).$$

Through $(X^\theta(t))_{t \geq 0}$ we define a $\Theta$ valued process such that

$$\{T_{X^\theta(t)}(\theta)\}_{t \geq 0}.$$ 

**Proposition 4.1** The following hold:

i) If $T_{X^\theta(0)}(\theta) = T_{X^\theta'(0)}(\theta')$, $P$ - a.s. $\omega \in \Omega$,

then

$$(X^\theta(t) - X^\theta(0))_{t \geq 0} = (X^{\theta'}(t) - X^\theta'(0))_{t \geq 0}, \quad P$ - a.s. $\omega \in \Omega,$

ii) For

$$\theta' = T_{X^\theta(0)}(\theta),$$
\[(T_{X^0(t)}(\theta))_{t \geq 0} = (T_{X_{0}^\theta(t)}(\theta'))_{t \geq 0}, \quad P \text{- a.s. } \omega \in \Omega, \quad (4.4)\]

where \(X^0(t)(\theta')\) is the diffusion defined by (4.2) with \(X^0(0) = 0\) and replacing \(\theta\) by \(\theta'\) in it.

By (4.4) such \((T_{X^\theta(t)}(\theta))_{t \geq 0}\) are represented by

\[(Y^\theta(t))_{t \geq 0} \equiv (T_{X_{0}^\theta(t)}(\theta'))_{t \geq 0}\]

iii) The process \((Y^\theta(t))_{t \geq 0}\) satisfies \(Y^\theta(0) = \theta'\) and is a Markov process.

\[\square\]

**Definition 4.1** By Proposition 4.1, we define Markovian semi-groups corresponding to \((X^\theta(t))_{t \geq 0}\) and \((Y(t))_{t \geq 0}\):

For bounded measurable \(f \in C(\prod_{k \in D} \mathbb{R}^{d_k} \rightarrow \mathbb{R})\) with some bounded \(D \subset \subset \mathbb{Z}^d\),

\[(p^X_t f)(x) \equiv E[f(X^\theta(t)) | X^\theta(0) = x], \quad x \in \prod_{k} \mathbb{R}^{d_k};\]

For \(\varphi \in \mathcal{D}\)

\[(p^Y_t \varphi)(y) \equiv E[\varphi(T_{X^\theta(t)}(\theta)) | T_{X^\theta(0)}(\theta) = y], \quad y \in \Theta.\]

\[\square\]

**5 Key assumption and the result**

As was done in [ABRY1,2], we assume the following:

There exist \(K < \infty\) and \(\gamma > 0\) such that

\[\sup_{y \in \Theta} |(p^Y_t \varphi)(y) - \langle \varphi, \mu \rangle| \leq Ke^{-\gamma t}(\|\nabla \varphi\|_{L^\infty} + \|\varphi\|_{L^\infty}), \quad \forall \theta \in \Theta, \quad \forall \varphi \in \mathcal{D}, \quad (5.1)\]

where

\[\|\nabla \varphi\|_{L^\infty} = \sup_{x, \theta} |\nabla \varphi(T_x(\theta))|, \quad \|\varphi\|_{L^\infty} = \sup_{\theta} |\varphi(\theta)|.\]

\[\square\]

**Definition 5.1** For each \(k \in \mathbb{Z}^d\) and \(i = 1, \ldots, d_k\), define an operator \(D^{k,i} : \mathcal{D} \rightarrow \mathcal{D}\) such that

\[(D^{k,i} \varphi)(y) \equiv \frac{\partial}{\partial x^k_i} \varphi(T_x(y))|_{x=0}, \quad \varphi \in \mathcal{D}, \quad y \in \Theta.\]

\[\square\]

**Proposition 5.1** For each \(k \in \mathbb{Z}^d, \ i = 1, \ldots, d_k\) let

\[b^k_i(y) \equiv \sum_{1 \leq j \leq d_k} \{ (D^{k,j} a^k_{ij}) (y) - a^k_{ij}(y)(D^{k,j} (\sum_{|k-k'| \leq L} U_{k'})(y)) \}, \quad y \in \Theta,\]

then

\[\chi_{i}^{\theta,k}(y) \equiv \int_0^\infty (p^Y_t b^k_i)(y) dt \in L^2(\mu).\]
Through the same discussions performed in [ABRY1,2], for the present general framework we are also able to show the following (cf. [ABRY1,2] for the exact statement and the terminologies)

**Theorem 5.1** By taking a subsequence of the scaling process with the scaling parameter $\epsilon > 0$ such that $\{\epsilon X^\theta(\frac{t}{\epsilon^2})\}_{t \geq 0}$, for $\mu - a.e. \quad \theta \in \Theta$ it converges weakly to a Gaussian process with a constant covariance matrix, characterized by $\sigma_{ij}^k, \chi_k^{\theta}, k \in \mathbb{Z}^d, 1 \leq i, j \leq d_k$, and $\theta \in \Theta$, as $\epsilon_n \downarrow 0$ where $\{\epsilon_n\}_{n \in \mathbb{N}}$ is the sequence of the parameter corresponding to the subsequence of $\{\epsilon X^\theta(\frac{t}{\epsilon^2})\}_{t \geq 0}$.

References
