Formulation of an Optimal Execution Problem with Random Market Impact [*]

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1   Introduction

Liquidity (or illiquidity) is significant theme in both financial theory and business, and recently, it draws more attentions after we experienced successive financial crises in 2007 and 2008. Sometimes decrease of liquidity causes serious damage for transactions, so a trader should be careful when he/she plans his/her portfolio and the schedule of execution. Besides it is hard to forecast effects of illiquidity beforehand.

In this paper we pay special attention to market impact (MI), which is the effect of the investment behavior of traders on security prices. MI plays an important role in portfolio theory, and is also significant when we consider the case of an optimal execution problem, where a trader has a certain amount of security holdings (shares of a security held) and tries to execute until the time horizon.

Tokyo Stock Exchange Inc. (TSE), the main securities exchange in Japan, launched a new trading system "arrowhead" (the Next Generation Trading System) on January 4, 2010. This system enables us to execute an order more rapidly and to shorten the execution lag of time as 10 milli-seconds or less. Simultaneously, TSE weakened the trading rule of price limit and shortened nominal price quotation. By this development, trading participants become able to execute securities more easily. Moreover, arrowhead gives us more information about limit order books. So considering appropriate execution policy becomes more significant task. According to speed-up transactions, the need of constructing algorithms of optimal execution becomes higher. To construct a model of optimal execution, it is important to consider not only application and practice but also theoretical formulation.

An optimal execution problem with MI has been studied by several papers ([1], [2], [8], [9], [10] and so on,) and [4]-[5] formulated such a problem mathematically as follows. First we construct the discrete-time model of an optimal execution problem with MI, and then characterize the corresponding continuous-time model as the limit of discrete-time model.

[4] assumed that MI function is deterministic like many other papers. This assumption means that we can get the information of MI beforehand. But in the real market it is hard to estimate the effect of MI. Moreover it often happens that the concentration of unexpected orders will cause the overfluctuation for the price.

Considering randomness of MI is more realistic task and few papers treat such a subject mathematically, so it is meaningful to construct the mathematical model of random MI. In

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this paper we generalize the framework of [4], especially considering random MI function. The approach is the same : we construct the discrete-time model first and derive the continuous-time model as their limit. We assume that noise of MI function is independent of both time and the amount of trading volume. More concretely, we set a MI function in the discrete-time model as a product of a positive i.i.d. random variable and a deterministic function (for details, see Section 2.) In this case we see that randomness of MI in the continuous-time model is described as a jump of a Lévy process.

We also consider the generalization of examples of [4] and investigate the effect of noise of MI for the optimal strategy of a trader by numerical experiments.

The rest of this paper is organized as follows. In Section 2, we present the mathematical formulation of our model. As our basic model, we set the discrete-time model of an optimal execution problem and define the corresponding value function. In Section 3, we give our main results. Similar to [4], we show that the discrete-time value functions converge to some function which can be regarded as a value function in the continuous-time model under some additional technical assumptions. Moreover we have the continuity property of the continuous-time value function like Theorem 2 of [4]. In Section 4 we treat some examples of our model and present the result of numerical experiments. Section 5 is the conclusion of this paper. Proofs are omitted in this paper but you can refer to [3].

2   The Model

In this section we present the details of the model. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $(B_t)_{0 \leq t \leq T}$ be a standard one-dimensional Brownian motion. $T > 0$ means a time horizon and we assume $T = 1$ for brevity. We suppose that the market consists of one risk-free asset (namely cash) and one risky asset (namely a security.) The price of cash is always equal to 1, which means that a risk-free rate is equal to zero. The price of a security fluctuates according to a certain stochastic flow, and is influenced by sales of a trader.

First we consider the discrete-time model with the time interval $1/n$. We consider a single trader who has an endowment $\Phi_0 > 0$ shares of a security. This trader executes the shares Φ_0 over a time interval $[0, 1]$ considering the effect of MI with noise. We assume that the trader executes only at time $0, 1/n, \ldots, (n-1)/n$ for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

For $l = 0, \ldots, n$, we denote by S_l^n the price of the security at time l/n and $X_l^n = \log S_l^n$. Let $s_0 > 0$ be an initial price (i.e. $S_0^n = s_0$) and $X_0^n = \log s_0$. If the trader sells the amount ψ_l^n at the time l/n, the log-price changes to $X_l^n - g_k^n(\psi_l^n)$ and this execution (selling) gives him/her the amount of cash $\psi_l^n S_l^n \exp(-g_k^n(\psi_l^n))$ as proceeds. Here the random function

$$g_k^n(\psi, \omega) = c_k^n(\omega) g_n(\psi), \quad \psi \in [0, \Phi_0], \ \omega \in \Omega$$

means MI with noise, which is given by the product of positive random variable c_k^n and a deterministic function $g_n : [0, \Phi_0] \longrightarrow [0, \infty)$. The function g_n is assumed to be non-decreasing, continuously differentiable and satisfying $g_n(0) = 0$. Moreover we assume that $(c_k^n)_k$ is i.i.d., so the noise of MI is time-homogeneous. We remark that if c_k^n is a constant (i.e. $c_k^n \equiv c$ for some $c > 0$,) then this setting is the same as in [4].

After the trading at the time l/n, X_{l+1}^n and S_{l+1}^n are given by

$$X_{l+1}^n = Y\left(\frac{l+1}{n}; \frac{l}{n}, X_l^n - g_k^n(\psi_l^n)\right), \quad S_{l+1}^n = e^{X_{l+1}^n}, \tag{2.1}$$

where $Y(t; r, x)$ is the solution of the following stochastic differential equation (SDE) on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t^B)_t, P)$

$$\begin{cases} dY(t; r, x) = \sigma(Y(t; r, x))dB_t + b(Y(t; r, x))dt, t \geq r, \\ Y(r; r, x) = x. \end{cases}$$

Here $(\mathcal{F}_t^B)_t$ is the Brownian filtration and $b, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ are Borel functions. We assume that b and σ are bounded and Lipschitz continuous. Then for each $r \geq 0$ and $x \in \mathbb{R}$ there exists a unique solution.

At the end of the time interval $[0, 1]$, The trader has the amount of cash W_n^n and the amount of the security φ_n^n, where

$$W_{l+1}^n = W_l^n + \psi_l^n S_l^n e^{-g_k^n(\psi_l^n)}, \quad \varphi_{l+1}^n = \varphi_l^n - \psi_l^n \tag{2.2}$$

for $l = 0, \ldots, n - 1$ and $W_0^n = 0$, $\varphi_0^n = \Phi_0$. We say that an execution strategy $(\psi_l^n)_{l=0}^{n-1}$ is admissible if $(\psi_l^n)_l \in \mathcal{A}_n^n(\Phi_0)$ holds, where $\mathcal{A}_k^n(\varphi)$ is the set of strategies $(\psi_l^n)_{l=0}^{k-1}$ such that ψ_l^n is $\mathcal{F}_{l/n}$-measurable, $\psi_l^n \geq 0$ for each $l = 0, \ldots, k - 1$ and $\sum_{l=0}^{k-1} \psi_l^n \leq \varphi$ almost surely.

Then the investor's problem is to choose an admissible strategy to maximize the expected utility $E[u(W_n^n, \varphi_n^n, S_n^n)]$, where $u \in \mathcal{C}$ is his/her utility function and $\mathcal{C}$ is the set of non-decreasing, non-negative and continuous functions on $D = \mathbb{R} \times [0, \Phi_0] \times [0, \infty)$ such that

$$u(w, \varphi, s) \leq C_u(1 + |w|^{m_u} + s^{m_u}), \quad (w, \varphi, s) \in D \tag{2.3}$$

for some constants $C_u > 0$ and $m_u > 0$.

For $k = 1, \ldots, n$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define the (discrete-time) value function $V_k^n(w, \varphi, s; u)$ by

$$V_k^n(w, \varphi, s; u) = \sup_{(\psi_l^n)_{l=0}^{k-1} \in \mathcal{A}_k^n(\varphi)} E[u(W_k^n, \varphi_k^n, S_k^n)]$$

subject to (2.1) and (2.2) for $l = 0, \ldots, k - 1$ and $(W_0^n, \varphi_0^n, S_0^n) = (w, \varphi, s)$. (For $s = 0$, we set $S_l^n \equiv 0$.) And we denote $V_0^n(w, \varphi, s; u) = u(w, \varphi, s)$. Then our problem is the same as $V_n^n(0, \Phi_0, s_0; u)$. We consider the limit of the value function $V_k^n(w, \varphi, s; u)$ as $n \to \infty$.

Let $h : [0, \infty) \longrightarrow [0, \infty)$ be a non-decreasing continuous function. We introduce the following condition for $g_n(\psi)$.

[A] $\lim_{n \to \infty} \sup_{\psi \in [0, \Phi_0]} \left| \frac{d}{d\psi} g_n(\psi) - h(n\psi) \right| = 0$.

Moreover we assume the following conditions for $(c_k^n)_k$.

[B1] For any $n \in \mathbb{N}$ and $x \geq 0$ it holds that $\gamma_n > 0$ and

$$\frac{h(x/\gamma_n)}{n} \longrightarrow 0, \quad n \to \infty, \tag{2.4}$$

where $\gamma_n = \mathrm{essinf}\, c_k^n$.

[B2] Let μ_n is the distribution of $\dfrac{c_0^n + \ldots + c_{n-1}^n}{n}$. Then μ_n has a weak limit μ as $n \to \infty$.

[B3] There is a sequence of infinitely divisible distributions $(p_n)_n$ on $\mathbb{R}$ such that $\mu_n = \mu * p_n$ and / either

[B3-a] $\displaystyle\int_{\mathbb{R}} x^2 p_n(dx) = O(1/n^3)$ as $n \to \infty$

or

[B3-b] There is a sequence $(K_n)_n \subset (0, \infty)$ such that $K_n = O(1/n)$, $p_n((-\infty, -K_n)) = 0$

(or $p_n((K_n, \infty)) = 0$) and $\displaystyle\int_{\mathbb{R}} x p_n(dx) = O(1/n)$ as $n \to \infty$,

where O denotes order notation (Landau's symbol.)

Let us give some remarks for condition [B1]. Since c_k^n, $k = 0, 1, 2, \ldots$, are identically distributed, γ_n is independent of k. Moreover, if $h(\infty) < \infty$, then (2.4) is always fulfilled. if $h(\infty) = \infty$, we have the following example :

$$h(\zeta) = \alpha\zeta^p, \quad \gamma_n = \frac{1}{n^{1/p-\delta}} \quad (p, \delta > 0, \ \delta < 1/p).$$

Since μ is an infinitely divisible distribution, there is some Lévy process $(L_t)_{0 \le t \le 1}$ on a certain probability space such that L_1 is distributed by μ. The process $(L_t)_t$ is independent of any Brownian motion, because $(L_t)_t$ is obviously non-decreasing and non-negative. Without loss of generality, we may assume that $(L_t)_t$ and $(B_t)_t$ are defined on the same filtered space. Let ν is a Lévy measure of $(L_t)_t$. We assume the following moment condition for ν.

[C] $\displaystyle\int_{(0,\infty)} (z + z^2)\nu(dz) < \infty$.

Now we define the function which gives the limit of the discrete-time value functions. For $t \in [0, 1]$ and $\varphi \in [0, \Phi_0]$ we denote by $\mathcal{A}_t(\varphi)$ the set of $(\mathcal{F}_r)_{0 \le r \le t}$-adapted and càglàd process (i.e. left continuous and has a right limit at each point) $(\zeta_r)_{0 \le r \le t}$ such that $\zeta_r \ge 0$ for each $r \in [0, t]$, $\displaystyle\int_0^t \zeta_r dr \le \varphi$ almost surely and $\sup_{r,\omega} \zeta_r(\omega) < \infty$. For $t \in [0, 1]$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define $V_t(w, \varphi, s; u)$ by

$$V_t(w, \varphi, s; u) = \sup_{(\zeta_r)_r \in \mathcal{A}_t(\varphi)} \mathrm{E}[u(W_t, \varphi_t, S_t)]$$

subject to

$$dW_r = \zeta_r S_r dr, \quad d\varphi_r = -\zeta_r dr,$$
$$dS_r = \hat{\sigma}(S_r)dB_r + \hat{b}(S_r)dr - g(\zeta_r)S_r dL_r$$

and $(W_0, \varphi_0, S_0) = (w, \varphi, s)$, where $\hat{\sigma}(s) = s\sigma(\log s)$, $\hat{b}(s) = s\left\{b(\log s) + \frac{1}{2}\sigma(\log s)^2\right\}$ for $s > 0$ $(\hat{\sigma}(0) = \hat{b}(0) = 0)$ and $g(\zeta) = \displaystyle\int_0^\zeta h(\zeta')d\zeta'$. When $s > 0$, we obviously see that the process of the log-price of the security $X_r = \log S_r$ satisfies

$$dX_r = \sigma(X_r)dB_r + b(X_r)dr - g(\zeta_r)dL_r.$$

3 Main Results

In this section we present main results. First we give the convergence theorem for value functions.

Theorem 1 *For each* $(w, \varphi, s) \in D$, $t \in [0, 1]$ *and* $u \in \mathcal{C}$ *it holds that*

$$\lim_{n \to \infty} V_{[nt]}^n(w, \varphi, s; u) = V_t(w, \varphi, s; u),$$

where $[nt]$ *is the greatest integer less than / equal to* nt.

By this theorem, we see that the function $V_t(w, \varphi, s; u)$ corresponds to the continuous-time model of an optimal execution problem with random MI. This result is almost the same as in [4], but we notice that the term of MI is given as an increment $g(\zeta_r)dL_r$. Let $L_t = \gamma t + \int_0^t \int_{(0,\infty)} zN(dr, dz)$ be the Lévy decomposition of $(L_t)_t$. Then $g(\zeta_r)dL_r$ can be divided into the following two terms :

$$g(\zeta_r)dL_r = \gamma g(\zeta_r)dr + g(\zeta_r) \int_{(0,\infty)} zN(dr, dz).$$

The last term in the right-hand side refers to the effect of noise of MI. This means that noise of MI appears as a jump of Lévy process.

As for continuity of the continuous-time value function, we have the following.

Theorem 2 *Let* $u \in \mathcal{C}$.
(i) *If* $h(\infty) = \infty$, *then* $V_t(w, \varphi, s; u)$ *is continuous in* $(t, w, \varphi, s) \in [0, 1] \times D$.
(ii) *If* $h(\infty) < \infty$, *then* $V_t(w, \varphi, s; u)$ *is continuous in* $(t, w, \varphi, s) \in (0, 1] \times D$ *and* $V_t(w, \varphi, s; u)$ *converges to* $Ju(w, \varphi, s)$ *uniformly on any compact subset of* D *as* $t \downarrow 0$, *where* $Ju(w, \varphi, s)$ *is given as*

$$\begin{cases} \sup_{\psi \in [0, \varphi]} u\left(w + \frac{1 - e^{-\gamma h(\infty)\psi}}{\gamma h(\infty)}s, \varphi - \psi, se^{-\gamma h(\infty)\psi}\right) \\ \hspace{5cm} (\gamma h(\infty) > 0), \\ \sup_{\psi \in [0, \varphi]} u(w + \psi s, \varphi - \psi, s) \hspace{1cm} (\gamma h(\infty) = 0). \end{cases}$$

This is also quite similar to [4]. The continuity in t at the origin is according to the state of the function h at the infinity point. When $h(\infty) < \infty$, the value function is not always continuous at $t = 0$ and has the right limit $Ju(w, \varphi, s)$. $Ju(w, \varphi, s)$ implies the utility of the profit of the execution of a trader who sells a part of the shares of a security ψ by dividing infinitely in infinitely short time (enough to neglect the fluctuation of the price of a security) and makes the amount $\varphi - \psi$ remain.

We pay attention to the fact that the noise part $g(\zeta_r) \int_{(0,\infty)} zN(dr, dz)$ makes no change for the result. We should notice that if $\gamma = 0$ and $h(\infty) < \infty$, then the effect of MI disappears in $Ju(w, \varphi, s)$. This situation may occur even if $\mathrm{E}[c_k^n] \geq \varepsilon_0$ (or $\mathrm{E}[L_1] \geq \varepsilon_0$) for some $\varepsilon_0 > 0$.

4 Examples

In this section we show two examples of our model, both of which are generalization of the ones in [4].

Let $b(x) \equiv -\mu$ and $\sigma(x) \equiv \sigma$ for some constants $\mu, \sigma \geq 0$ and suppose $\tilde{\mu} = \mu - \sigma^2/2 > 0$. We assume that a trader has a risk-neutral utility function $u(w, \varphi, s) = w$.

For the noise part of MI, we consider the Gamma distribution.

$$P(c_k^n - \gamma \in dx) = \text{Gamma}(\alpha_1/n, n\beta_1)(dx)$$
$$= \frac{1}{\Gamma(\alpha_1/n)(n\beta_1)^{\alpha_1/n}} x^{\alpha_1/n-1} e^{-x/(n\beta_1)} 1_{(0,\infty)}(x) dx,$$

where $\Gamma(x)$ is the Gamma function. Here $\alpha_1, \beta_1, \gamma > 0$ are constants.

For the deterministic part of MI, we consider two patterns : (log-)linear case and (log-)quadratic case i.e. $g_n(\psi) = \alpha_0\psi$ or $g_n(\psi) = n\alpha\psi^2$ for $\alpha > 0$. In each cases the assumption [A] is satisfied, and so are [B1] - [B5] and [C].

4.1 Log-Linear Impact & Gamma Distribution

In this case we have the following theorem.

Theorem 3 *It holds that*

$$V_t(w, \varphi, s; u) = w + \frac{1 - e^{-\gamma\alpha\varphi}}{\gamma\alpha} s \tag{4.1}$$

for each $t \in (0, 1]$ and $(w, \varphi, s) \in D$.

An implication of this result is the same as in [4] : the right-hand side of (4.1) is equal to $Ju(w, \varphi, s)$ and converges to $w + \varphi s$ as $\alpha \downarrow 0$ or $\gamma \downarrow 0$, which is the profit gained by choosing the execution strategy of so-called block liquidation such that a trader sells all shares φ at $t = 0$ when there is no MI. So the optimal strategy in this case is to execute all shares dividing infinitely in infinitely short time at $t = 0$ (we call such a strategy an almost block liquidation at the initial time.) We see that the effect of (pure) noise part of MI does not influence the result. That is, the value of $V_t(w, \varphi, s; u)$ and the corresponding optimal strategy do not vary with respect to α_1 and β_1.

4.2 Log-Quadratic Impact & Gamma Distribution

In [4], we have the analytical solution of the problem partially : when φ is small enough or large enough, we have the explicit form of optimal strategies. But noise of MI makes the problem complicated and hard to solve, so deriving the explicit solutions is more difficult. Thus we rely on numerical experiments. Thanks to the assumption that a trader is risk-neutral, we see that an optimal strategy is determinsitic. We assume the following additional condition.
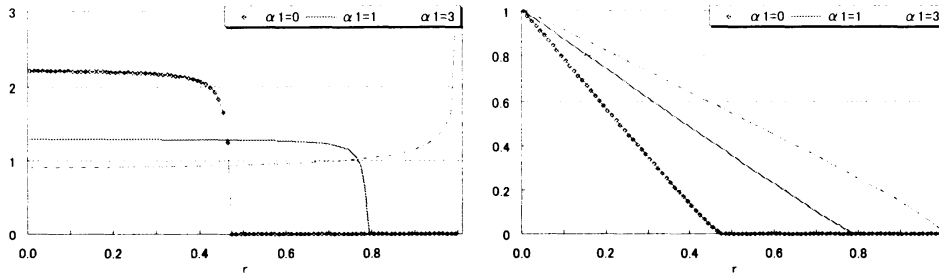
[D] $\gamma \geq \dfrac{\alpha_1\beta_1}{8}$.

Figure 1: The result with $\varphi = 1$. The top graph : An optimal strategy ζ_r, The buttom graph : an amount of the security holdings φ_r. The horizontal axis means time r.

Then we can replace our optimization problem with the deterministic control problem

$$f(t, \varphi) = \sup_{(\zeta_r)_r} \int_0^t \exp\left(-\int_0^r q(\zeta_v)dv\right) \zeta_r dr,$$

$$q(\zeta) = \tilde{\mu} + \gamma\alpha_0\zeta^2 + \alpha_1 \log(\alpha_0\beta_1\zeta^2 + 1)$$

for deterministic process $(\zeta_r)_r$. Indeed we have the following theorem.

Theorem 4 $V_t(w, \varphi, s; u) = w + sf(t, \varphi)$ *under* $[D]$.

So we can solve this example numerically by considering the deterministic control problem in the discrete-time model for large enough n. We set each parameters as follows : $\alpha_0 = 0.01, t = 1, \tilde{\mu} = 0.05, \beta_1 = 2, \gamma = 1, w = 0, s = 1$ and $n = 100$. For value of α_1 and φ, we examine the following patterns : $\alpha_1 = 0, 1, 3$ and $\varphi = 1, 10, 100$. This is because forms of optimal strategies vary according to these values.

4.2.1 The case of $\varphi = 1$

If a trader has a fiew amount of the security holdings, the result is given as in Figure 1. In fact, if MI function has no noise, that is $\alpha_1 = 0$, then the optimal strategy is "to sell up all the amount at the same speed." (We remark that the roundness at the corner in the top graph of Figure 1 expresses the discritization error and this is no essential.) The same tendency is found in the case of $\alpha_1 = 10$, but in this case the execution time is longer than the case of $\alpha_1 = 0$. When we consider the case of $\alpha_1 = 3$, the situation undergoes a complete change. In this case the optimal strategy is to increase the execution speed as the time horizon comes near.

4.2.2 The case of $\varphi = 10$

When the amount of the security holdings is little larger than the case of Section 4.2.1, then we have the result of Figure 2. In this case a trader's optimal strategy is to increase the execution speed as the trading time comes to end like the case of Section 4.2.1 with $\alpha_1 = 3$. We see that the larger the value of α_1, the higher the speed of execution near the time horizon. We add that a trader cannot finish the liquidation when $\alpha_1 = 3$.
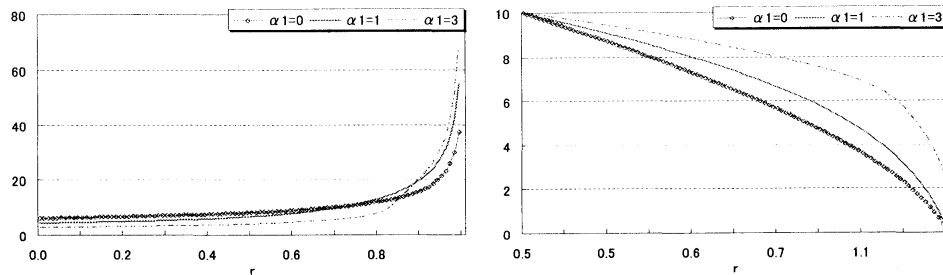
Figure 2: The result with $\varphi = 10$. The top graph : An optimal strategy ζ_r, The buttom graph : an amount of the security holdings φ_r. The horizontal axis means time r.
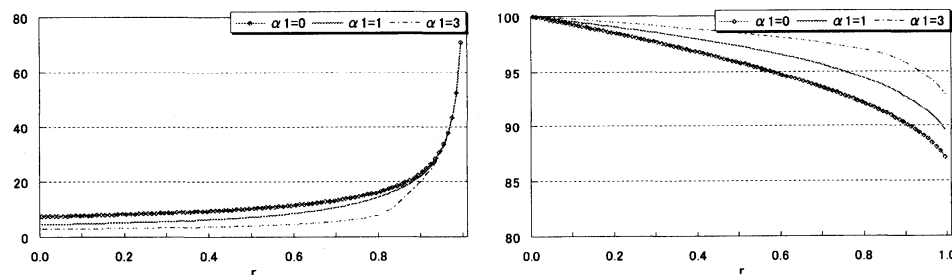

Figure 3: The result with $\varphi = 100$. The top graph : An optimal strategy ζ_r, The buttom graph : an amount of the security holdings φ_r. The horizontal axis means time r.

4.2.3 The case of $\varphi = 100$

When the amoun ot the security holdings is too large, a trader cannot finish the liquidation regardless of the value of α_1, as Figure 3 implies. Remaining is similar to the case of Section 4.2.2 with $\alpha_1 = 3$. The rest shares of the security at the time horizon is larger as noise of MI becomes larger.

5 Concluding Remarks

In this paper we generalize the framework of [4] and consider an optimal execution problem with random MI. We define the (one-shot) MI function as a product of i.i.d. positive random variable and a deterministic function in discrete-time model. We derive the continuous-time model of an optimization problem as the limit of discrete-time models, and find out that the noise of MI in continuous-time model is described as a jump of a Lévy process.

It is interesting that our main results discussed in Section 3 are almost the same as in [4] and our numerical experiments suggest that the bigger the noise of MI, the longer we need liquidation time.

For getting deeper insight, we should investigate the structure of the MI function punctually. In Theorem 2 (ii) and Theorem 3, the important parameter is γ, which is the infimum of L_1

and is smaller than (or equal to) $E[L_1]$. We can give the interpretation that the (almost) block liquidation disappears the effect of positive jumps of $(L_t)_t$. But we have another decomposition of L_t such that

$$L_t = \tilde{\gamma}t + \int_0^t \int_{(0,\infty)} z\tilde{N}(dr, dz),$$

where $\tilde{\gamma} = \gamma + \int_{(0,\infty)} z\nu(dz)$ and $\tilde{N}(dr, dz) = N(dr, dz) - \int_{(0,\infty)} z\nu(dz)dr$. This representation is essential from the view of martingale theory and the value of $\tilde{\gamma}$ is also important. Considering the meaning of this value may present significant findings.

In our settings, MI function is stationary in time, but in the real market, characteristics of MI is found to change according to the time zone. So it is meaningful to study the case where MI function is inhomogeneous in time. It is one of our further developments.

A characterization of the continuous-time value function as a viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation like [4], [5] is also a remaining task and significant in mathematical finance. It may be difficult to consider viscosity solutions corresponding to a stochastic control problem with jumps, but we surmise that it is possible if we give a suitable technical assumptions.

[6] and [7] treat a case where a security price has mean-reverting property and they list properties which cause gradual liquidation of a trader. In their examples, we can also regard the parameter σ as volatility of strength of mean-reverting. By considering more generalized model, we may induce randomness of transient MI into our model. On the other hand, this paper treats randomness of one-shot MI itself. By mixing them, we may be able to construct finer model which is subject of our future research.

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