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The Valuation of Discrete-Time Contingent Claims with Upper and Lower Bounds; Revisited with Refinements

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Abstract

In this paper we deal with the valuation of callable-putable contingent claims with general payoff functions under the setting of an optimal stopping problem between the seller and the buyer. The seller can cancel the claim issued by him/her as well as the buyer can exercise the right. Such claims may provide the upper bound of the loss to the seller and the lower of the payoff to the buyer, respectively. We derive the valuation formula of the callable-putable claims where the asset price follows a random walk. Some analytical properties of optimal stopping rules and the value function are investigated in more detail.

1 Introduction

We consider a financial market consisting of a riskless asset and of a risky asset over the discrete time horizon $t = 0, 1, 2 \cdots, T$. Suppose that a new callable contingent claim (hereafter abbreviated by CC) has been issued by the firm into the market. The callable CC enable the seller to cancel by paying an extra penalty to the buyer. On the other hand, the buyer can exercise the right at any time up to the maturity. The game option introduced by Kifer [11] is one of such securities. Callable convertible bonds, liquid yield option notes and callable stock options are examples of such financial derivatives (see [1], [8], [14], [19]and [20]).

In this paper we deal with a valuation model of such callable CC where the payoff functions are more general and different from the payoff if both of the buyer and seller do not exercise their right before the maturity. The decision making related to callable CC consists of the selection of the cancellation time by the seller and the exercise time by the buyer, that is, a pair of two stopping times. When the seller stops at a time before the buyer does, the seller must pay to the buyer more than when the buyer stops before the seller does. When either of them do not stop before the maturity, then the payoff would turn out to be intermediate.

This paper is organized as follows. Section 2 sets up a discrete time valuation model for callable CC whose payoff functions are more general. In section 3 we derive optimal policies and investigate their analytical properties by using contraction mappings. In section 4 we discuss a special case of binominal price processes to derive the specific stop and continue regions. In section 5, concluding remarks are given together with some directions for the future research.

2 Pricing Model

We consider the discrete time case where the capital market consists of riskless bond B_t with interest rate r_t at time t, so that

$$B_t = \Pi_{k=1}^{t}(1 + r_k)B_0 \tag{2.1}$$

and of a risky asset whose price S_t at time t equals

$$S_t = S_0 \Pi_{k=1}^{t}(1+\rho_k) = S_{t-1}(1+\rho_t) \tag{2.2}$$

where $\rho_k(\omega) = \frac{1}{2}(d + u_k + \omega_k(u_k - d_k)), \ = \omega(\omega_1, \omega_2, \cdots, \omega_T) \in \Omega\{1, -1\}^T$ which is the sample space of finite sequences ω with the product probability $P = \{(p_k, 1-p_k)\}^T$.
To exclude an arbitrage opportunity as usual, we assume for each k

$$-1 < d_k < r_k < u_k, \ 0 < p_k < 1. \tag{2.3}$$

The equivalent martingale probability p^* with respect to p is given by

$$p_k^* = \frac{r_k - d}{u - d}, \ q_k^* = 1 - p_k^*.$$

It is clear that $E^*(\rho_k) = r_k$
Given an initial wealth w_0, an investment strategy is a sequence of portfolios $\pi = (\pi_1, \pi_2, \cdots, \pi_T)$ at each time where a portfolio π_t is a pair of (α_t, β_t) with α_t and β_t representing the amount of risky asset and of riskless bond at time t, respectively. The wealth formed by the portfolio π at time t is given by

$$W_t^\pi = \alpha_t S_t + \beta_t B_t, \ t \geq 1 \tag{2.4}$$

with $W_0 = w$ is given.
An investment strategy π is called self-financing if

$$\alpha_1 S_0 + \beta_1 B_0 = w$$

and

$$S_{t-1}(\alpha_t - \alpha_{t-1}) + B_{t-1}(\beta_t - \beta_{t-1}) = 0, \quad t > 1$$

which means no cash-in and no cash-out from or to the external sources.
Let $\hat{W}_t^\pi = B_t^{-1} W_t^\pi$. Then, for a self-financing strategy π we have $\hat{W}_t^\pi = w_0 + \Sigma_{k=1}^{t} B_k^{-1} \alpha_k S_{k-1}(\rho_k - r_k)$ which is a martingale $u.r.t.p^*$. Denote by $\mathcal{J}_{t,T}$ the finite set of stopping times taking values in $\{t, t+1, \cdots, T\}$. A callable contingent claim is a contract between an issuer A and an investor B addressing the asset with a maturity T. The issuer can choose a stopping time σ to call back the claim with the payoff function Y_σ and the investor can also choose a stopping time τ to exercise his/her right with the payoff function X_τ at any time before the maturity. Should neither of them stop before the maturity, the payoff should be Z_t. The payoff always goes from the issuer to the investor. We assume

$$0 \leq X_t \leq Z_t \leq Y_t, \ 0 \leq t < T$$

and

$$X_T = Z_T \tag{2.5}$$

The investor wishes to exercise the right so as to maximize the expected payoff. On the other hand, the issuer wants to call the contract so as to minimize the payment to the investor. Then, for any pair of the stopping times (σ, τ), define the payoff function by

$$R(\sigma, \tau) = Y_\sigma 1_{\{\sigma < \tau \leq T\}} + X_\tau 1_{\{\tau < \sigma \leq T\}} + Z_T 1_{\{\sigma \wedge \tau = T\}} \tag{2.6}$$

A hedge against a callable CC with a maturity T is a pair (σ, π) of a stopping time σ and a self-financing investment strategy π such that

$$W^{\pi}_{\sigma \wedge t} \geq R(\sigma, t), \ t = 0, 1, \cdots, T.$$

The price v^* of a callable CC is the infinum of $v \geq 0$ such that there exists a hedge (σ, π) against this callable CC with $W^{\pi}_{o} = v$.

Theorem 1 *(Kifer [11]) Let $P^* = \{p^*_t, 1 - p^*_t\}^T$ be the probability on the space Ω with $p^*_t = \frac{r_t - u_t}{r_t - d_t}$, $t \leq T < \infty$, and E^* be the expectation with respect to P^*. Then, the price v^* of the callable CC equals $v^*_{0,T}$ which can be obtained from the recursive equations as follows;*

$$v^*_{T,T} = \Pi^T_{t=1}(1 + r_t)^{-1} Z_T$$

and

$$v^*_{t,T} = \min\{\Pi^t_{k=1}(1 + r_k)^{-1} Y_t, \ \max[\Pi^t_{k=1}(1 + r_k)^{-1} X_t, \ E^*(v^*_{t+1,N})]\} \tag{2.7}$$

Moreover, for $t = 0, 1, \cdots, T$

$$\begin{aligned} v^*_{t,T} &= \min_{\sigma \in \mathcal{J}_{t,T}} \max_{\tau \in \mathcal{J}_{t,T}} E^*[\Pi^{-\sigma \wedge \tau}_{k=1}(1 + r_k)^{-1} R(\sigma, \tau) | \Im_t] \\ &= \max_{\tau \in \mathcal{J}_{t,T}} \min_{\sigma \in \mathcal{J}_{t,T}} E^*[\Pi^{-\sigma \wedge \tau}_{k=1}(1 + r_k)^{-1} R(\sigma, \tau) | \Im_t], \end{aligned} \tag{2.8}$$

for each $t = 0, 1, \cdots, T$, the stopping times

$$\sigma^*_{t,T} = \min\{k \geq t | \Pi^k_{l=1}(1 + r_l)^{-1} Y_k = v^*_{k,T}\} \tag{2.9}$$

and

$$\tau^*_{t,T} = \min\{k \geq t | \Pi^k_{l=1}(1 + r_l)^{-1} X_k = v^*_{k,T}\} \tag{2.10}$$

belong to $\mathcal{J}_{t,T}$ and $v^*_{T,T} = \Pi^T_{t=1}(1 + r_t)^{-1} Z_T$.
The inequalities

$$\begin{aligned} E^*[\Pi^{\sigma^*_{t,T} \wedge \tau}_{k=1}(1 + r_k)^{-1} R(\sigma^*_{t,T}, \tau) | \Im_t] &\leq v^*_{t,T} \\ &\leq E^*[\Pi^{\sigma \wedge \tau^*_{t,T}}_{k=1}(1 + r_k)^{-1} R(\sigma, \tau^*_{t,T}) | \Im_t] \end{aligned} \tag{2.11}$$

hold for any $\sigma, \tau \in \mathcal{J}$

Remark 1 *The model can be extended to the infinite case $T \to \infty$, provided that $r_k = r$ for all k*

$$\lim_{T \to \infty} (1 + r)^{-T} Y_T = 0 \quad \text{with} \quad v_{T,T} = Z_T \tag{2.12}$$

with p^*-probability 1. If $Y_t = (K - S_t) + \delta_t$, then equation (2.12) can be replaced by

$$\lim_{t \to \infty} (1 + r)^{-t} \delta_t = 0 \tag{2.13}$$

which means that the penalty does not grow too fast as $t \to \infty$.

Remark 2 *Defining* $Z_t = \Pi_{k=1}^t (1+r_k)^{-1} W_t^\pi$, *then we obtain*

$$Z_t = w + \Sigma_{k=1}^t \Pi_{l=1}^k (1+r_l)^{-1} \alpha_k S_{k-1}(\rho_k - r_k) \tag{2.14}$$

which is a martingale w.r.t. $P^* = \{p^*, 1-p^*\}^T$

Corollary 1 *Assume that equation (2.12) holds. Then, the limit value*

$$v^* = \lim_{T\to\infty} v_{0,T}^* \tag{2.15}$$

exists.

3 Optimal Policies and their Analytic Properties

In this section we propose a different approach from Kifer [11] and Dynkin [6]. Since the asset price process follows a (non-stationary) binominal process, the payoff processes of X_t and Y_t are both Markov processes. So we formulate this optimal stopping problem as a Markov decision process. In this section, we assume $r_k = r$ for all k and put $\beta = (1+r)^{-1}$. Let $X_t = \beta^t X(S_t)$, $Y_t = \beta^t Y(S_t)$ and $Z_t = \beta^T Z(S_T)$. It follows from these new notations that $\prod_{k=1}^t (1+r_k)^{-k} X_t = X(S_t)$, $\prod_{k=1}^t (1+r_k)^{-k} Y_t = Y(S_t)$ and $\prod_{k=1}^t (1+r_k)^{-k} Z_T = Z(S_T)$.
Put $V_0(s) = Z(s)$ and define for $n \geq 1$

$$\begin{aligned} v^{n+1}(s) &\equiv (\mathcal{U}v^n)(s) \\ &\equiv \min(Y(s), \max(X(s), \beta E_s[v^n(\tilde{S}_{n+1})])) \end{aligned} \tag{3.1}$$

where E_s is the expectation $w.r.S_n = s$
Let V be the set of all bounded functions and its limit with the norm $\|v\| = \sup_{s\in\Omega} |v(s)|$. For $u, v \in V$ we write $u \leq v$ if $u(s) \leq v(s)$ for all $s \in \Omega$. A mapping is called a contraction mapping if

$$\|\mathcal{U}u - \mathcal{U}v\| \leq \beta \|u - v\|$$

for some $\beta < 1$ and for all $u, v \in V$.

Lemma 1 *The mapping* $\mathcal{U}$ *as defined by equation(3.1) is a contraction mapping*

Proof. For any $u,\ v \in V$ we have

$$\begin{aligned} (\mathcal{U}u)(s) - (\mathcal{U}v)(s) &= \min(Y(s), \max(X(s), \beta E_s[u(\tilde{S})])) \\ &- \min(Y(s), \max(X(s), \beta E_s[v(\tilde{S})])) \end{aligned}$$

$$\begin{aligned} (\mathcal{U}u)(s) - (\mathcal{U}v)(s) &\leq \min(Y(S), \beta E_s[u_s(\tilde{S})]) \\ &- \max(X(S), \beta E_s[v_s(\tilde{S})]) \\ &\leq \beta E_s[u(\tilde{S})] - \beta E_s[v_s(\tilde{S})] \\ &\leq \beta E_s[\sup(u(\tilde{S}) - v(\tilde{S}))] \\ &= \beta \|u - v\| \end{aligned}$$

Hence, we obtain

$$\sup_{s\in\Omega}(\mathcal{U}u)(s) - (\mathcal{U}v)(s) \leq \beta\|u - v\|. \tag{3.2}$$

By taking the roles of u and v reversely, we have

$$\sup_{s\in\Omega}(\mathcal{U}v)(s) - (\mathcal{U}u)(s) \leq \beta\|v - u\| \tag{3.3}$$

Putting equation(3.2) and (3.3) together we obtain

$$\|\mathcal{U}u - \mathcal{U}v\| \leq \beta\|u - v\|$$

□

Corollary 2 *There exists a unique function $v \in V$ such that*

$$(\mathcal{U}v)(s) = v(s) \quad for\ all\ s \tag{3.4}$$

Furthermore, for all $u \in V$

$$(\mathcal{U}^T u)(s) \rightarrow v(s)\ as\ T \rightarrow \infty$$

where $v(s)$ is equal to the fixed point defined by equation(3.4), that is, $v(s)$ is a unique solution to

$$v(s) = \min\{Y(s), \max(X(s), \beta E_s[v(\tilde{S})])\}$$

Since $\mathcal{U}$ is a contraction mapping from corollary 1, the optimal value function v for the perpetual contingent claim can be obtain as the limit by successively applying an operator $\mathcal{U}$ to any initial value function v for a finite lived contingent claim.

Remark 3 *When we specialize the price process into the binominal process, the probability space can be reduced to $\Omega = \{0, 1, 2\cdots\}$ with a $\sigma - field\ \Im_t = i$ which represents the number of up-jumps by time t and $P = (p, 1-p)$*

Assumption 2 *If $v(s)$ is monotone in s, then $E_s v(\tilde{S})$ is monotone in s.*

Lemma 2 *Suppose that Assumption 2 holds. Then,*

i) *$(\mathcal{U}^n v)(s)$ is monotone in s for $v \in V$.*

ii) *v satisfying $v = \mathcal{U}v$ is monotone in s.*

iii) *there exists a pair (s_n^*, s_n^{**}) of the optimal boundaries such that*

$$v^{n+1}(s) \equiv (\mathcal{U}v^{n-1})(s) = \begin{cases} Y(s) & if\ s \geq s_n^* \\ \max(X(s), \beta E_s[v^{n-1}(\tilde{S})]) & if\ s \leq s_n^{**}, n = 1, 2, \cdots, T \end{cases}$$

with $v_0 = Z$.

Proof.

i) The proof follows by induction on n. For $n = 1$, we have

$$(\mathcal{U}v^0)(s) = \min\{Y(s), \max[X(s), \beta E_s Z(\tilde{S})]\}$$

Suppose that $X(s), Y(s)$ and $Z(s)$ is increasing in s.
which is monotone in s, provided that Assumption 2 holds. Suppose that v_n is monotone for $n > 1$. Then,

$$v^{n+1}(s) \equiv (\mathcal{U}v^n)(s) = \min\{X(s), \max(Y(s), \beta E_s[v^n(\tilde{S})])\}$$

which is again monotone in s since the maximum of the monotone functions is monotone.

ii) Since $\lim_{n\to\infty}(\mathcal{U}^n v_o)(s)$ point-wisely converges to the limit $v(s)$ from corollary 2, the limit function $v(s)$ is also monotone in s.

iii) Should $v^n = (\mathcal{U}^{n-1}v_0)(s)$ be monotone in s, then there exists at least one pair of boundary values s^* and s^{**}_n such that

$$v^n = \begin{cases} Y(s) & \text{if } s \geq s^* \\ \max[X(s), \beta E_s(v^{n-1}(\tilde{S})] & \text{otherwise} \end{cases}$$

$$\max(X(s), E_s[v^{n-1}(\tilde{S})]) = \begin{cases} X(s) & \text{for } s \leq s^{**} \\ E_s[v^{n-1}(\tilde{S})] & \text{otherwise} \end{cases}$$

□

From equation(2.11), v^n is monotone increasing in n since $X_n(s) \leq v^n(s) \leq Y_n(s)$. Define for the issuer

$$S^I_n = \{s | V^n(s) = Y(s)\} \tag{3.5}$$

$$s^*_n = \inf\{s | s \in S^I_n\} \tag{3.6}$$

and for the investor

$$S^{II}_n = \{s | V^n(s) = X(s)\} \tag{3.7}$$

$$s^{**}_n = \inf\{s | s \in S^{II}_n\} \tag{3.8}$$

It is easy to show that

$$s^*_n \geq s^{**}_n \; for\ each\ n \tag{3.9}$$

Remark 4 *In game put options (Kifer [11],Kyprianou [13]) it is assumed that $X_n(S_n) \equiv \beta^n X(S_n)$ and $Y_n(S_n) = \beta^n(X(S_n) + \delta)$ with $\delta > 0$ where $X(S_n) = (K - S_n)^+$. It is easy to show that $v^n = v^n(s)$ is continuous and decreasing in s and increasing in δ.*

4 A Simple Random Walk Case

Suppose that the process $\{S_t, t = 1, 2, \cdots\}$ is a random walk, that is,

$$S_{t+1} = S_t \cdot \tilde{X}_{t+1}$$

where $\tilde{X}_1, \tilde{X}_2 \cdots$ are independently distributed random variables with the finite mean.

i) We consider the case of a callable call option where $X(s) = (s - K)^+$ and $Y(s) = X(s) + \delta, \delta > 0$

$$\beta E_s(\tilde{S}) = \beta s(1 + p^* u + (1 - p^*)d) = \beta(1 + r)s = s$$

which is a martingale. So $\beta^n X(S_n) = \max(\beta^n S - \beta^n K, 0)$ is a submartingale. Applying the Optimal Sampling Theorem, we obtain that

$$\begin{aligned} v_t(s) &= \min_{\sigma \in \mathcal{J}_{t,T}} \max_{\tau \in \mathcal{J}_{t,T}} E_s^*[\beta^{\sigma \wedge \tau} R(\sigma, \tau)] \\ &= \min_{\sigma \in \mathcal{J}_{t,T}} \max_{\tau \in \mathcal{J}_{t,T}} E_s^*[\beta^{\sigma \wedge \tau}(Y(S_{\sigma \wedge \tau})1_{\{\sigma < \tau\}} + X(S_{\sigma \wedge \tau})1_{\{\tau < \sigma\}} + Z_T 1_{\{\sigma \wedge \tau = T\}})] \\ &= \min_{\sigma \in \mathcal{J}_{t,T}} E_s^*[\beta^\sigma X(S_\sigma)1_{\{\sigma < T\}} + \beta^T Z_T 1_{\{\sigma = T\}}] \end{aligned} \tag{4.1}$$

which can be represented in the following corollary;

Corollary 3 *Callable-Putable contingent claims with the maturity $T < \infty$ can be degenerated into callable ones.*

This corollary corresponds to the well known result that American call options are identical to the corresponding European call options. In the case of callable-putable call claims it follows that the investor should exercise his/her putable right at the maturity. However, the issuer should choose an optimal call stopping time which minimize the expected payoff function given by equation(4.1). From equation(2.11) we know that

$$X_t \le v_t \le Y_t \quad for\ 0 \le t \le T.$$

and the optimal stopping times for each $t = 0, 1, \cdots, T$ are

$$\sigma_t^* = \min\{n \ge t : v_{t,T} = \beta^n Y_n(s)\} \wedge T$$

and

$$\tau_t^* = \{n \ge t : v_{t,T}(s) = \beta^n X_n(s)\}.$$

Lemma 3 *$V_t(s) - s$ is decreasing in s for each t and decreasing in t for each s.*

$$\begin{aligned} S_t^I &= \{s | v_t(s) - s \ge -k + \delta\} \ \ for\ t < T.\ S_T^I = \phi \\ S_t^{II} &= \{s | v_t(s) - s \le -k\} = \phi \ \ for\ t < T\ if\ E(\tilde{X}) \ge 1. \end{aligned}$$

ii) We consider the case of a callable put option where $X(s) = \max\{K - s, 0\}$ and $Y(t) = X(t) + \delta$

Lemma 4 *Let $X(s) = \max\{K - s, 0\}, Y(s) = X(s) + \delta$ and $E(\tilde{S}_t) > -1$. $V_t(s) + s$ is increasing in s for each t.*

Proof. For $t = T$ we have

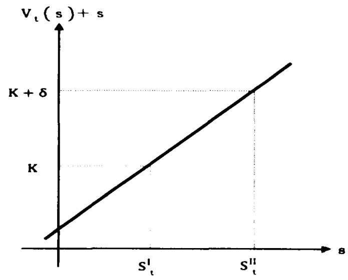
$$\begin{aligned} V_T(s) + s &= \max\{X(s), 0\} + s \\ &= \max\{K - s, 0\} + s = \max\{K, s\} \end{aligned}$$

which is increasing in s. Suppose the assertion for $t + 1$.
Then, putting $\mu = E(\tilde{S}_t)$

$$\begin{aligned} V_t(s) + s &= \min\{(K - s)^+ + \delta, \max\{(K - s)^+, \beta E_s V_{t+1}(s\tilde{S}) + s\tilde{S}\} + \beta s(1 + \mu)\} \\ &= \min\{K + \delta, \max[K, \beta E(V_{t+1}(s\tilde{S}) + s\tilde{S})] + \beta s(1 + \mu)\} \end{aligned}$$

$V_{t+1}(s\tilde{S}) + s\tilde{S}$ is increasing in s for all $\tilde{S} > 0$ and $s(1+\mu)$ is also increasing in s for $\mu > -1$. So is $V_t(s) + s$. □

For each t, define



$$\begin{aligned} s_t^I &= \inf\{s : V_{t+1}(s) + s \geq K + \delta\} \\ s_t^{II} &= \inf\{s : V_{t+1}(s) + s \geq K\} \end{aligned}$$

where s_t^I and s_t^{II} equal ∞ when these sets are empty.

Lemma 5 *s_t^I and s_t^{II} are increasing in t*

Proof.

$$\begin{aligned} s_t^I &= \inf\{s | V_t(s) + s \geq K + \delta\} \\ &\leq \inf\{s | V_{t+1}(s) + s \geq K + \delta\} \\ &= s_{t+1}^I \\ s_t^{II} &= \inf\{s | V_t(s) + s \geq K\} \\ &\geq \inf\{s | V_{t+1}(s) + s \geq K\} \\ &= s_t^{II} \end{aligned}$$

Lemma 6 *If $\frac{K}{S}F(\frac{K}{S}) > 1 - \int_{\frac{K}{S}}^{\infty} x dF(x)$, it is never optimal for the investor to exercise before the maturity. It is never optimal for the issuer to c all at the maturity.*

Proof.

$$\begin{aligned}
S_T^I &\equiv \{s|V_T(s) = K - s + \delta\} \\
&= \{s|\max(K-s,0) + s = K + \delta\} \\
&= \{\phi\} \\
S_T^{II} &\equiv \{s|V_\tau(s) = K - s\} \\
&= \{s|\max(K-s,0) = K - s\} \\
&= \{K\}
\end{aligned}$$

$$\begin{aligned}
V_{T-1}(s) &= \min\{K - s + \delta, \max(K-s, \beta E[V_T(sX_t)])\} \\
&= \min\{K - s + \delta, \max(K-s, \beta E \max(K-s,0))\} \\
&= \min\{K - s + \delta, \max(K-s, \beta K \int_0^{\frac{K}{S}} dF(x) - \beta \int_0^{\frac{K}{S}} sxdF(x)\} \\
&= \min\{K - s + \delta, \max(K-s, \beta K F(\frac{K}{S}) - \beta \int_0^{\frac{K}{S}} sxdF(x)\}
\end{aligned}$$

$$\begin{aligned}
\beta K F(\frac{K}{S}) - \beta s \int_0^{\frac{K}{S}} sxdF(s) &= \beta K F(\frac{K}{S}) - \beta s(\int_0^{\infty} xdF(x) - \int_{\frac{K}{S}}^{\infty} xdF(x)) \\
&\leq \beta K F(\frac{K}{S}) - \beta s\mu + \beta s \int_{\frac{K}{S}}^{\infty} \frac{K}{s} dF(x)) \\
&\leq \beta K F - \beta s\mu + \beta K(1-F) \\
&= \beta K - \beta s\mu \quad \mu > -1 \\
&= \beta(K - s\mu) \quad -\mu > 1 \\
&\leq \beta(K-s)
\end{aligned}$$

Hence,

$$\begin{aligned}
V_{T-1}(s) &= \min\{K - s + \delta, \beta(K-s)\} \\
&< K - s + \delta \ \ if \ (K - s + \delta) < \beta(K-s) \\
S_{T-1}^I &= \{\phi\}
\end{aligned}$$

□

Theorem 3

i) *There exists an optimal call policy for the issuer as follows;*
If the asset price is s at time t and $s > s_t^I$, then the issuer call the contingent claim.

ii) *There exists an optimal exercise policy for the investor as follows;*
If the asset price is s at time t and $s \leq s_t^{II}$, the investor exercises the contingent claim, otherwise, either of them do not exercise.

Since $X \leq v_{t,T} \leq Y$, for each $t \leq T$, the issuer should stop or call if $s \in D_t^I$ and the investor should exercise if $s \in D_t^{II}$

Lemma 7

$$C_t^I \supset C_{t+1}^I, \quad C_t^{II} \subset C_{t+1}^{II}$$
$$C_t^I \subset C_t^I \ \ and \ \ C_t^{II} \supset C_{t+1}^{II}$$

The proof directly follows from the result that $v_{t,T}$ is increasing in t.
Let

$$(\mathcal{U}^t v)(s) = \min\{Y(s), \max(X(s), \beta^t E_s[v(s\tilde{S}_t)])\}$$

Lemma 8 *If there exists $\theta > 1$ and $\delta > 0$ for which $E[X_t^\theta | X_0, \cdots, X_{t-1}] \leq e^\delta$ for $t = 1, 2, \cdots, T$, then we obtain*

$$V_t(s) \geq f(s)$$

where $f(s)$ is given by

$$f(s) = \begin{cases} \frac{s^\theta(\theta-1)^{\theta-1}}{\theta^\theta} & if\ s \leq \frac{\theta}{\theta-1} \\ K - S & otherwise \end{cases}$$

5 Concluding Remark

In this paper we consider the discrete time valuation model for callable contingent claims in which the asset price follows a random walk including a binominal process as a special case. It is shown that such valuation model can be formulated as a coupled optimal stopping problem of a two person game between the issuer and the investor. We show under some assumptions that these exists a simple optimal call policy for the issuer and optimal exercise policy for the investor which can be described by the control limit values. Also, we investigate analytical properties of such optimal stopping rules for the issuer and the investor, respectively, possessing a monotone property.

It is of interest to extend it to the three person games among the issuer, investor and the their party like stake holders. Furthermore, we might analyze a dynamic version of the model by introducing the state of the economy which follows a Markov chain. In this extended dynamic version the optimal stopping rules as well as their value functions should depend on the state of the economy. We shall discuss such a dynamic valuation model somewhere in a near future.

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