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Recent topics on a class of linear systems

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1 Introduction

This is a survey of [9, 10, 11]. We consider a class of continuous-time stochastic growth models on d-dimensional lattice $\mathbb{Z}^d$ with non-negative real numbers as possible values per site, so that the configuration at time t can be written as $\eta_t = (\eta_{t,x})_{x\in\mathbb{Z}^d}$, $\eta_{t,x} \geq 0$. We interpret the coordinate $\eta_{t,x}$ as the "population" at time-space (t,x), though it need *not* be an integer. The class of growth models considered here is a reasonably ample subclass of the one considered in [8, Chapter IX] as "linear systems". For example, it contains examples such as binary contact path process and potlatch process. The basic feature of the class is that the configurations are updated by applying the random linear transformation of the following form, when the Poisson clock rings at time-space (t,z):

$$\eta_{t,x} = \begin{cases} K_0\eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K_{x-z}\eta_{t-,z} & \text{if } x \neq z, \end{cases}$$

where $K = (K_x)_{x\in\mathbb{Z}^d}$ is a random vector with non-negative entries, and independent copies of K are used for each update (See next section for more detail). These models are known to exhibit, roughly speaking, the following phase transition [8, Chapter IX, sections 3–5]:

i) If the dimension is high $d \geq 3$, *and* if the vector K is not too random, then, with positive probability, the growth of the population is as fast as its expected value as time t tends to infinity, as such the *regular growth phase*.

ii) If the dimension is low $d = 1, 2$, *or* if the vector K is random enough, then, almost surely, the growth of the population strictly slower than its expected value as the time t tends to infinity, as such the *slow growth phase*.

In this paper, we review following: In the case i) above we see the equivalent conditions for the spatial distribution of the population,

$$\rho_{t,x} = \frac{\eta_{t,x}}{|\eta_t|}\mathbf{1}_{\{|\eta_t|>0\}}, \quad t > 0, x \in \mathbb{Z}^d,$$

obeys the central limit theorem, where $|\eta_t| = \sum_{x\in\mathbb{Z}^d} \eta_{t,x}$. In the case ii) above we see the equivalence between slow growth and localization property. Furthermore, under the reasonable condition, strong localization property holds, i.e., the spatial distribution $\rho_{t,x}$ does not decay uniformly in space as time t tends to infinity.

It should be mentioned that the central limit theorem in the same manner is discussed in [12, 14] and the localization/delocalization in the same spirit is discussed in [1, 2, 3, 4, 7, 13, 15, 16].

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2 Model and results

We introduce a random vector $K = (K_x)_{x\in\mathbb{Z}^d}$ such that

$$0 \le K_x \le b_K \mathbf{1}_{\{|x|\le r_K\}} a.s. \text{ for some constants } b_K, r_K \in [0,\infty),$$
$$\text{the set } \{x \in \mathbb{Z}^d; E[K_x] \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d.$$

The first condition amounts to the standard boundedness and the finite range assumptions for the transition rate of interacting particle systems. The second condition makes the model "truly d-dimensional".

Let $\tau^{z,i}$, ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}$) be i.i.d. mean-one exponential random variables and $T^{z,i} = \tau^{z,1} + \cdots + \tau^{z,i}$. Let also $K^{z,\cdot} = (K_x^{z,i})_{x\in\mathbb{Z}^d}$, ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}$) be i.i.d. random vectors with the same distributions as K, independent of $\{\tau^{z,i}\}_{z\in\mathbf{Z},i\in\mathbf{N}}$. We suppose that the process (η_t) starts from a deterministic configuration $\eta_0 = (\eta_{0,x})_{x\in\mathbb{Z}^d} \in [0,\infty)^{\mathbf{Z}^d}$ with $|\eta_0| < \infty$. At time $t = T^{z,i}$, η_{t-} is replaced by η_t, where

$$\eta_{t,x} = \begin{cases} K_0^{z,i}\eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K_{x-z}^{z,i}\eta_{t-,z} & \text{if } x \neq z. \end{cases} \tag{1}$$

We also consider the *dual process* $\zeta_t \in [0,\infty)^{\mathbb{Z}^d}$, $t \ge 0$ which evolves in the same way as $(\eta_t)_{t\ge0}$ except that (1) is replaced by its transpose:

$$\zeta_{t,x} = \begin{cases} \sum_{y\in\mathbb{Z}^d} K_{y-x}^{z,i}\eta_{t-,y} & \text{if } x = z, \\ \zeta_{t-,x} & \text{if } x \neq z. \end{cases}$$

We shall give typical examples which fall into the above set-up after the main results.

We recall the following facts. Let $\mathcal{F}_t$ be the σ-field generated by η_s, $s \le t$. Let $(\eta_t^x)_{t\ge0}$ be the process $(\eta_t)_{t\ge0}$ starting from one particle at the site x: $\eta_0^x = \delta_x$. Similarly, let $(\zeta_t^x)_{t\ge0}$ be the dual process starting from one particle at the site x: $\zeta_0^x = \delta_x$. We set

$$\begin{aligned} k &= (k_x)_{x\in\mathbb{Z}^d} = (E[K_x])_{x\in\mathbb{Z}^d} \\ \bar{\eta}_t &= (e^{-(|k|-1)t}\eta_{t,x})_{x\in\mathbb{Z}^d}. \end{aligned}$$

Proposition 2.1 *([8], Chapter IX, Theorem 2.2 and 2.4)*
a) $(|\bar{\eta}_t|, \mathcal{F}_t)_{t\ge0}$ *is a non-negative martingale, and therefore, the following limit exists a.s.*

$$|\bar{\eta}_\infty| = \lim_{t\to\infty} |\bar{\eta}_t|.$$

b) Either

$$E[|\bar{\eta}_\infty^0|] = 1 \text{ or } 0.$$

Moreover, $E[|\bar{\eta}_\infty^0|] = 1$ *if and only if the martingale* $|\bar{\eta}_t|$ *is uniformly integrable.*
c) The above (a)-(b), with η *replaced* ζ *are true for the dual process.*

We introduce some notations. For $\eta, \zeta \in \mathbb{R}^{\mathbb{Z}^d}$, the inner product and the discrete convolution are defined respectively by

$$\langle \eta, \zeta \rangle = \sum_{x\in\mathbb{Z}^d} \eta_x \xi_x \quad \text{and} \quad (\eta * \zeta)_x = \sum_{y\in\mathbb{Z}^d} \eta_{x-y}\zeta_y$$

provided the summations converge. We define $\beta \in \mathbb{R}^{\mathbb{Z}^d}$ by

$$\beta_x = \sum_{y \in \mathbb{Z}^d} E[(K - \delta_0)_{x+y}(K - \delta_0)_y].$$

We define G_S by

$$G_S(x) = \int_0^\infty P_S^0(S_t = x)dt$$

where $((S_t)_{t\geq 0}, P_S^x)$ is the continuous-time random walk on $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$, with the generator

$$L_S f(x) = \sum_{y \in \mathbb{Z}^d} \frac{k_{x-y} + k_{y-x}}{2}(f(y) - f(x))$$

Theorem 2.2 *Suppose $d \geq 3$. Then, the following conditions are equivalent:*
a) $\langle \beta, G_s \rangle < 2$
b) There exists a bounded function $h : \mathbb{Z}^d \to [1, \infty)$ such that

$$(L_S h)(x) + \frac{1}{2}\delta_{0,x}\langle \beta, h \rangle \leq 0, \quad x \in \mathbb{Z}^d$$

c) $\sup_{t\geq 0} E[|\overline{\eta}_t|^2] < \infty$
d) $\lim_{t\to\infty} \sum_{x \in \mathbb{Z}^d} f((x - mt)/\sqrt{t})\overline{\eta}_{t,x} = |\overline{\eta}_\infty| \int_{\mathbb{R}^d} f d\nu$ *in* $\mathbb{L}^2(P)$ *for all* $f \in C_b(\mathbb{R}^d)$
where $m = \sum_{x \in \mathbb{Z}^d} x k_x \in \mathbb{R}^d$, ν *is the Gaussian measure with*

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \sum_{x \in \mathbb{Z}^d} x_i x_j k_x, \quad i, j = 1, \ldots, d,$$

and $C_b(\mathbb{R}^d)$ denotes the set of bounded continuous function on $\mathbb{R}^d$.
b') There exists a bounded function $h : \mathbb{Z}^d \to [1, \infty)$ such that

$$(L_S h)(x) + \frac{1}{2}h(0)\beta_x \leq 0, \quad x \in \mathbb{Z}^d$$

c') $\sup_{t\geq 0} E[|\overline{\zeta}_t|^2] < \infty$
d') $\lim_{t\to\infty} \sum_{x \in \mathbb{Z}^d} f((x - mt)/\sqrt{t})\overline{\zeta}_{t,x} = |\overline{\zeta}_\infty| \int_{\mathbb{R}^d} f d\nu$ *in* $\mathbb{L}^2(P)$ *for all* $f \in C_b(\mathbb{R}^d)$

In order to see the slow growth phase, we present a sufficient condition.

Proposition 2.3 *a) For $d = 1, 2$, $|\overline{\eta}_\infty| = 0$ a.s. In particular for $d = 1$, there exists a constant $c > 0$ such that*

$$|\overline{\eta}_t| = O(e^{-ct}), \quad \textit{as } t \to \infty, \ \textit{a.s.} \tag{2}$$

b) For any $d \geq 1$, suppose that

$$\sum_{x \in \mathbb{Z}^d} E[K_x \ln K_x] > |k| - 1$$

then, again there exists a constant $c > 0$ such that (2) holds.

Recall that we have defined the spatial distribution of the population by

$$\rho_{t,x} = \frac{\eta_{t,x}}{|\eta_t|}\mathbf{1}_{\{|\eta_t|>0\}}, \quad t > 0, x \in \mathbb{Z}^d.$$

Interesting object related to the density would be

$$\rho_t^* = \max_{x\in\mathbb{Z}^d} \rho_{t,x}, \quad \text{and} \quad \mathcal{R}_t = \sum_{x\in\mathbb{Z}^d} \rho_{t,x}^2.$$

It is easy to see that $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. These quantities convey information on localization/delocalization of particles.

Theorem 2.4 *a) Suppose that* $P(|\overline{\eta}_\infty| > 0) > 0$. *Then,*

$$\int_0^\infty \mathcal{R}_s ds < \infty \quad a.s.$$

b) Suppose that $P(|\overline{\eta}_\infty| = 0) = 1$. *Then,*

$$\{\text{survival}\} = \{\int_0^\infty \mathcal{R}_s ds = \infty\}, \quad a.s.$$

where $\{\text{survival}\} = \{|\eta_t| \neq 0 \text{ for all } t \geq 0\}$. *Moreover, there exists a constant* $c > 0$ *such that*

$$|\overline{\eta}_t| \leq \exp(-c\int_0^t \mathcal{R}_s ds) \text{ for all large enough } t\text{'s}, a.s.$$

Theorem 2.5 *Suppose either*
a) $d = 1, 2$,
b) $d \geq 3$, $P(|\overline{\eta}_\infty| = 0) = 1$ *and* $\langle \beta, G_s \rangle > 2$
Then there exists a constant $c \in (0, 1]$ *such that*

$$\{\text{survival}\} = \{\int_0^\infty \mathbf{1}_{\{\mathcal{R}_s \geq c\}} ds = \infty\}, \quad a.s.$$

Here are some typical examples:
The extended binary contact path process: The extended binary contact path process is a special case of our set-up, in which

$$K_\alpha = \begin{cases} (\delta_{x,0} + \alpha\delta_{x,e})_{x\in\mathbb{Z}^d} & \text{with probability } \frac{\lambda}{2d\lambda+1} \\ & \text{for each } 2d \text{ neighbor } e \text{ of } 0, \\ 0 & \text{with probability } \frac{1}{2d\lambda+1}, \end{cases}$$

for $\alpha > 0$. The process is interpreted as the spread of an infection, with $\eta_{t,x}$ infected individuals at time t at site x. All the infected individuals at site $x - e$ are duplicated, multiplied α and added to those on the site x with probability $\frac{\lambda}{2d\lambda+1}$. On the other hand, all the infected individuals at a site become healthy with probability $\frac{1}{2d\lambda+1}$. If $\alpha = 1$, then this gives the binary contact path process (BCPP), originally introduced by D. Griffeath [5]. A motivation to study the BCPP comes from the fact that projected process

$$(\mathbf{1}_{\{\eta_{t,x}>0\}})_{x\in\mathbb{Z}^d}, \quad t \geq 0$$

is the basic contact process. Note that this relation is valid for our extended binary contact path process. Let π_d be the return probability for the simple random walk on $\mathbb{Z}^d$. Then we have

$$\langle \beta, G_S \rangle = \frac{2d\alpha^2\lambda + 1}{2d\alpha\lambda}\frac{1}{1-\pi_d}.$$

Hence

$$\begin{array}{lll} \langle \beta, G_S \rangle > 2 \iff \lambda < \dfrac{1}{2d(2\alpha(1-\pi_d)-\alpha^2)}, & & \text{if } 0 < \alpha < 2(1-\pi_d), \\ \langle \beta, G_S \rangle > 2 \quad \text{for all } \lambda > 0, & & \text{if } \alpha \geq 2(1-\pi_d). \end{array}$$

We can improve [5, Corollary], by taking $\alpha = 1 - \pi_d$:

Proposition 2.6 *Let $d \geq 3$. Then we have*

$$\lambda_c \leq \frac{1}{2d}\frac{1}{(1-\pi_d)^2}$$

where λ_c is the critical value of the basic contact process.

We also have

$$\begin{array}{ll} \displaystyle\sum_{x\in\mathbb{Z}^d} E[K_x \ln K_x] > |k| - 1 \iff \lambda < \frac{1}{2d\alpha(1-\ln\alpha)}. & \text{if } 0 < \alpha < e \\ \displaystyle\sum_{x\in\mathbb{Z}^d} E[K_x \ln K_x] > |k| - 1 \quad \text{for all } \lambda > 0, & \text{if } \alpha \geq e \end{array}$$

Suppose $\alpha = 1$. Then it is known that if $\lambda < \frac{1}{2d}$, then $\eta_t \equiv 0$ for large enough t's a.s. In fact, we do not know if there is a value λ for which BCPP with $d \geq 3$ is in slow growth phase, without getting extinct a.s.

The potlatch/smoothing processes: The potlatch process discussed in e.g. [6] and [8] is also a special case of our set-up, in which

$$K_x = Wk_x, \quad x \in \mathbb{Z}^d$$

Here $k = (k_x)_{x\in\mathbb{Z}^d} \in [0,\infty)^{\mathbb{Z}^d}$ is a non-random vector and W is a non-negative, bounded mean-one random variables such that $P(W=1) < 1$ (so that the notation k is consistent with the definition above). The smoothing process is the dual process of the potlatch process. Then we have

$$\begin{array}{ll} \langle \beta, G_S \rangle > 2 \iff E[W^2] > \dfrac{(2|k|-1)G_S(0)}{\langle G_S * k, k\rangle}, & \text{for } d \geq 3, \\ \displaystyle\sum_{x\in\mathbb{Z}^d} E[K_x \ln K_x] > |k| - 1 \iff E[W\ln W] > \frac{|k| - 1 - \sum_x k_x \ln k_x}{|k|}. & \end{array}$$

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