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Statistical mechanics of fluid turbulence taking into account fluid velocity

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I. Introduction

The research on fluid turbulence has its long history. J. Bousinessq first considered the transport coefficient of fluid turbulence as

$$-\rho\, \overline{u_i u_j} = \rho K \frac{\partial u_i}{\partial x_j}, \qquad (1)$$

where x_j is j component of space and u_i is i component of fluid velocity. $\bar{a}$ is the average of quantity a with respect to space and is described as

$$\bar{a}(\underset{\sim}{r}) = \int d\underset{\sim}{r}_1\, G(\underset{\sim}{r}, \underset{\sim}{r}_1)\, a(\underset{\sim}{r}_1). \qquad (2)$$

$G(\underset{\sim}{r}, \underset{\sim}{r}_1)$ is the kernel of integration and spatial average in this proceeding. K is the eddy viscosity coefficient and was referred in [1, 2].

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22 03 2009

u_i is decomposed as

$$u_i = \overline{u_i} + u_i' , \qquad (3)$$

where u_i is i component of fluid velocity at space x and time t.

II. Large-eddy simulation and its expansion

The transport equation of fluid turbulence is described as

$$\frac{\partial}{\partial t} U + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} F_i = 0 , \qquad (4)$$

where

$$U = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho e \end{pmatrix} \qquad (5)$$

and

$$F_i = \begin{pmatrix} \rho u_i \\ \rho u_i u_1 + p\delta_{i1} - 2\mu A_{i1} \\ \rho u_i u_2 + p\delta_{i2} - 2\mu A_{i2} \\ \rho u_i u_3 + p\delta_{i3} - 2\mu A_{i3} \\ (\rho e + p) u_i - 2\mu \sum_{j=1}^{3} A_{ij} u_j - \lambda \frac{\partial T}{\partial x_i} \end{pmatrix}. \qquad (6)$$

ρ, u_i, p, μ, A_{ij}, λ, and T are density, i component of velocity, pressure, molecular viscosity coefficient, ij element of stress tensor, thermal diffusion coefficient, and temperature of fluid at space r, respectively.

A_{ij} is described as

$$A_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\,\delta_{ij}\right]. \qquad (7)$$

Temperature T is defined in the relation as

$$\frac{3}{2} k_B T = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{3} m(j)\left(u(j)_i - \overline{u}_i\right)^2, \qquad (8)$$

where

$$\overline{u}_i = \frac{\sum_{j=1}^{N} m(j)\, u(j)_i}{\sum_{j=1}^{N} m(j)} \qquad (9)$$

and k_B is Boltzmann constant.

T is defined at space $\mathbf{r}$ over Kolmogorov length l_K. $m(j)$ is mass of j th molecule and $u(j)_i$ is i component of j th molecule's velocity.

Spatial average of eq. (4) is described as

$$\frac{\partial}{\partial t}\overline{U} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\overline{F_i} = 0, \qquad (10)$$

where

$$\overline{U} = \begin{pmatrix} \bar\rho \\ \bar\rho\tilde u_1 \\ \bar\rho\tilde u_2 \\ \bar\rho\tilde u_3 \\ \bar\rho\tilde e \end{pmatrix} \qquad (11)$$

and

$$\overline{F_i} = \begin{pmatrix} \bar\rho\tilde u_i \\ \bar\rho\tilde u_i\tilde u_1 + \varpi\delta_{i1} - 2\mu_t\overline{A_{i1}} \\ \bar\rho\tilde u_i\tilde u_2 + \varpi\delta_{i2} - 2\mu_t\overline{A_{i2}} \\ \bar\rho\tilde u_i\tilde u_3 + \varpi\delta_{i3} - 2\mu_t\overline{A_{i3}} \\ (\bar\rho\tilde e + \varpi)\tilde u_i - 2\mu\sum_{j=1}^{3}A_{ij}u_j - \lambda_t\frac{\partial\vartheta}{\partial x_i} \end{pmatrix}. \qquad (12)$$

Spatial average $\bar{a}$ of quantity a at space $\underset{\sim}{r}$ is described as

$$\bar{a}(\underset{\sim}{r}) = \int d\underset{\sim}{r}_1 \, G(\underset{\sim}{r}, \underset{\sim}{r}_1) \, a(\underset{\sim}{r}_1) , \qquad (13)$$

where $G(\underset{\sim}{r}, \underset{\sim}{r}_1)$ is the kernel of integration. $G(\underset{\sim}{r}, \underset{\sim}{r}_1)$ is the weight of spatial average and the weight is constant in this proceeding.

Eq. (12) is the spatial average of quantity a over space $\underset{\sim}{r}_1$. $\tilde{b}$ is defined as

$$\tilde{b} \equiv \frac{\overline{\rho b}}{\bar{\rho}} . \qquad (14)$$

ϖ is defined as

$$\varpi \equiv \bar{p} - \frac{1}{3} \sum_{l=1}^{3} \Gamma_{ll} , \qquad (15)$$

and Γ_{lm} is

$$\Gamma_{lm} = - \overline{\rho u_l u_m} + \bar{\rho} \tilde{u}_l \tilde{u}_m . \qquad (16)$$

ϑ is defined as

$$\vartheta \equiv \tilde{T} - \frac{1}{2 \bar{\rho} C_v} \sum_{l=1}^{3} \Gamma_{ll} . \qquad (17)$$

μ_t and λ_t are eddy viscosity coefficient and thermal diffusion coefficient, respectively.

It is possible to expand eq. (10) as

$$\frac{\partial^2}{\partial t^2}\overline{U} + 2a_{\overline{U}}\frac{\partial}{\partial t}\overline{U} = W_{\overline{U}}\frac{\partial^2}{\partial X_1^2}\overline{U}. \qquad (18)$$

Spatial dimension is one for simplicity.
$a_{\overline{U}}$ and $W_{\overline{U}}$ are matrices as

$$a_{\overline{U}} = \begin{pmatrix} a_{\bar\rho} & 0 & 0 & 0 & 0 \\ 0 & a_{\overline{\rho u_1}} & 0 & 0 & 0 \\ 0 & 0 & a_{\overline{\rho u_2}} & 0 & 0 \\ 0 & 0 & 0 & a_{\overline{\rho u_3}} & 0 \\ 0 & 0 & 0 & 0 & a_{\overline{\rho e}} \end{pmatrix} \qquad (19)$$

and

$$W_{\overline{U}} = \begin{pmatrix} w^2_{\bar\rho} & 0 & 0 & 0 & 0 \\ 0 & w^2_{\overline{\rho u_1}} & 0 & 0 & 0 \\ 0 & 0 & w^2_{\overline{\rho u_2}} & 0 & 0 \\ 0 & 0 & 0 & w^2_{\overline{\rho u_3}} & 0 \\ 0 & 0 & 0 & 0 & w^2_{\overline{\rho e}} \end{pmatrix}, \qquad (20)$$

respectively. $w_{\bar\rho}$, $w_{\overline{\rho u_1}}$, $w_{\overline{\rho u_2}}$, $w_{\overline{\rho u_3}}$, and $w_{\overline{\rho e}}$ are propergation velocity of waves for density, 1 component of momentum, 2 component, 3 component, and energy, respectively. The propergation velocities are considered fluid velocity. The solution of eq. (18) is

$$u_j(X_1 | x_j, t) = \exp(-a_{1j}\tau_{1j})$$
$$\Big[\varepsilon_{1j}(X_1, x_j)\,\delta(\varsigma_j - w_{1j}\tau_{1j})$$
$$+ \frac{a_{1j}}{2w_{1j}} \Big\{ I_0\Big(a_{1j}\tau_{1j}\sqrt{1-\frac{\varsigma_j^2}{w_{1j}^2\tau_{1j}^2}} \Big)$$
$$+ \frac{1 + \dfrac{\varsigma_j\,\varepsilon_{1j}(X_1, x_j)}{w_{1j}\tau_{1j}}}{\sqrt{1-\dfrac{\varsigma_j^2}{w_{1j}^2\tau_{1j}^2}}}\, I_1\Big(a_{1j}\tau_{1j}\sqrt{1-\frac{\varsigma_j^2}{w_{1j}^2\tau_{1j}^2}} \Big) \Big\} \Big] \qquad (21)$$

for j th element of vector $\overline{U}$. X_1 is the coordinate of direction chosen. $\varsigma_j = X_j - x_j$, $\tau_{1j} = t - t_0$, and w_{1j} is the j th element of diagonal part of matrix (20). $I_i(y)$ is i th order Bessel's function of argument y.

Initial conditions

$$u_j(X_1 | x_j, t_0) = \varepsilon_{1j}(X_1, x_j)\,\delta(X_j - x_j) \quad (22)$$

and

$$\varepsilon_{1j}(X_1, x_j) = P\{u_j(X_j | x_j, t_0) \,|\, X_j(x_j, t_0) = x_j\},$$

(23)

where the variable $\varepsilon_{1j}(X_1, x_j)$ is the probability distribution of $u_j(X_1 | x_1, t)$ at coordinate X_1 and time t under the conditions of $u_j(x_1 | x_1, t)$ and $u_1(x_j | x_j, t_0)$ at time t and t_0.

III. Statistical mechanics of fluid turbulence

In section II solution of spatial average of transport equation is expressed. Eddy transport coefficients are obtained in this section.

The average $\langle X_i \rangle_j$ and the variance $\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j$ of X_j with respect to measure are defined as

$$\langle X_i \rangle_j \equiv \int dX_i \left(\bar{\rho}\,\tilde{u}_j(X_i)\right) X_i \quad (23)$$

and

$$\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j \equiv \int dX_i \left(\bar{\rho} \tilde{u}_j (X_i) \right) (X_i - \langle X_i \rangle_j)^2,$$

respectively. They are calculated with (24)
the result (21) as

$$\langle X_i \rangle_j = x_i + \frac{w_{ij}\, \varepsilon_{ij}(x_i, x_j)}{2a_{ij}} \left(1 - \exp[-2a_{ij}\tau_{ij}] \right) \tag{25}$$

and

$$\begin{aligned} \langle (X_i - \langle X_i \rangle_j)^2 \rangle_j &= \frac{w_{ij}^2 \tau_{ij}}{a_{ij}} \left[1 - \frac{1 - \exp[-2a_{ij}\tau_{ij}]}{2a_{ij}\tau_{ij}} \right. \\ &\quad + \varepsilon_{ij}^2 (x_i, x_i) \frac{(1 - \exp[-2a_{ij}\tau_{ij}])^2}{4a_{ij}\tau_{ij}} . \end{aligned} \tag{26}$$

The coefficient of eddy viscosity is obtained as

$$\begin{aligned} D_{ij} &= \frac{\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j}{2\tau_{ij}} = \frac{w_{ij}^2}{2a_{ij}} \left[1 - \frac{\exp[-2a_{ij}\tau_{ij}]}{2a_{ij}\tau_{ij}} \right. \\ &\quad \left. + \varepsilon_{ij}^2 (x_i, x_j) \frac{(1 - \exp[-2a_{ij}\tau_{ij}])^2}{4a_{ij}\tau_{ij}} \right], \end{aligned} \tag{27}$$

10

22 03 2009

IV. Conclusions.

The eddy transport coefficient is calculated as

$$D_{ij} = \frac{w_{ij}^2}{2a_{ij}}\left[1 - \frac{\exp[-2a_{ij}\tau_{ij}]}{2a_{ij}\tau_{ij}} + \varepsilon_{ij}^2(x_i, x_j)\frac{(1-\exp[-2a_{ij}\tau_{ij}])^2}{4a_{ij}\tau_{ij}}\right] \quad (28)$$

and substituting the relation of $\tau_{ij} = L/w_{ij}$ leads to the relation as

$$D_{ij} = \frac{w_{ij}^2}{2a_{ij}}\left[1 - \frac{\exp[-2a_{ij}L/w_{ij}]}{2a_{ij}L/w_{ij}} + \varepsilon_{ij}^2(L, L)\frac{(1-\exp[-2a_{ij}L/w_{ij}])^2}{4a_{ij}L/w_{ij}}\right]. \quad (29)$$

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