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Statistical mechanics of fluid turbulence taking into account fluid velocity (Multiplicity and hierarchical nature of turbulence, their mathematical structures)

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Statistical mechanics of fluid turbulence
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I. Introduction

The research on fluid turbulence has a long history. J. Boussinesq first considered the transport coefficient of fluid turbulence as

$$-\rho \overline{u_i u_j} = \rho K \frac{\partial \overline{u_i}}{\partial x_j}, \quad (1)$$

where $x_j$ is the j component of space and $u_i$ is the i component of fluid velocity. $\overline{a}$ is the average of quantity $a$ with respect to space and is described as

$$\overline{a}(x) = \int d^3 x G(x,y) a(y). \quad (2)$$

$G(x,y)$ is the kernel of integration and spatial average in this proceeding. $K$ is the eddy viscosity coefficient and was referred in [1,2].
\( u_i \) is decomposed as
\[
    u_i = U_i + u_i^*, \tag{3}
\]
where \( u_i \) is \( i \) component of fluid velocity at space \( x \) and time \( t \).

II. Large-eddy simulation and its expansion

The transport equation of fluid turbulence is described as
\[
    \frac{\partial}{\partial t} \mathbf{U} + \sum_{\hat{n}=1}^{3} \frac{\partial}{\partial x_{\hat{n}}} \mathbf{F}_{\hat{n}} = 0, \tag{4}
\]
where
\[
    \mathbf{U} = \begin{pmatrix}
        \rho \\
        \rho u_1 \\
        \rho u_2 \\
        \rho u_3 \\
        \rho e
    \end{pmatrix} \tag{5}
\]
and

\[ F_i = \begin{pmatrix} \rho u_i \\ \rho u_i u_1 + p \delta_{i1} - 2\mu A_{i1} \\ \rho u_i u_2 + p \delta_{i2} - 2\mu A_{i2} \\ \rho u_i u_3 + p \delta_{i3} - 2\mu A_{i3} \\ (\rho e + p) u_i - 2\mu \sum_{j=1}^{3} A_{ij} u_j - \lambda \frac{\partial T}{\partial x_i} \end{pmatrix} \quad (b) \]

\( \rho, u_i, p, \mu, A_{ij}, \lambda, \) and \( T \) are density, \( i \) component of velocity, pressure, molecular viscosity coefficient, \( ij \) element of stress tensor, thermal diffusion coefficient, and temperature of fluid at space \( x \), respectively.

\( A_{ij} \) is described as

\[ A_{ij} = \frac{1}{\rho} \left[ \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\varepsilon}{3} (\nabla \cdot \mathbf{u}) \delta_{ij} \right]. \quad (c) \]

Temperature \( T \) is defined in the relation as

\[ \frac{3}{2} k_B T = \frac{1}{\varepsilon} \sum_{j=1}^{N} \sum_{i=1}^{3} m_j (u_{ij} - \bar{u}_i)^2, \quad (8) \]

where

\[ \bar{u}_i = \frac{\sum_{j=1}^{N} m_j u_{ij}}{\sum_{j=1}^{N} m_j} \quad (9) \]
and $k_B$ is Boltzmann constant.

$T$ is defined at space $x$ over Kolmogorov length $l_k$. $m_{ij}$ is mass of $j$th molecule and $u_{ij}^i$ is $i$ component of $j$th molecule's velocity.

Spatial average of eq. (4) is described as

$$\frac{\partial}{\partial t} \overline{\mathbf{U}} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \overline{F}_i = 0,$$

where

$$\overline{\mathbf{U}} = \begin{pmatrix}
\overline{\rho} \\
\overline{\rho} \overline{u}_1 \\
\overline{\rho} \overline{u}_2 \\
\overline{\rho} \overline{u}_3 \\
\overline{\rho} \overline{\varepsilon}
\end{pmatrix}$$

and

$$\overline{F}_i = \begin{pmatrix}
\overline{\rho} \overline{u}_i \\
\overline{\rho} \overline{u}_i \overline{u}_1 + \bar{v} \bar{S}_{i1} - 2 \mu \bar{A}_{i1} \\
\overline{\rho} \overline{u}_i \overline{u}_2 + \bar{v} \bar{S}_{i2} - 2 \mu \bar{A}_{i2} \\
\overline{\rho} \overline{u}_i \overline{u}_3 + \bar{v} \bar{S}_{i3} - 2 \mu \bar{A}_{i3} \\
(\overline{\rho} \overline{\varepsilon} + \bar{v}) \overline{u}_i - 2 \mu \sum_{j=1}^{3} \bar{A}_{ij} \overline{u}_j - \lambda \frac{\partial \overline{\rho}}{\partial x_i}
\end{pmatrix}.$$
Spatial average \( \overline{a} \) of quantity \( a \) at space \( x \) is described as
\[
\overline{a}(x) = \frac{1}{V_1} \int dV_1 G(x, \xi) a(\xi),
\]
where \( G(x, \xi) \) is the kernel of integration. \( G(x, \xi) \) is the weight of spatial average and the weight is constant in this proceeding.

Eq. (12) is the spatial average of quantity \( a \) over space \( V_1 \). \( \overline{\gamma} \) is defined as
\[
\overline{\gamma} = \frac{\overline{\rho b}}{\overline{\rho}}.
\]
\( \overline{\nu} \) is defined as
\[
\overline{\nu} \equiv \overline{\rho} - \frac{1}{3} \sum_{l=1}^{3} \Gamma_{le l},
\]
and \( \Gamma_{le l} \) is
\[
\Gamma_{le l} = -\overline{\rho u_e u_l} + \overline{\rho u_l u_e}.
\]
\( \nu \) is defined as
\[
\nu = \overline{\gamma} - \frac{1}{2 \overline{\rho} c_v} \sum_{l=1}^{3} \Gamma_{le l}.
\]
\( \mu_t \) and \( \lambda_t \) are eddy viscosity coefficient and thermal diffusion coefficient, respectively.
It is possible to expand eq. (10) as
\[ \frac{d^2}{dt^2} \overline{t} + 2a_{\overline{t}} \frac{dt}{dx} \overline{t} = W_{\overline{t}} \frac{d^2}{dx^2} \overline{t}. \]  
(18)

Spatial dimension is one for simplicity.

\( a_{\overline{t}} \) and \( W_{\overline{t}} \) are matrices as
\[ a_{\overline{t}} = \begin{pmatrix}
  a_{\overline{t}} & 0 & 0 & 0 & 0  \\
  0 & a_{\overline{t}u_1} & 0 & 0 & 0  \\
  0 & 0 & a_{\overline{t}u_2} & 0 & 0  \\
  0 & 0 & 0 & a_{\overline{t}u_3} & 0  \\
  0 & 0 & 0 & 0 & a_{\overline{t}e}
\end{pmatrix} \]  
(19)

and
\[ W_{\overline{t}} = \begin{pmatrix}
  w_{\overline{t}}^{\frac{2}{p}} & 0 & 0 & 0 & 0  \\
  0 & w_{\overline{t}u_1}^{\frac{2}{p}} & 0 & 0 & 0  \\
  0 & 0 & w_{\overline{t}u_2}^{\frac{2}{p}} & 0 & 0  \\
  0 & 0 & 0 & w_{\overline{t}u_3}^{\frac{2}{p}} & 0  \\
  0 & 0 & 0 & 0 & w_{\overline{t}e}^{\frac{2}{p}}
\end{pmatrix} \]  
(20)

respectively. \( w_{\overline{t}}, w_{\overline{t}u_1}, w_{\overline{t}u_2}, w_{\overline{t}u_3}, \) and \( w_{\overline{t}e} \) are propagation velocity of waves for density, 1 component of momentum, 2 component,
3 component, and energy, respectively. The propagation velocities are considered fluid velocity. The solution of eq. (18) is

\[ u_j (x_j, t) = \exp (-a_{2j} z_j) \]

\[ \left[ \varepsilon_j (x_j) S (z_j - w_j \tau_j) \right] + \frac{a_{2j}}{2 w_j} \left\{ I_0 \left( a_{2j} \sqrt{1 - \frac{z_j^2}{w_j^2 \tau_j^2}} \right) \right. \]

\[ + \frac{1}{\sqrt{1 - \frac{z_j^2}{w_j^2 \tau_j^2}}} \left[ I_1 \left( a_{2j} \sqrt{1 - \frac{z_j^2}{w_j^2 \tau_j^2}} \right) \right] \]

(21)

for j th element of vector \( \mathbf{U} \). \( x_j \) is the coordinate of direction chosen. \( \xi_j = x_j - x_j^* \), \( \tau_j = t - t_0 \), and \( w_j \) is the j th element of diagonal part of matrix (20). \( I_i (y) \) is i th order Bessel's function of argument y.
Initial conditions
\[ u_j (x_1 | x_j, t_0) = \psi_j (x_1, x_j) S (x_j - x_j) \] (22)
and
\[ \psi_j (x_1, x_j) = P \{ u_j (x_j | x_j, t_0) | \chi_j (x_j, t_0) = x_j \}, \] (23)
where the variable \( \psi_j (x_1, x_j) \) is the probability distribution of \( u_j (x_1 | x_1, t) \) at coordinate \( x_1 \) and time \( t \) under the conditions of \( u_j (x_j | x_j, t) \) and \( u_1 (x_j | x_j, t_0) \) at time \( t \) and \( t_0 \).

III. Statistical mechanics of fluid turbulence

In section II solution of spatial average of transport equation is expressed. Eddy transport coefficients are obtained in this section.

The average \( \langle x_i \rangle_j \) and the variance
\[ \langle (x_i - \langle x_i \rangle_j)^2 \rangle_j \] of \( x_j \) with respect to measure are defined as
\[ \langle x_i \rangle_j = \int dx_i \left( \tilde{\rho} \tilde{u}_j (x_i) \right) x_i \] (23)
and
\[ \langle (X_i - \langle X_i \rangle_j)^2 \rangle_j = \int dx_i \left( \bar{P}_j (X_i) \right) (X_i - \langle X_i \rangle_j)^2, \]
respectively. They are calculated with the result (24) as
\[ \langle X_i \rangle_j = x_i + \frac{w_{ij} \varepsilon_{ij} (x_i, x_i)}{2 a_{ij}} \left( 1 - \exp \left[ -2 a_{ij} \bar{z}_{ij} \right] \right) \]
and
\[ \langle (X_i - \langle X_i \rangle_j)^2 \rangle_j = \frac{w_{ij} \varepsilon_{ij}}{a_{ij}} \left[ 1 - \frac{1 - \exp \left[ -2 a_{ij} \bar{z}_{ij} \right]}{2 a_{ij} \bar{z}_{ij}} \right] \]
\[ \quad + \varepsilon_{ij}^2 (x_i, x_i) \frac{(1 - \exp \left[ -2 a_{ij} \bar{z}_{ij} \right])^2}{4 a_{ij} \bar{z}_{ij}}. \]
The coefficient of eddy viscosity is obtained as
\[ D_{ij} = \frac{\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j}{2 \bar{z}_{ij}} = \frac{w_{ij}^2}{2 a_{ij}} \left[ 1 - \frac{\exp \left[ -2 a_{ij} \bar{z}_{ij} \right]}{2 a_{ij} \bar{z}_{ij}} \right] \]
\[ \quad + \varepsilon_{ij}^2 (x_i, x_i) \frac{(1 - \exp \left[ -2 a_{ij} \bar{z}_{ij} \right])^2}{4 a_{ij} \bar{z}_{ij}}. \]
IV. Conclusions.

The eddy transport coefficient is calculated as

$$D_{ij} = \frac{\omega_{ij}^2}{2a_{ij}} \left[ 1 - \frac{\exp[-2a_{ij}L_{ij}]}{2a_{ij}L_{ij}} \right]$$

$$+ \varepsilon_{ij}^{(x)} \frac{(1 - \exp[-2a_{ij}L_{ij}])^2}{4a_{ij}L_{ij}}$$

and substituting the relation of $2\tau_{ij} = L_{ij}/\omega_{ij}$

leads to the relation as

$$D_{ij} = \frac{\omega_{ij}^2}{2a_{ij}} \left[ 1 - \frac{\exp[-2a_{ij}L_{ij}/\omega_{ij}]}{2a_{ij}L_{ij}/\omega_{ij}} \right]$$

$$+ \varepsilon_{ij}^{(x)} \frac{(1 - \exp[-2a_{ij}L_{ij}/\omega_{ij}])^2}{4a_{ij}L_{ij}/\omega_{ij}} \right].$$

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