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A universal framework of the homogenization problem of infinite dimensional diffusions

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Abstract

By generalizing the ccncrete formulations in [ABRY1,2], A universal frame work of the homogenization problem of infinite dimensional diffusions is proposed. The corresponding general structure is considered.

1 Probability space $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ on which the random coefficients are defined

Suppose that we are given the following:

$\{(\Theta_{\mathbf{k}}, \mathcal{B}_{\mathbf{k}}, \lambda_{\mathbf{k}})\}_{\mathbf{k}\in\mathbf{Z}^d}$: a system of complete probability spaces, where d is a given natural number.

$(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$: the probability space that is the completion of $(\prod_{\mathbf{k}} \Theta_{\mathbf{k}}, \bigotimes_{\mathbf{k}} \mathcal{B}_{\mathbf{k}}, \prod_{\mathbf{k}} \lambda_{\mathbf{k}})$, i.e., the completion of the direct product probability space.

$(\Theta, \overline{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows: for $\forall D \subset\subset \mathbb{Z}^d$ and for any bounded measurable function φ defined on $\prod_{\mathbf{k}\in D'} \Theta_{\mathbf{k}}$ with some $\forall D' \subset\subset \mathbb{Z}^d$, μ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu),$$

where

$$\begin{aligned}(\mathbb{E}^D \varphi)(\theta) &\equiv \int_\Theta \mathbb{E}^D(d\theta'|\theta) \\ &\equiv \int_\Theta \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \overline{\lambda}(d\theta'),\end{aligned}$$

with

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{\mathbf{k}\in D} U_{\mathbf{k}},$$

$$\Theta \ni \theta \longmapsto \theta_D \in \prod_{\mathbf{k}\in D} \Theta_{\mathbf{k}}$$

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is the natural projection, $\theta'_D \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta''_D = \theta'_D, \qquad \theta''_{D^c} = \theta_{D^c},$$

also, for each $\mathbf{k} \in \mathbb{Z}^d$, $U_\mathbf{k}$ is a given bounded measurable function of which support is in $\prod_{|\mathbf{k}'-\mathbf{k}|\le L} \Theta_{\mathbf{k}'}$, where the number L (the range of interactions) does not depend on $\mathbf{k}$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

2 The ergodic flow

On $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ we are given an ergodic flow $T_\mathbf{x}$ (which is also a map on $(\Theta, \overline{\mathcal{B}}, \mu)$, but is not a measure preserving map on it) as follows:
Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{Z}^d \quad \text{there exists a } d_\mathbf{k} \text{ such that} \quad d_\mathbf{k} < M_1. \tag{2.1}$$

For each $\mathbf{x} \in \prod_\mathbf{k} \mathbb{R}^{d_\mathbf{k}}$ such that $\mathbf{x} = (\mathbf{x}^\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^d}$ with $\mathbf{x}^\mathbf{k} = (x_1^\mathbf{k}, \dots, x_{d_\mathbf{k}}^\mathbf{k})$
the map $T_\mathbf{x}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ is defined by
i)

$$T_\mathbf{x} : \Theta \longrightarrow \Theta$$

that is a measure preserving transformation with respect to the measure $\overline{\lambda}$;
ii)

$$T_\mathbf{0} = \text{the identity},$$

$$\text{for } \mathbf{x},\ \mathbf{x}' \in \mathbf{x} \in \textstyle\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_\mathbf{k}} \qquad T_{\mathbf{x}+\mathbf{x}'} = T_\mathbf{x} \circ T_{\mathbf{x}'},$$

where

$$\mathbf{x} + \mathbf{x}' \equiv (\mathbf{x}^\mathbf{k} + \mathbf{x}'^\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^d},$$

with

$$\mathbf{x}^\mathbf{k} + \mathbf{x}'^\mathbf{k} = (x_1^\mathbf{k} + x'^\mathbf{k}_1, \dots, x_{d^\mathbf{k}}^\mathbf{k} + x'^\mathbf{k}_{d^\mathbf{k}}),$$

for

$$\mathbf{x} = (\mathbf{x}^\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}^\mathbf{k} = (x_1^\mathbf{k}, \dots, x_{d_\mathbf{k}}^\mathbf{k}),$$

$$\mathbf{x}' = (\mathbf{x}^\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}'^\mathbf{k} = (x'^\mathbf{k}_1, \dots, x'^\mathbf{k}_{d_\mathbf{k}}),$$

and

$$\mathbf{0} \equiv (\mathbf{0}^\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{0}^\mathbf{k} = (0, \dots, 0) \in \mathbb{R}^{d_\mathbf{k}};$$

iii)

$$(\mathbf{x}, \theta) \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_\mathbf{k}}) \times \theta \longrightarrow T_\mathbf{x}(\theta) \in \Theta$$

is $\mathcal{B}(\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_\mathbf{k}}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}}$−measurable, where, $\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_\mathbf{k}}$ is assumed to be the topological space with the direct product topology;

iv) A function which is $T_{\mathbf{x}}$ invariant for all $\mathbf{x} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is a constant function;

v) For $D \subset \mathbb{Z}^d$, let

$$\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \ni \mathbf{x} \longmapsto \mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$$

be the natural *projection*. If $\mathbf{x}_{D^c} = \mathbf{0}_{D^c}$, then

$$(T_{\mathbf{x}}(\theta))_{D^c} = \theta_{D^c}, \qquad \forall \theta \in \Theta, \quad \forall D \subset\subset \mathbb{Z}^d.$$

□

3 The core

We assume that an existence of a *core* $\mathcal{D}$. Namely, there exists $\mathcal{D}$ which is a dense subset of both $L^2(P)$ and $L^1(P)$, and $\forall\varphi \in \mathcal{D}$ satisfies
($\mathcal{D}$-1) φ is a bounded measurable function having only a finite number of variables θ_D for some $D \subset\subset \mathbb{Z}^d$,
($\mathcal{D}$-2)

$$\varphi(T_{\mathbf{x}_L}(\theta)) \in C^\infty(\prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}} \to \mathbb{R}), \quad \forall\theta \in \Theta,$$

(cf. v) in the previous section) where we identify $\mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$ with an $\mathbf{x} \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$ of which projection to $\prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$ is $\mathbf{x}_D$,
($\mathcal{D}$-3) in ($\mathcal{D}$-2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T_{\cdot}(\theta))$ (with the variables $\mathbf{x}_D$) are bounded and

$$\forall\varphi \in \mathcal{D}, \ \exists M < \infty; \quad |\nabla_{\mathbf{k}}\varphi(T_{\mathbf{x}}(\theta))| < M, \quad \forall\theta \in \Theta, \ \forall\mathbf{x}, \ \forall\mathbf{k} \in \mathbb{Z}^d, \tag{3.1}$$

where

$$\nabla_{\mathbf{k}} = (\frac{\partial}{x_1^{\mathbf{k}}}, \dots, \frac{\partial}{x_{d_{\mathbf{k}}}^{\mathbf{k}}}).$$

□

4 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ on which the infinite dimensional diffusions are defined

Suppose that we are given a system of family of functions $a_{ij}^{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d$, $1 \le i,j \le d_{\mathbf{k}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $\mathbf{k} \in \mathbb{Z}^d$ and each $1 \le i,j \le d_{\mathbf{k}}$, $a_{ij}^{\mathbf{k}}$ is a measurable function on $\Theta_{\mathbf{k}}$ and there exists $M_2 \in (0,\infty)$ and

$$M_2^{-1} \le \sum_{1\le i,j\le d_{\mathbf{k}}} a_{ij}^{\mathbf{k}}(\theta_{\mathbf{k}}) x_i x_j \le M_2, \quad \forall\mathbf{k} \in \mathbb{Z}^d, \ \forall\theta_{\mathbf{k}} \in \Theta_{\mathbf{k}}, \ \forall(x_1, \dots, x_{d_{\mathbf{k}}}) \in \mathbb{R}^{d_{\mathbf{k}}}. \tag{4.1}$$

We assume that

$$U_{\mathbf{k}}, \ a_{ij}^{\mathbf{k}} \in \mathcal{D}, \quad \mathbf{k} \in \mathbb{Z}^d, \ 1 \le i,j \le d_{\mathbf{k}}.$$

Also, we assume that there exists a common $M < \infty$ by which the evaluation (3.1) holds for all $a_{i,j}^{\mathbf{k}}$ and $U_{\mathbf{k}}$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtoration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbb{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^{\theta,\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbb{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

be the unique solution of

$$\begin{aligned} X^{\theta,\mathbf{k},i}(t) = X^{\theta,\mathbf{k},i}(0) &+ \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \{\frac{\partial}{\partial x_j^{\mathbf{k}}} a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \\ &-a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta))(\frac{\partial}{\partial x_j^{\mathbf{k}}}(\sum_{|\mathbf{k}-\mathbf{k}'|\leq L} U_{\mathbf{k}'}(T_{X^\theta(s)}(\theta))))\}ds \\ &+ \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \sigma_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta))dB^{\mathbf{k},j}(s), \qquad t\geq 0, \end{aligned} \tag{4.2}$$

where

$$(\sigma_{ij}^{\mathbf{k}}) = (2a_{ij}^{\mathbf{k}})^{\frac{1}{2}},$$

and

$$X^{\theta,\mathbf{k}}(t) = (X^{\theta,\mathbf{k},1}(t), \ldots, X^{\theta,\mathbf{k},d_{\mathbf{k}}}(t)),$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,\mathbf{k}}(t))_{\mathbf{k}\in\mathbb{Z}^d} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}.$$

Then, the random variable on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in

$$C([0,\infty) \to \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}).$$

Through $(X^\theta(t))_{t\geq 0}$ we define a Θ valued process such that

$$\{T_{X^\theta(t)}(\theta)\}_{t\geq 0}.$$

Proposition 4.1 *The following hold:*
i) If $\quad T_{X^\theta(0)}(\theta) = T_{X^{\theta'}(0)}(\theta'), \quad P-a.s. \quad \omega\in\Omega,$
then

$$(X^\theta(t) - X^\theta(0))_{t\geq 0} = (X^{\theta'}(t) - X^{\theta'}(0))_{t\geq 0}, \qquad P-a.s. \quad \omega\in\Omega,$$

ii) For

$$\theta' = T_{X^\theta(0)}(\theta), \tag{4.3}$$

$$(T_{X^\theta(t)}(\theta))_{t\geq 0} = (T_{X_0^{\theta'}(t)}(\theta'))_{t\geq 0}, \qquad P-a.s. \quad \omega\in\Omega, \tag{4.4}$$

where $X_0^{\theta'}(t)(\theta')$ *is the diffusion defined by (4.2) with* $X_0^{\theta'}(0) = \mathbf{0}$ *and replacing* θ *by* θ' *in it.*
By (4.4) such $(T_{X^\theta(t)}(\theta))_{t\geq 0}$ *are represented by*

$$(Y_{\theta'}(t))_{t\geq 0} \equiv (T_{X_0^{\theta'}(t)}(\theta'))_{t\geq 0}$$

iii) *The process* $(Y_{\theta'}(t))_{t\geq 0})$ *satisfies* $Y_{\theta'}(0) = \theta'$ *and is a Markov process.*

□

Definition 4.1 *By Proposition 4.1, we define Markovian semi-groups corresponding to* $(X^\theta(t))_{t\geq 0}$ *and* $(Y_\cdot(t))_{t\geq 0}$*:*
For bounded measurable $f \in C(\prod_{\mathbf{k}\in D}\mathbb{R}^{d_{\mathbf{k}}} \to \mathbb{R})$ *with some dounded* $D \subset\subset \mathbb{Z}^d$*,*

$$(p_t^{X,\theta} f)(\mathbf{x}) \equiv E[f(X^\theta(t)) \mid X^\theta(0) = \mathbf{x}], \qquad \mathbf{x}\in\prod_{\mathbf{k}}\mathbb{R}^{d_{\mathbf{k}}};$$

For $\varphi\in\mathcal{D}$

$$(p_t^Y\varphi)(\mathbf{y}) \equiv E[\varphi(T_{X^\theta(t)}(\theta)) \mid T_{X^\theta(0)}(\theta) = \mathbf{y}], \qquad \mathbf{y}\in\Theta.$$

□

5 Key assumption and the result

As was done in [ABRY1,2], we assume the following:
There exist $K<\infty$ and $\gamma>0$ such that

$$\sup_{\mathbf{y}\in\Theta}|(p_t^Y\varphi)(\mathbf{y}) - <\varphi,\mu>| \leq K\,e^{-\gamma t}(\|\nabla\varphi\|_{L^\infty} + \|\varphi\|_{L^\infty}), \quad \forall\theta\in\Theta, \quad \forall\varphi\in\mathcal{D}, \tag{5.1}$$

where

$$\|\nabla\varphi\|_{L^\infty} = \sup_{\mathbf{x},\theta}|\nabla\varphi(T_{\mathbf{x}}(\theta))|, \qquad \|\varphi\|_{L^\infty} = \sup_\theta|\varphi(\theta)|.$$

□

Definition 5.1 *For each* $\mathbf{k}\in\mathbb{Z}^d$ *and* $i = 1,\dots,d_{\mathbf{k}}$*, define an operator* $D^{\mathbf{k},i}:\mathcal{D}\to\mathcal{D}$ *such that*

$$(D^{\mathbf{k},i}\varphi)(\mathbf{y}) \equiv \frac{\partial}{\partial x_i^{\mathbf{k}}}\varphi(T_{\mathbf{x}}(\mathbf{y}))|_{\mathbf{x}=0}, \qquad \varphi\in\mathcal{D}, \quad \mathbf{y}\in\Theta.$$

□

Proposition 5.1 *For each* $\mathbf{k}\in\mathbb{Z}^d$*,* $i = 1,\dots,d_{\mathbf{k}}$ *let*

$$b_i^{\mathbf{k}}(\mathbf{y}) \equiv \sum_{1\leq j\leq d_{\mathbf{k}}}\{(D^{\mathbf{k},j}a_{ij}^{\mathbf{k}})(\mathbf{y}) - a_{ij}^{\mathbf{k}}(\mathbf{y})(D^{\mathbf{k},j}(\sum_{|\mathbf{k}-\mathbf{k}'|\leq L}U_{\mathbf{k}'})(\mathbf{y}))\}, \quad \mathbf{y}\in\Theta,$$

then

$$\chi_i^{\theta,\mathbf{k}}(\mathbf{y}) \equiv \int_0^\infty(p_t^Y b_i^{\mathbf{k}})(\mathbf{y})dt \in L^2(\mu).$$

□

Through the same discussions performed in [ABRY1,2], for the present general framework we are also able to show the following (cf. [ABRY1,2] for the exact statement and the terminologies)

Theorem 5.1 *By taking a subsequence of the scaling process with the scaling parameter* $\epsilon > 0$ *such that* $\{\epsilon X^{\theta}(\frac{t}{\epsilon^2})\}_{t\geq 0}$, *for* $\mu - a.e.$ $\theta \in \Theta$ *it converges weakly to a Gaussian process with a constant covariance matrix, characterized by* $\sigma^{\mathbf{k}}_{ij}, \chi^{\theta,\mathbf{k}}_i, \mathbf{k} \in \mathbb{Z}^d, 1 \leq i,j \leq d_{\mathbf{k}}$, *and* $\theta \in \Theta$, *as* $\epsilon_n \downarrow 0$ *where* $\{\epsilon_n\}_{n\in\mathbb{N}}$ *is the sequence of the parameter corresponding to the subsequence of* $\{\epsilon X^{\theta}(\frac{t}{\epsilon^2})\}_{t\geq 0}$.

□

References

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