数理解析研究所講究録
第 1676 巻 2010 年 66-70

Minimum Implicational Bases of Affine Convex Geometries

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Abstract

Originally, a rooted circuit is defined for a convex geometry. However the definition itself is valid in closure systems generally, and works very well. We shall show that the system of rooted circuits of a closure system gives rise to a natural implicational base. In case of an affine convex geometry, it gives a canonical minimum base.

1 Closure Systems and Implicational Systems

A closure system affords an underlying foundation for a number of systems, such as implicational systems (or functional dependencies in relational database theory) [1], knowledge systems [2], formal concept analysis [3], logic [5], and so on. Here we shall show that the concept of rooted circuits plays a good role in determining the minimum implicational bases of implicational systems.

Let E be a finite set, and $\mathcal{K} \subseteq 2^E$ a family of subsets of E. $(\mathcal{K}, E)$ is a *closure system* if it is closed under intersection and $E \in \mathcal{K}$.

A map $\tau : 2^E \to 2^E$ is a closure operator if the followings hold.

$$(1)\ A \subseteq \tau(A), \qquad (2)\ A \subseteq B \Rightarrow \tau(A) \subseteq \tau(B), \qquad (3)\ \tau(\tau(A)) = \tau(A) \qquad (A, B \subseteq E).$$

A closure system gives a closure operator $\tau(A) = \bigcap\{X | A \subseteq X, X \in \mathcal{K}\}$, and conversely $\mathcal{K} = \{X \subseteq E : \tau(X) = X\}$ holds. So there is a one-to-one correspondence between them. $\tau(X)$ is often denoted $\overline{X}$.

Note that $A \subset B$ implies a proper inclusion and we write $X \cup e$ to denote $X \cup \{e\}$.

An ordered pair $(A, B) \in 2^E \times 2^E$, written as $A \to B$, is an *implication* on E with *premise* X and *conclusion* Y. Let S be a family of implications on E, which is called an *implicational system*. A subset $A \subseteq E$ *satisfies* $X \to Y$ if $X \subseteq A$ implies $Y \subseteq A$, i.e. either $X \not\subseteq A$ or $Y \subseteq A$ holds.

Lemma 1.1 If $A, B \subseteq E$ satisfies $X \to Y$, then $A \cap B$ satisfies $X \to Y$.

Hence $\mathcal{K}_S = \{A \subseteq E : A$ satisfies all the implications of $S\}$ is a closure system, which is automatically an lattice called an *implicational lattice*, and S is said to be an *implicational base* of a closure system $\mathcal{K}_S$. If a pair of implicational bases determine the same closure system, they are called *equivalent*. Fig. 1 shows an example of an implicational system and the implicational lattice

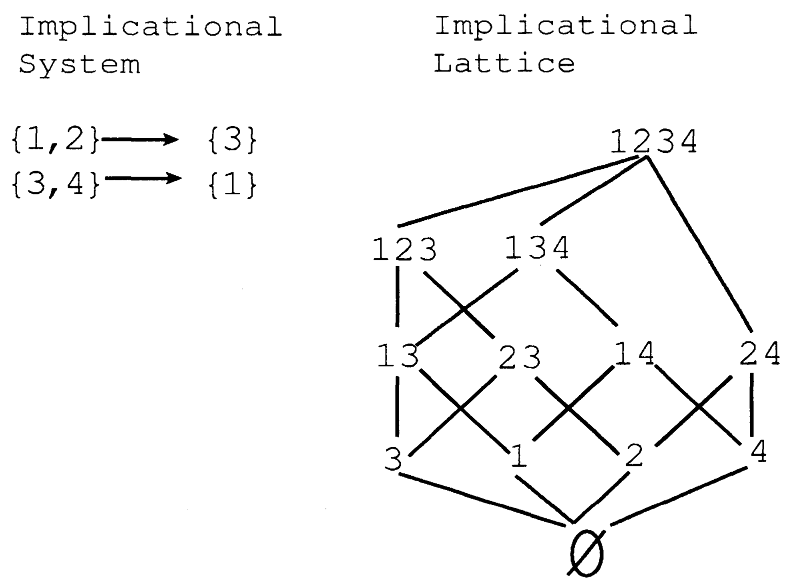

Figure 1: An implicational system and the implicational lattice

We define an equivalence relation $A\theta B \ \ (A, B \subseteq E)$ by $\tau(A) = \tau(B)$, where $\tau(A)$ is the largestelement in the equivalence class $[A]$.

For a closure system $(\mathcal{K}, E)$, a pair (X, e) with $e \in E$ and $X \subseteq E \backslash e$, is called a *rooted circuit* if $e \in \tau(X)$ and X is minimal with respect to this property. X and e are called its *stem* and its *root*, respectively. We will later describe that the collection of rooted circuits automatically provides an implicational base of any closure system. We define an *extreme function* ex by $ex(X) = \{x \in X \ : \ x \notin \tau(X - x)\}$. A subset $A \subseteq E$ is *independent* if $ex(A) = A$ and *dependent* otherwise. A minimal dependent set is said to be a *circuit.*

An *(abstract) convex geometry* is a closure system $(\mathcal{K}, E)$ if for any $X \in \mathcal{K}$ with $X \neq E$, there exists an element $e \in E \setminus X$ such that $X \cup e \in \mathcal{K}$. Equivalently, a closure system $(\mathcal{K}, E)$ is a convex geometry if the corresponding closure operator τ meets the anti-exchange property.

In a convex geometry, a rooted circuit was so far defined as follows [4]. Each circuit C of a convex geometry has a unique non-extreme element, i.e. $C \setminus ex(C) = \{e\}$ for some $e \in C$, and $(C \setminus e, e)$ is a *rooted circuit.* (Other authors usually designate a rooted circuit by(C, e) rather than the form $(C \setminus e, e)$ [4] et al.) It is shown in [6] that our general definition of rooted circuits for closure systems coincides with the so far known definition of rooted circuits in case of convex geometries.

An implicational system S is *nonredundant* if for any $X \to Y \in S$, $S' = S \setminus \{X \to Y\}$ is not equivalent to S. S is *minimum* if $|S| \leqslant |S'|$ for any implicational system S' equivalent to S.

S is *optimal* if $s(S) \leqslant s(S')$ for any S' which is equivalent to S where $s(S)$ implies $\sum_{X \to Y \in S}(|X| + |Y|)$.

Now we shall define quasiclosed sets and pseudoclosed sets.

$$A^\circ = A \cup \bigcup\{\tau(X) \mid X \subset A,\ \tau(X) \subset \tau(A)\} \tag{1}$$

$$A^\bullet = A^\circ \cup A^{\circ\circ} \cup A^{\circ\circ\circ} \cdots \tag{2}$$

Then $A \mapsto A^\bullet$ is a closure operator on E, and $A^\bullet \subseteq \tau(A)$.

$W \subseteq E$ is *quasiclosed* (or *q-closed*) if $W = W^\bullet$ and $W \neq \tau(W)$ [7]. A quasiclosed set W is *pseudoclosed* (or *p-closed*) if W is a minimal q-closed set in the θ-class $[W]$.

A closed set $T \in \mathcal{K}$ is *essential* if $[T]$ contains a q-closed set, i.e. there exists a nonredundant generating set F of $[T]$ with $F^\bullet \subseteq T$. Let $Es(\mathcal{K})$ denote the collection of the essential sets of $\mathcal{K}$.

Theorem 1.1 (Wild [7]) Let $(\mathcal{K}, E)$ be a closure system.

(1) Let S be a nonredundant base of $\mathcal{K}$. Then $Es(\mathcal{K}) = \{\overline{X} | X \to Y \in S\}$. If S is a nonredundant implicational base of $\mathcal{K}$ and all the implications of S are of the form $X \to \overline{X}$, S is minimum.

(2) $S_\mathcal{K} = \{P \to (\overline{P} - P) | P \text{ is p-closed}\} = \cup \{P \to (T - P) \mid T \in Es(\mathcal{K}), P \in [T], \text{ and } P \text{ is p-closed }\}$ is a canonical minimum base of $\mathcal{K}$. That is, let S' be an arbitrary base of $\mathcal{K}$. Then for each implication $P \to (\overline{P} - P)$ in $S_\mathcal{K}$, S' contains an implication $X_P \to Y_P$ such that $X_P \subseteq P$ and $\overline{X_P} = \overline{P}$.

(3) Each optimal base of $\mathcal{K}$ is minimum, and the cardinality of X_P is uniquely determined as $c_P = \min\{|X| \mid X \subseteq P,\ \overline{X_P} = \overline{P}\} = \min\{|X| \mid X \subseteq P,\ X^\bullet = P\}$

2 Rooted Circuits and Implicational Bases

Let $\mathcal{K}$ be a closure system on E. Let $\mathbb{C}_\mathcal{K}$ be the collection of all the rooted circuits of $\mathcal{K}$, and $S_\mathcal{K} = \{X \to \{r\} \mid (X, r) \in \mathbb{C}_\mathcal{K}\}$ is an implicational system given by the rooted circuits.

Lemma 2.1 $S_\mathcal{K}$ is an implicational base of $\mathcal{K}$.

(Proof) Suppose A is A satisfies $X \to \{r\}$ for every $(X, r) \in \mathbb{C}_\mathcal{K}$, i.e. $X \not\subseteq A$ or $r \in A$, and $A \notin \mathcal{K}$. Then there exists an element $e \in \overline{A} \setminus A$. Hence there exists $X \subseteq A$ with $e \in \overline{X}$. Take a minimal X. Then (X, e) is a rooted circuit in $\mathbb{C}_\mathcal{K}$. Then A does not satisfy $X \to \{e\}$, a contradiction.

Suppose contrarily that $A \in \mathcal{K}$ and A is not $S_\mathcal{K}$-closed. Then there is a rooted circuit $(X, r) \in \mathbb{C}_\mathcal{K}$ such that $X \subseteq A$ and $r \notin A$. By definition, $r \in \overline{X} \subseteq \overline{A} = A$, a contradiction. □

Let $T_\mathcal{K}$ denote the collection of all the stems, i.e. $T_\mathcal{K} = \{X \subseteq E \mid \exists e \in E :\ (X, e) \in \mathbb{C}_\mathcal{K}\}$.

For each stem $X \in \mathcal{T}_{\mathcal{K}}$, let us define $R_X = \cup\,\{e \| \, (X, e) \in \mathbb{C}_{\mathcal{K}}\}$. Obviously $S_T = \{X \to R_X \mid X \in \mathcal{T}_{\mathcal{K}}\}$ is an implicational base equivalent to $S_{\mathcal{K}}$, and $|S_T| \leqslant |S_{\mathcal{K}}|$. We shall call S_T a *circuit-induced base* of $\mathcal{K}$, which seems to be one of the standard forms of implicational bases.

Let P be a finite poset. A subset A of P is said to be closed if for any $a, b \in A$ and c, $a \leqslant c \leqslant b$, then $c \in A$. The collection of all the closed sets of P forms a convex geometry, called a *poset bishelling convex geometry*.

Let E be a finite set in $\mathbb{R}^n$. $\mathcal{K}_A = \{X \subseteq E \ : \ conv.hull(X) \cap E = X\}$ is a convex geometry, called an *affine convex geometry*.

Proposition 2.1 In a poset double shelling convex geometry, the circuit-induced base $S_T = \{X \to R_X \mid X \in \mathcal{T}_{\mathcal{K}}\}$ is an optimal implicational base.

(Proof) In a poset double shelling convex geometry, a p-closed set takes the form of $\{a, b\}$ with $a < c < b$ for some $a, b, c \in E$, while a rooted circuit is $(\{x, y\}, z)$ such that $x < z < y$ and $x, y, z \in E$. Namely, a p-closed set is equal to the stem of a rooted circuit. Hence the circuit-induced base is a canonical minimum base by Theorem 1.1 (1). Furthermore, this base fulfills the condition (3) of Theorem 1.1 and the conclusions of the implications are all singletons, which implies this base is optimal. □

3 Affine Convex Geometries

Let E be a finite set in $\mathbb{R}^n$. $\tau(A) = conv.hull(A) \cap E(A \subseteq E)$ is naturally an anti-exchange closure operator, and hence defines a closure system $\mathcal{K}$, is a convex geometry, called an *affine convex geometry.*

For a rooted circuit (X, e) of an affine convex geometry $(\mathcal{K}, E)$, a stem X is said to be *pure* if X is equal to the vertex set of the convex hull of X.

Lemma 3.1 In an affine convex geometry $(\mathcal{K}, E)$, a set $S \subseteq E$ is p-closed if an only if S is a pure stem.

(Proof) Suppose $E \subseteq \mathbb{R}^n$ is a nonempty finite set. Let $(\mathcal{K}, E)$ and τ the affine convex geometry on E and the associated closure operator, respectively.

First, we suppose S is a p-closed set, and show that S is necessarily a pure stem. Let P_S be the convex hull of S which being a d-dimensional polytope, and T be the set of vertices of P_S. If P_S has a simplicial decomposition by T into a multiple number of d-simplexes $S_1, \ldots, S_k (k \geqslant 2)$, then $S_i \subset S$, $\tau(S_i) \subset S$ for each S_i, and $S = S^\circ = S^\bullet = \bigcup_{i=1}^{k} \tau(S_i) = \tau(S)$, which contradicts that S is q-closed. Hence P_S is a d-simplex. Similarly, if P_S contains an element a in S as a proper interior point of P_S, then $S^\circ = S^\bullet = \tau(S)$, a contradiction. Let $F_1, \ldots, F_d$ be the facets of P_S. If P_S does not contain any element in $E \setminus S$, then $\cup_{i=1,\ldots,d} F_i = S^\circ = \tau(S)$, which contradicts the assumption. Hence P_S is a simplex, and contains at least one proper interior point which is not in S.

Let us denote the set of vertices of P_S by $T = \{v_1, \ldots, v_{d+1}\} \subseteq S$. Suppose contrarily that the boundary $\partial(P_S)$ of P_S contains a non-vertex elements $w_1, \ldots, w_l \notin T$.
(i) If $w_q \in S$ for some q, then we can assume without loss of generality that a facet $F = \{v_1, \ldots, v_d\}$ contains w_q. Then Let $V_j^F = (F \setminus v_j) \cup \{v_{d+1}, w_q\}$ for $j = 1, \ldots, d$. Then for $V_j^F \subset S$ for $j = 1, \ldots, d$, we have $S^\circ = S^\bullet = \bigcup_{j=1}^d \tau(V_j^F) = \tau(S)$, which contradicts our assumption.
(ii) We suppose $w_j \notin S$ for every $j = 1, \ldots, l$. Then it holds that $S \neq S^\bullet$, which contracting our assumption.

Hence S must be the set of vertices of a simplex and there exisits no element on the boundary $\partial(P_S)$ except the vertices. Furthermore, P_S contains at least one point not in S, say $e \in E \setminus S$. Then it is clear that (S, e) is a rooted circuit and S is a pure stem.

Conversely, it is easy to see that a pure stem is necessarily p-closed. □

Theorem 3.1 In an affine convex geometry $(\mathcal{K}, E)$, $S_{pstem} = \{X \to R_X \mid X \in T_\mathcal{K}$ is a pure stem.$\}$ is a canonical minimum base.

(Proof) It immediately follows from Lemma 3.1 and Theorem 1.1 (2). □

Wild [7] presented $\Sigma_\mathcal{K} = \{ex(X) \to ex(X - ex(X)^\bullet) \ : \ X \in \mathcal{K}\}$ as a natural implicational base for a convex geometry. In contrast $S_{pstem} = \{X \to R_X \mid X \in T_\mathcal{K}$ is a pure stem.$\}$ fulfills the necessary conditions of optimality in Theorem 1.1 (3). Actually, suppose P to be a p-closed set. Then Lemma 3.1 implies $X_P = P$, and it is obvious that $|X_P| = \min\{ |X| \mid \overline{X_P} = \overline{P} \} = \min\{ |X| \mid X \subseteq P,\ X^\bullet = P \}$.

References

[1] N. Caspard and B. Monjardet, "The lattice of closure systems, closure operators, and implicational systems on a finite set: A survey," Disc. Appl. Math. 127 (2003) 241–269.

[2] J-P. Doignon and J-C. Falmage, Knowledge Spaces, Springer, 1999.

[3] B. Ganter and R. Wille, Formal Concept Analysis, Springer, 1999.

[4] B. Korte, L. Lovász and R. Schrader, Greedoids, Springer, 1991.

[5] N. M. Martin and S. Pollard, Closure Spaces and Logic, Kluwer, 1996.

[6] M. Nakamura, "On the NBC-complex and β-invariants of abstract convex geometries," Disc. Apppl. Math. 157 (2009) 1799–1805.

[7] M. Wild, "A theory of finite closure spaces based on implications," Advances in Math. 108 (1994) 118–139.