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The tensor structure of the original Navier-Stokes equations

SHIGERU MASUDA

Graduate school of Tokyo Metropolitan University, doctoral cource in mathematics
E-mail: `masuda-sigeru@ed.tmu.ac.jp`

Abstract

The two-constants theory introduced first by Laplace in 1805 is currently accepted theory describing isotropic, linear elasticity. The original, macroscopically-descriptive Navier-Stokes equations [MDNS equations] were derived in the course of the development the two-constants theory. From the viewpoint of MDNS equations, we trace the evolution of the equations and the notion of tensor following in historical order the various contributions of Navier, Cauchy, Poisson, Saint-Venant and Stokes[1], and note the concordance between each.

Keywords : the microscopically descriptive equation, the Navier-Stokes equations, mathematical history.

1 Preliminary Remarks

In this report, we use the following definition of the stress tensor, due to I. Imai[7, p.178]: we call a P of 3×3 matrix such as P a stress tensor that returns a new vector $\mathbf{P}_n$ when multiplied from the right by the column vector of directional cosines :

$$\begin{bmatrix} P_{nx} \\ P_{ny} \\ P_{nz} \end{bmatrix} = \begin{bmatrix} p_{xx} & p_{yx} & p_{zx} \\ p_{xy} & p_{yy} & p_{zy} \\ p_{xz} & p_{yz} & p_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_n = \mathrm{P}\cdot\mathbf{n}$$

Moreover, if $p_{ij} = p_{ji}$ for all $i, j = x, y, z$ then this tensor is said to be symmetric. If we suppose for example t_{ij} is the (i, j) element of a matrix, and $t_{ij} = -t_{ji}$ then anti-symmetric or skew-symmetric. Troughout the paper, we display for brevity a tensor by specifying its components, such as δ_{ij} of the well-known Kronecker δ. Furthermore, we write $v_{k,k} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \cdots$ where we have the Einstein convention[2]. Simplifications occur as, for example, in Navier's elasticity of (1-1) in Table 4 where the tensor can be expressed as follows:

$$-\varepsilon \begin{bmatrix} \left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) & \left(\frac{du}{dy} + \frac{dv}{dx}\right) & \left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ \left(\frac{du}{dy} + \frac{dv}{dx}\right) & \left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right) & \left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ \left(\frac{dw}{dx} + \frac{du}{dz}\right) & \left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right) \end{bmatrix} = -\varepsilon \begin{bmatrix} \epsilon + 2\frac{du}{dx} & \frac{du}{dy} + \frac{dv}{dx} & \frac{dw}{dx} + \frac{du}{dz} \\ \frac{du}{dy} + \frac{dv}{dx} & \epsilon + 2\frac{dv}{dy} & \frac{dv}{dz} + \frac{dw}{dy} \\ \frac{dw}{dx} + \frac{du}{dz} & \frac{dv}{dz} + \frac{dw}{dy} & \epsilon + 2\frac{dw}{dz} \end{bmatrix}, \quad \text{where} \quad \epsilon = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \qquad (1)$$

Expressions in Poisson's elasticity (3-1) in Table 4 are also of similar style.

Moreover, we can easily express Navier's stress tensor t_{ij} of elasticity in the form: $t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$. Stokes' fluid theory (20) or (5) in Table 4 affords a second illustration: $t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, or the equivalent expression $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$.[3] In what follows, "tensor"means the stress tensor as defined by I. Imai. [4] When referring to a "fluid", an "elastic fluid" is implied.

2 Introduction

We have studied the original MDNS equations as the progenitors[5], Navier, Cauchy, Poisson, Saint-Venant and Stokes, and endeavor to acertain their aims and conceptual thoughts in formulations these new equations. "The two-constants theory" was introduced first introduced in 1805 by Laplace[6] in regard to capillary action with constants denoted by H and K (cf. Table 2, 3). Thereafter, various pairs of constants have been proposed by their originators in formulating MDNS equations or equations describing equilibrium or capillary situations. It is commonly accepted that this theory describes isotropic, linear elasticity.[7] We argue that Poisson had already pointed out the special aspect deduced by Laplace when, in 1831, he states, 'elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure' [18, p.4]. Poisson was, we think, one of the persons who were aware of this issue.

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[1] Navier(1785-1836), Cauchy(1789-1857), Poisson(1781-1840), Saint-Venant(1797-1886), Stokes(1819-1903).

[2] Remark: in general, $v_{k,k} \neq v_{i,j}$, because the summation convention is in force when there is a repetition of indices.

[3] c.f. Schlichting [20], in our footnote(19).

[4] Numbers on the Left-hand-side of equations refer to those given by the author in the original paper while numbers on the right-hand-side correspond to our indexing. The subscript to the original indexing, for example N^e/N^f, refer to author and type of theory, such as "elastic/fluid by Navier". For equations indexed by section, the citation is then in the format "section no.-no. by author".

[5] The order followed is by date of proposal or publication.

[6] Of capillary action, Laplace[8, V.4, Supplement p.2] achnowledges Clailaut[3, p.22], and Clailaut cites Maupertuis[10].

[7] Darrigol [4, p.121].

3 A universal method for the two-constants theory

Now, we would like to propose the uniformal methods to describe the kinetic equations for isotropic, linear elasticity such as :

- The partial differential equations of the elastic solid or elastic fluid are expressed by using one or the pair of C_1 and C_2 such that :
in the elastic solid : $\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}$,
In the elastic fluid : $\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f}$,
where $T_1,\ T_2, \cdots$ are tensors or terms consisting our equations, where we suppose the tensor as the first kind. For example, the MDNS equationsions corresponding to incompressible fluid is composed of the kinetic equation along with the continuity equation and are conventionally written, in modern vector notation, as follows :

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{2}$$

- C_1 and C_2 are the two coefficients of the two-constants theory, for example, ε and E by Navier, or R and G by Cauchy, k and K by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, or μ and $\frac{\mu}{3}$ by Stokes. Moreover C_1 and C_2 can be expressed in the following form:

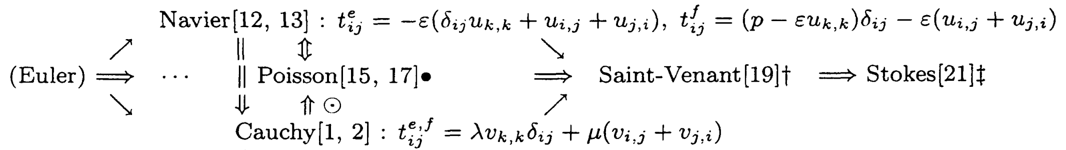
$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1, \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

- The two coefficients are expressible in terms of the operator $\mathcal{L}$ ($\sum_0^\infty$ or $\int_0^\infty$) by personal principles or methods, where r_1 and r_2 are the radial functions related to the radius of the active sphere of the molecules.
- g_1 and g_2 are the certain functions which depend on r and are described with attraction &/or repulsion.
- S_1 and S_2 are the two expressions which describe the surface of active unit-sphere at the center of a molecule by the double integral (or single sum in case of Poisson's fluid).
- g_3 and g_4 are certain compound trigonometric functions to calculate the momentum in the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients from the invariant tensor. Except for Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are calculated from the total momentum of the active sphere of the molecules in computing only by integral, and which are independent on personal manner. In Poisson's case, after multiplying by $\frac{1}{4\pi}$, we get the same as above.
- The ratio of the two coefficients including Poisson's case is always same as : $\frac{C_3}{C_4} = \frac{1}{5}$.

4 A genealogy and convergence of stress tensor

We show in the figure 1, a genealogy tracing in paticular the form of the tensor t_{ij} appearing in the Navier-Stokes equations. In Table 4, we differentiate the tensors associated with elastic solids or elastic fluids. From this genealogy, it could be asserted that Cauchy[1, 2] was the inventor or the first user of tensors, a view supported by the admission of Poisson[17] that he received the idea of symmetric tensor from Cauchy. Moreover, the idea of Saint-Venant reappears in the work of Stokes. Here, we denotethe two routes as NCP and PSS, both of which are portrayed in our figure, and by which, we can explain the genealogy of tensor as it applies to the MDNS equations. cf. Table 4.

(fig. 1) A genealogy of stress tensors in the prototypical Navier-Stokes equations



• Poisson : $t^e_{ij} = -\frac{a^2}{3}(\delta_{ij} u_{k,k} + u_{i,j} + u_{j,i}),\ t^f_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
† Saint-Venant : $t^f_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3} v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}),\ \frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) = -p$
‡ Stokes : $t^f_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,
$\odot$ Poisson says his reducing of tensor elements to 6 from 9 is due to Cauchy. (cf.§5.2).

5 Deductions of two constants and tensor

Recently Darrigol [4, p.121] has concluded: 'it is called that the two-constants theory is the one now accepted for isotropic, linear elasticity,' but Poisson [18, p.4] has stated already in 1831: 'elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure.' Moreover, we believe that the first proposer of "two-constants" theory was Laplace [9] in Table 3.

Table 1: C_1, C_2, C_3, C_4 : the constant of definitions and computing of total momentum of molecular actions by Navier, Cauchy, Poisson, Saint-Venant & Stokes

no	name/problem	elastic solid	elastic fluid	remark
1	Navier elasticity:[12] fluid:[13]	(Navier[12] only :) $C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi \;\; g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = \frac{2\pi}{15}$	(Navier[13] only :) $C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3} \int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \;\; g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \;\; g_4 \Rightarrow \frac{2\pi}{3}$	$\alpha = \rho\cos\psi\cos\varphi,$ $\beta = \rho\cos\psi\sin\varphi,$ $\gamma = \rho\sin\psi$
2	Cauchy elastic and fluid[2]	(Cauchy[2] :) $C_1 = R = \frac{2\pi\Delta}{15} \int_0^\infty r^3 f(r) dr$ $= \pm\frac{2\pi\Delta}{15} \int_0^\infty [r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)] dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3} \int_0^\infty r^3 \mathrm{f}(r) dr$ $C_3 = \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2\alpha\cos^2\beta dp,$ $= \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2 p \sin^2 p \sin p dp = \frac{2\pi}{15},$ $C_4 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi \cos^2\alpha \sin p dq dp$ $= \pi\int_0^\pi \cos^2 p \sin \; p dp = \frac{2\pi}{3},$	(Cauchy[2] :) samely as in elastic solid	$\cos\alpha = \cos p,$ $\cos\beta = \sin p\cos q,$ $\cos\gamma = \sin p\sin q$ $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
3	Poisson elasticity:[15, 17] fluid:[17]	(Poisson[15] only :) $C_1 = k \equiv \frac{2\pi}{15}\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3}\sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta\sin\beta d\beta \;\; g_3 \Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta\sin\beta d\beta \;\; g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot,\cdot\}$	(Poisson[17] : both elastic and fluid) $C_1 = -k \equiv -\frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15}\sum \frac{r^3}{4\pi\varepsilon^3}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3}\sum rfr$ $= -\frac{2\pi}{3}\sum \frac{r}{4\pi\varepsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3\text{-}2)_{Pf} \quad N = \frac{1}{6\varepsilon^3}\sum rfr \Rightarrow \frac{1}{6}$	In Poisson[17], he treates as the same as both elastic and fluid. $x_1 = r\cos\beta\cos\gamma,$ $y_1 = r\sin\beta\sin\gamma,$ $\zeta = -r\cos\beta$
4	Saint-Venant[19]		$C_1 = \varepsilon, \quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes[21]	$C_1 = A, \quad C_2 = B$	$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$	

Table 2: The two constants in the kinetic equations

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier [12]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier [13]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy [2]	system of particles in elastic and fluid	R		$\frac{2\pi}{15}\Delta$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}\Delta$	$\int_0^\infty dr$		r^3		$\pm\mathrm{f}(r)$	$f(r) \neq \mathrm{f}(r)$, $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
4	Poisson [15]	elastic solid	k		$\frac{2\pi}{15}$		$\sum\frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum\frac{1}{\alpha^5}$		r^3		fr	
5	Poisson [17]	elastic solid and fluid	k		$\frac{1}{30}$		$\sum\frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum\frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Saint-Venant [19]	fluid	ε	$\frac{\varepsilon}{3}$								
7	Stokes [21]	fluid	μ	$\frac{\mu}{3}$								
8	Stokes [21]	elastic solid	A	B								$A = 5B$

Table 3: The two constants in equations describing equilibrium or capillary situations

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Laplace [8, V.4, Supplement p.9] [9, V.4, p.700]	capillary action	H		2π		$\int_0^\infty dz$	z		$\Psi(z)$		z : distance
				K		2π	$\int_0^\infty dz$				$\Psi(z)$	cf.Gauss
2	Poisson [18]	capillary action	H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[18, p.14]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	[18, p.12]
3	Navier fluid [13]	equiliblium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
4	Poisson [17, §5, ¶46, p.104]	equiliblium of fluid	p		$\frac{1}{6}$		$\sum \frac{1}{\varepsilon^3}$		r		R	$C_3 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
				q		$\frac{1}{4}$		$\sum \frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2 z' R$	$C_4 = \frac{1}{4\pi}\pi = \frac{1}{4}$

5.1 Navier's two constants and tensor

In his theory of elasticity, Navier deduced the single constant ε in(1). The corresponding Navier-Stokes equations by Navier himself for the incompressible fluid (2) are as follow :

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\,; \end{cases} \tag{3}$$

along with the equation of continuity: $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$. Navier supposes two constants as follows :

$$(3\text{-}10)_{Nf} \quad \varepsilon \equiv \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) = \frac{4\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \quad E \equiv \frac{4\pi}{6}\int_0^\infty d\rho\rho^2 F(\rho) = \frac{2\pi}{3}\int_0^\infty d\rho\rho^2 F(\rho). \tag{4}$$

In the case of fluid, Navier was well aware of necessity for the equation of continuity, because from (3) he obtained $\varepsilon\Delta$, by defferentiating the equation of continuity with $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$. For example, the ε-terms in (3) , as well as (5) are reduced to $\varepsilon\Delta\mathbf{u}$ in (6). This is solely due to the mass conservative law, according to the explaination given by Navier.

As an aside, Navier always used his well-used mathematical methods involving a four-steps procedure to solve the three equations such as the equilibrium equation for the fluid [13], the kinetic equation for the elastic [12], and the kinetic equation for the fluid [13] with the general methods as follows:

- initially, to deduce one or two constants including uncomputable functions: g_1, g_2 i.e. $f\rho$, $f(\rho)$ or $F(\rho)$ in Table 2,
- then, to construct the indeterminate equation, which he denoted the nomencrature of "equation undeterminant" (cf. §5.1.1),
- then, to make Taylor series expansion and partial integration, exchanging d and δ, and pairing with the same integral operator,
- and finally, to solve the indeterminate equation from the two points of view, the interior and the boundary. We present more details of this procedure by outlining Navier's analysis of fluid flow[13].

5.1.1 Indeterminate equation

The indeterminate equation, so-called then by Navier, is as follows:

$$\begin{aligned} (3\text{-}24)_{Nf} \quad 0 \quad = \quad & \iiint dxdydz \begin{cases} \left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\right]\delta u \\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\right]\delta v \\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\right]\delta w \end{cases} \\ - \quad & \varepsilon\iiint dxdydz \begin{cases} \left(3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz}\right) + \left(\frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy}\right) + \left(\frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\right) \\ \left(\frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx}\right) + \left(\frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz}\right) + \left(\frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\right) \\ \left(\frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx}\right) + \left(\frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy}\right) + \left(\frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\right) \end{cases} \\ + \quad & \mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w). \end{aligned} \tag{5}$$

5.1.2 Determinated equation operated by Taylor expansion and partial integral

Putting $\mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w)=0$ of indeterminate equation (5) and performing a Taylor series expansion to first-order and neglecting higher-order terms, we get as follows:

$$(3\text{-}29)_{Nf}\quad 0=\iiint dxdydz\begin{cases}\left[P-\frac{dp}{dx}-\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)+\varepsilon\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)\right]\delta u\\ \left[Q-\frac{dp}{dy}-\rho\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)+\varepsilon\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\right]\delta v\\ \left[R-\frac{dp}{dz}-\rho\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right)+\varepsilon\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)\right]\delta w\end{cases}\qquad(6)$$

From (6) we get (3) i.e. the kinetic equation which is the first expression of (2).

5.1.3 Determinated equation deduced from boundary condition

As the boundary condition, Navier uses two constants in one equation. In this aspect, his method is the unique among the original formulators. Navier explains as follows: regarding the conditions which react at the points of the surface of the fluid, if we substitute

- $dydz \longrightarrow ds^2\cos l$, where l : the angles by which the tangent plane makes with the yz-plane on the surface frame,
- $dxdz \longrightarrow ds^2\cos m$, where m : similarly m is the angles with the xz-plane,
- $dxdy \longrightarrow ds^2\cos n$, where n : samilarly, n is the angles with the xy-plane,
- $\iint dydz, \iint dxdz, \iint dxdy \longrightarrow \mathbf{S}ds^2$, where $\mathbf{S}$ is the unit normal to the surface at this point,

then because the factors multiply $\delta u, \delta v$ and δw respectively reduce to zero, the following determinated equations should hold for any points of the surface of the fluid element:

$$(3\text{-}32)_{Nf}\quad\begin{cases}Eu+\varepsilon\left[\cos l2\frac{du}{dx}+\cos m\left(\frac{du}{dy}+\frac{dv}{dx}\right)+\cos n\left(\frac{du}{dz}+\frac{dw}{dx}\right)\right]=0,\\ Ev+\varepsilon\left[\cos l\left(\frac{du}{dy}+\frac{dv}{dx}\right)+\cos m2\frac{dv}{dy}+\cos n\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\right]=0,\\ Ew+\varepsilon\left[\cos l\left(\frac{dw}{dx}+\frac{du}{dz}\right)+\cos m\left(\frac{dw}{dy}+\frac{dv}{dz}\right)+\cos n2\frac{dw}{dz}\right]=0.\end{cases}\qquad(7)$$

Here the value of the constant E must vary in accordance with the nature of solid with which the fluid is in contact. The equations of (7) are an expression of conditions prevailing on the boundary condition of the surface and constitute the so-called boundary conditions. The first terms of the left-hand-side of (7) are defined in (4) for the expression that we seek for the sum of the momenta ofall interactions arising between the molecules on the boundary and the fluid, while the second terms are the normal derivatives. Here, derivative terms on the left-hand-side of (7) are expressible as $v_{i,j}+v_{j,i}$.

5.2 Cauchy's two constants and tensor

(Definition) We suppose that :
• a, b, c : the coordinate values of a molecule $\mathbf{m}$ in the rectanglar axes by x, y, z ; • $a+\Delta a$, $b+\Delta b$, $c+\Delta c$: the coordinates of an arbitrary molecule m ; • ξ, η, ζ : the functions of a, b, c, which represent the infinitesimal displacements, and are parallel to the axes of a molecule $\mathbf{m}$; • $(x,\ y,\ z)$, $(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$: the coordinates of the molecules $\mathbf{m}$ and m in the new state of the system ; • $r(1+\varepsilon)$: the distance between the molecule $\mathbf{m}$ and m ; • ε : the dilatation of the length r in the path from the first state to the second, and then we have $x=a+\xi$, $y=b+\eta$, $z=c+\zeta$; • X, Y, Z : the quantities of the algebraic projections.

Cauchy deduces the three elements X, Y, Z in the sysytem of meterial points of elasticity after calculating the interactions of molecules, the details of which are omitted for sake of brevity. Moreover we start with the following equation of elasticity

$$(40)_C\quad\begin{cases}X=(L+G)\frac{\partial^2\xi}{\partial a^2}+(R+H)\frac{\partial^2\xi}{\partial b^2}+(Q+I)\frac{\partial^2\xi}{\partial c^2}+2R\frac{\partial^2\eta}{\partial a\partial b}+2Q\frac{\partial^2\zeta}{\partial c\partial a},\\ Y=(R+G)\frac{\partial^2\eta}{\partial a^2}+(M+H)\frac{\partial^2\eta}{\partial b^2}+(P+I)\frac{\partial^2\eta}{\partial c^2}+2P\frac{\partial^2\zeta}{\partial b\partial c}+2R\frac{\partial^2\xi}{\partial a\partial b},\\ Z=(Q+G)\frac{\partial^2\zeta}{\partial a^2}+(P+H)\frac{\partial^2\zeta}{\partial b^2}+(N+I)\frac{\partial^2\zeta}{\partial c^2}+2Q\frac{\partial^2\xi}{\partial c\partial a}+2P\frac{\partial^2\eta}{\partial b\partial c}\end{cases}$$

(The invariants of the tensor are represented by the two constants of G and R.)
Cauchy says about the elements of tensor i.e. the invariable values : $G, H, I, L, M, N, P, Q, R$:

If we suppose that the molecules $m, m', m'', \cdots$ are originally allocated by the same way in relation to the three planes made by the molecule m in parallel with the plane coordinates, then the values of these quantities come to remain invariable, even though a series of changes are made among the three angles : α, β, γ.

Cauchy considers symmetric tensor such that :

$$(41)_C\quad G=H=I,\quad L=M=N,\quad P=Q=R,\quad (45)_C\quad L=3R.$$

$$(46)_C \begin{cases} X = (R+G)\left(\frac{\partial^2 \xi}{\partial a^2} + \frac{\partial^2 \xi}{\partial b^2} + \frac{\partial^2 \xi}{\partial c^2}\right) + 2R\frac{\partial \nu}{\partial a}, \\ Y = (R+G)\left(\frac{\partial^2 \eta}{\partial a^2} + \frac{\partial^2 \eta}{\partial b^2} + \frac{\partial^2 \eta}{\partial c^2}\right) + 2R\frac{\partial \nu}{\partial b}, \\ Z = (R+G)\left(\frac{\partial^2 \zeta}{\partial a^2} + \frac{\partial^2 \zeta}{\partial b^2} + \frac{\partial^2 \zeta}{\partial c^2}\right) + 2R\frac{\partial \nu}{\partial c}, \end{cases} \qquad (47)_C \quad \nu = \frac{\partial \xi}{\partial a} + \frac{\partial \eta}{\partial b} + \frac{\partial \zeta}{\partial c}$$

Cauchy may be the inventor of the nomenclature[8] of "tensor", and Poisson backs up the structure of symmetry such that his idea reducing from 9 to 6 elements is due to Cauchy, as follows :

> D'un autre côté, il faut, pour l'equilibre d'un parallélépipède rectangle d'une étendue insensible, que les neuf composantes des pressions appliquées à ses trois faces non-parallélles, se réduisent à six forces qui peuvent être inégales. Cette proposition est due à M.Cauchy, et se déduit de la considération des momens. [17, §38, p.83]

Continuing, we define the density of molecules as: $(48)_C \quad \Delta = \frac{\mathcal{M}}{\mathcal{V}}$, where, $\mathcal{M}$ is the sum of the mass of molecules contained in the sphere and $\mathcal{V}$ is the volume of the sphere. We the find expression for the two constants, G and R:

$$(50)_C \quad \begin{cases} G = \pm\frac{\Delta}{2} \int_0^\infty \int_0^{2\pi} \int_0^\pi r^3 \mathrm{f}(r) \cos^2\alpha \sin p dr dq dp = \pm\frac{2\pi\Delta}{3} \int_0^\infty r^3 \mathrm{f}(r) dr, \\ R = \frac{\Delta}{2} \int_0^\infty \int_0^{2\pi} \int_0^\pi r^3 f(r) \cos^2\alpha \cos^2\beta \sin p dr dq dp = \frac{2\pi\Delta}{15} \int_0^\infty r^3 f(r) dr = \pm\frac{2\pi\Delta}{15} \int_0^\infty \left[r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)\right] dr \end{cases} \tag{8}$$

When we calculate these values in the general case then (8) yieldsthe following expressions:

$$(56)_C \begin{cases} A \equiv \left[(L+G)\frac{\partial \xi}{\partial a} + (R-G)\frac{\partial \eta}{\partial b} + (Q-G)\frac{\partial \zeta}{\partial c}\right]\Delta, \\ B \equiv \left[(R-H)\frac{\partial \xi}{\partial a} + (M+H)\frac{\partial \eta}{\partial b} + (P-H)\frac{\partial \zeta}{\partial c}\right]\Delta, \\ C \equiv \left[(Q-I)\frac{\partial \xi}{\partial a} + (P-I)\frac{\partial \eta}{\partial b} + (N+I)\frac{\partial \zeta}{\partial c}\right]\Delta, \end{cases} \qquad (57)_C \begin{cases} D \equiv \left[(P+I)\frac{\partial \eta}{\partial c} + (P+H)\frac{\partial \zeta}{\partial b}\right]\Delta, \\ E \equiv \left[(Q+G)\frac{\partial \zeta}{\partial a} + (Q+I)\frac{\partial \xi}{\partial c}\right]\Delta, \\ F \equiv \left[(R+H)\frac{\partial \xi}{\partial b} + (R+G)\frac{\partial \eta}{\partial a}\right]\Delta, \end{cases}$$

By $(41)_C$ and $(45)_C$, we obtain the following reduced form:
$\frac{A}{\Delta} = 2(R+G)\frac{\partial \xi}{\partial a} + (R-G)v, \quad \frac{B}{\Delta} = 2(R+G)\frac{\partial \eta}{\partial b} + (R-G)v, \quad \frac{C}{\Delta} = 2(R+G)\frac{\partial \zeta}{\partial c} + (R-G)v,$
$\frac{D}{\Delta} = (R+G)\left(\frac{\partial \eta}{\partial b} + \frac{\partial \zeta}{\partial c}\right), \quad \frac{E}{\Delta} = (R+G)\left(\frac{\partial \zeta}{\partial a} + \frac{\partial \xi}{\partial c}\right), \quad \frac{F}{\Delta} = (R+G)\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)$
For convenience' sake, in the paticular case when both $(41)_C$ and $(45)_C$ hold, it is sufficient to have: $(59)_C \quad (R+G)\Delta \equiv \frac{1}{2}k, \quad (R-G)\Delta \equiv K, \quad \Rightarrow \quad 2R = \frac{k+2K}{2\Delta}$. Equations $(56)_C$ and $(57)_C$ can be displayed in a more convenient manner

$$(60)_C \quad \Rightarrow \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial \xi}{\partial a} + Kv & \frac{1}{2}k\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) & \frac{1}{2}k\left(\frac{\partial \zeta}{\partial a} + \frac{\partial \xi}{\partial c}\right) \\ \frac{1}{2}k\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) & k\frac{\partial \eta}{\partial b} + Kv & \frac{1}{2}k\left(\frac{\partial \eta}{\partial c} + \frac{\partial \zeta}{\partial b}\right) \\ \frac{1}{2}k\left(\frac{\partial \zeta}{\partial a} + \frac{\partial \xi}{\partial c}\right) & \frac{1}{2}k\left(\frac{\partial \eta}{\partial c} + \frac{\partial \zeta}{\partial b}\right) & k\frac{\partial \zeta}{\partial c} + Kv \end{bmatrix} \tag{9}$$

Here, we must remark that the layout of symmetric tensor of $(58)_C$ or $(60)_C$ is the Cauchy's invention. If, moreover, the condition $(54)_C$: $R = -G$ holds, then $k = 0$ holds, thus yielding the following identities: $(61)_C \quad A = B = C = Kv, \quad D = E = F = 0$.

5.2.1 Equilibrium and kinetic equation of fluid by Cauchy

In what follows, equations referring to Cauchy's work on fluids will be designated in the form $(\cdot)_{C^*}$ instead by $(\cdot)_C$ to distinguish these from equations appearing in his work on elasticity above.
(Verification of equations in fluid.)
By replacing $(a,\ b,\ c)$ of $(56)_C$ and $(57)_C$ with $(x,\ y,\ z)$, we derive an equvalent set of equations for fluid as for elasticity. We omite for the sake of brevity the pricese processes in leading to the two constants or equations and present the final form

$$(76)_{C^*} \quad \begin{cases} \frac{\partial A}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial E}{\partial z} + X\Delta = 0, \\ \frac{\partial F}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial D}{\partial z} + Y\Delta = 0, \\ \frac{\partial E}{\partial x} + \frac{\partial D}{\partial y} + \frac{\partial C}{\partial z} + Z\Delta = 0, \end{cases} \quad \Rightarrow \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} + \Delta \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{0}$$

We follow the layout of Cauchy's symmetric tensor as presented originally in $(76)_{C^*}$. By replacing $R+G$ and $2R$ with Cauchy's usage $C_1 \equiv R+G = \frac{k}{2\Delta}, \quad C_2 \equiv 2R = \frac{k+2K}{2\Delta}$, we can reduce these equations of fluids in motion and in equilibrium to the same form $(46)_C$ found for elasticity. However, here, we would like to adopt not Cauchy's C_1 and C_2, but $C_1 = R$ and $C_2 = G$, because it is more rational to do so, as we can seen by checking the reciprocal coincidence in Table 2.[9]

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[8] The editors of Hamilton's papers [6, p.237, footnote] say, " The writer believes that what originally led him to use the terms 'modulus' and 'amplitude,' was a recollection of M. Cauchy's nomenclature respecting the usual imaginaries of algebra."

[9] Here, C_1 and C_2 are not the two-constants by ours but named temporarily by Cauchy himself.

(Comparison with and commentd on Navier's equation in elasticity.)

Cauchy states: for the reduction of the equations $(79)_{C'}$ and $(80)_{C'}$ to Navier's equations([12]) to determine the law of equilibrium and elasticity, it is necessary to assume such as the condition which we have mentioned above : $k = 2K$. According to Cauchy's assertion, if $G = 0$ then we get as the equations of equiliblium and the kinetic equations in equal elasticity, then the tensor is equivalent with the tensor not only of the elastic but also of ε in Navier's fluid equation (3) (c.f. Table 4).

5.3 Poisson's two constants and tensor

5.3.1 Principle and equations in elastic solid

Below, we deduce K and k according to Poisson[15, pp.368-405, §1-§16]. For brevity, we introduce the following definitions:

$$\begin{cases} ax_1 + by_1 + c(z_1 - \zeta_1) \equiv \phi, \\ a'x_1 + b'y_1 + c'(z_1 - \zeta_1) \equiv \psi, \\ a''x_1 + b''y_1 + c''(z_1 - \zeta_1) \equiv \theta, \end{cases} \qquad \begin{cases} \phi\frac{du}{dx} + \psi\frac{du}{dy} + \theta\frac{du}{dz} \equiv \phi', \\ \phi\frac{dv}{dx} + \psi\frac{dv}{dy} + \theta\frac{dv}{dz} \equiv \psi', \\ \phi\frac{dw}{dx} + \psi\frac{dw}{dy} + \theta\frac{dw}{dz} \equiv \theta' \end{cases} \tag{10}$$

We assume that α is the average molecular distance, ω presents a finite surface area, and $\frac{\omega}{\alpha^2}$ is the average number of molecules in ω. We then get the pressure terms.

$$P = \sum \frac{(\phi + \phi')\zeta}{\alpha^3 r'} fr', \quad Q = \sum \frac{(\psi + \psi')\zeta}{\alpha^3 r'} fr' \quad R = \sum \frac{(\theta + \theta')\zeta}{\alpha^3 r'} fr'. \tag{11}$$

By using his so-called *effective transformation*,[10], we get from (11) the following:

$$\begin{cases} P = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(g + g')\sum \frac{r^3}{\alpha^5} fr + (gg' + hh' + ll')g \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(h + h')\sum \frac{r^3}{\alpha^5} fr + (gg' + hh' + ll')h \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(l + l')\sum \frac{r^3}{\alpha^5} fr + (gg' + hh' + ll')l \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases} \qquad \Delta := \cos\beta \cdot \sin\beta \; d\beta \; d\gamma, \tag{12}$$

Later, Poisson recalculates this problem in another book [17][11], in which he deduces the general principles behind elasticity and fluid, and hence derives the representive two-constants with K and k for both elasticity and fluids as follows:

$$\begin{cases} P = \left[K(1 + \frac{du}{dx}) + k\left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right]c + \left[K\frac{du}{dy} + k\left(\frac{du}{dy} + \frac{dv}{dx}\right)\right]c' + \left[K\frac{du}{dz} + k\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right]c'', \\ Q = \left[K(1 + \frac{dv}{dy}) + k\left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right)\right]c' + \left[K\frac{dv}{dx} + k\left(\frac{dv}{dx} + \frac{du}{dy}\right)\right]c + \left[K\frac{dv}{dz} + k\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right]c'', \\ R = \left[K(1 + \frac{dw}{dz}) + k\left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right)\right]c'' + \left[K\frac{dw}{dy} + k\left(\frac{dw}{dy} + \frac{dv}{dz}\right)\right]c' + \left[K\frac{dw}{dx} + k\left(\frac{dw}{dx} + \frac{du}{dz}\right)\right]c, \end{cases} \tag{13}$$

where, for abbreviation, he uses similarly K and k. Moreover, instead of α in (11), he introduces ε as the average distance between molecules, and from the following considerations:

- on voit que la pression N restera la même en tous sens autour de ce point : elle sera normale à ce plan et dirigée de dehors en dedans de A, ou de dedans en dehors, selon que sa value sera positive ou negative, [⇒ we see that the pressure N orients omnidirectionally around an arbitrary point : A, and from outward into inward or from inward to outward, according to that the value will be positive or negative, (then we ought to consider as $\frac{1}{2}$) ;]
- from the supposition of isotropy and homogeneity, $r^2 = x^2 + y^2 + z^2, \quad \Rightarrow \quad \Sigma\frac{z^2}{r}fr = \Sigma\frac{1}{3}rfr$, (cf. Poisson [17], pp. 32-34) :

$$(3\text{-}8)_{Pe} \qquad K \equiv \frac{1}{6\varepsilon^3}\sum rfr = \frac{2\pi}{3}\sum \frac{rfr}{4\pi\varepsilon^3}, \quad k \equiv \frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} = \frac{2\pi}{15}\sum \frac{1}{4\pi\varepsilon^3} r^3 \frac{d.\frac{1}{r}fr}{dr}, \tag{14}$$

et étendant les sommes Σ à tous les points matériels du corps qui sont compris dans la sphère d'activité de M. [⇒ and extending the summation Σ to all the material points contained in the active sphere by M.] (cf. Poisson [17], p. 46) :

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[10] $\frac{1}{r'}fr' = \frac{1}{r}fr + (\phi\phi' + \psi\psi' + \theta\theta')\frac{d.\frac{1}{r}fr}{rdr}$ ([17, p.42]).

[11] In Poisson [17], the title of the chaper 3 reads "*Calcul des Pressions dans les Corps élastiques ; équations defferentielles de l'équilibre et du mouvement de ces Corps.*"

5.3.2 Fluid pressure in motion

[12] Poisson's tensor of the pressures in a fluid, which he assumes compressible, reads as follows :

$$(7\text{-}7)_{Pf}\quad \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix},$$

$$(k+K)\alpha=\beta,\quad (k-K)\alpha=\beta',\quad p=\psi t=K,\quad \Rightarrow\quad \beta+\beta'=2k\alpha,$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here K and k are the same one as in $(3\text{-}8)_{Pe}$(=(14)) of the elastic body. The velocity and pressure are defined as follows :

$$\mathbf{u}=(u,v,w),\quad \frac{dx}{dt}=u,\quad \frac{dy}{dt}=v,\quad \frac{dz}{dt}=w,\quad \varpi\equiv p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt},\quad (\varpi\equiv p,\quad \text{if incompressible.})$$

which substituted into the equation yields

$$\begin{cases} \frac{d^2x}{dt^2}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}, \\ \frac{d^2y}{dt^2}=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2}=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}. \end{cases} \quad\Rightarrow\quad (7\text{-}9)_{Pf}\quad \begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}), \\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}), \\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}). \end{cases} \qquad (15)$$

5.4 Saint-Venant's tensor

Saint-Venant[13] explains that the object of his paper [19] is to simplfy the description and calculation of molecular interactions without specifying the molecular function. We show Saint-Venant's tensor, which from the extract [19] seems to hint Stokes[21]. For this section we introduce the following parameters: ξ,η,ζ are the velocity components at the arbitrary point m of a fluid in motion in the coordinate directions x,y,z respectively, P_{xx},P_{yy},P_{zz} are the normal pressures and P_{yz},P_{zx},P_{xy} are the tangential pressures with sub-index pair indicating the perpendicular plane and direction of decomposition. His expressions are:

$$(1)_{SV}\quad \frac{P_{xx}-P_{yy}}{2(\frac{d\xi}{dx}-\frac{d\eta}{dy})}=\frac{P_{zz}-P_{xx}}{2(\frac{d\zeta}{dx}-\frac{d\eta}{dz})}=\frac{P_{yy}-P_{zz}}{2(\frac{d\eta}{dy}-\frac{d\zeta}{dz})}=\frac{P_{yz}}{\frac{d\eta}{dz}+\frac{d\zeta}{dy}}=\frac{P_{zx}}{\frac{d\zeta}{dx}+\frac{d\xi}{dz}}=\frac{P_{xy}}{\frac{d\xi}{dy}+\frac{d\eta}{dx}}=\varepsilon,$$

where, $\frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big)-\frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)=\pi$. From this last equation, we solve for normal pressure respectively as follows: $(2)_{SV}\quad P_{xx}=\pi+2\varepsilon\frac{d\xi}{dx},\quad P_{yy}=\pi+2\varepsilon\frac{d\eta}{dy},\quad P_{zz}=\pi+2\varepsilon\frac{d\zeta}{dz}$. From $(1)_{SV}$, we then obtain the tangential pressures : $P_{yz},\ P_{zx},\ P_{xy}$, which then reduces the tensor to symmetric form

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) \\ \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) \\ \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}, \qquad (16)$$

Saint-Venant says that by using his theory, we can obtain concordance with Navier, Cauchy and Poisson:

> Si l'on remplace π par $\varpi-\varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$, et si l'on substitue les équations $(2)_{SV}$ et $(3)_{SV}$ dans les relations connues entre les pressions et les forces accélératrices, on obtient, en supposant ε le même en tous les points du fluide, les équations différentielles données le 18 mars 1822 par M.Navier (*Mémoires de l'Institut*, t.VI), en 1828 par M.Cauchy (*Exercices de Mathématiques*, p.187)[14], et le 12 octobre 1829 par M.Poisson (même *Mémoire*, p.152)[15]. La quantité variable ϖ ou π n'est autre chose, dans les liquides, que la *pression normale moyenne* en chaque point. [19, p.1243]

Saint-Venant's paper[19] seems to provide Stokes a clue to the notion of tensor (20) and his principle, because we can see the close correspondence by comparing[16] Saint-Venant's t_{ij}:

$$\begin{aligned} t_{ij} &= \big(\pi+2\varepsilon v_{i,j}-\gamma\big)\delta_{ij}+\gamma,\quad (\text{where},\quad \gamma\equiv\varepsilon(v_{i,j}+v_{j,i})), \\ &= \Big(\frac{1}{3}(P_{xx}+P_{yy}+P_{zz})-\frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)+2\varepsilon v_{i,j}-\gamma\Big)\delta_{ij}+\gamma \\ &= (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz})-\frac{2\varepsilon}{3}v_{k,k})\delta_{ij}+\varepsilon(v_{i,j}+v_{j,i}) \quad\Leftarrow\quad 2\varepsilon v_{i,j}\delta_{ij}=\varepsilon(v_{i,j}+v_{j,i})\delta_{ij}=\gamma\delta_{ij} \end{aligned} \qquad (17)$$

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[12] In Poisson [17], the title of the chaper 7 reads "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

[13] Adhémar Jean Claude Barré de Saint-Venant (1797-1886).

[14] Cauchy [1, p.226]

[15] Poisson [17, p.152] $(7\text{-}9)_{Pf}$.

[16] In our paper, we cite the source of the tensorial description of t_{ij} of the tensor : of Poisson and Cauchy from C.Truesdell[23], of Navier from G.Darrigol [4], and othewise by ourself or Schlichting[20].

with Stokes's t_{ij} (21). Here, using (17), if we put[17] $P_{xx} = P_{yy} = P_{zz} = -p$ by assuming isotropy and homogeneity, which Stokes similarly takes as his principle in § 5.5, then (17) is equivalent to Stokes' t_{ij} as follows. For example, if we put $\varepsilon \equiv \mu$, and choose t_{xx} component of Saint-Venant's tensor form (16):

$$\begin{aligned} \pi + 2\varepsilon\frac{d\xi}{dx} &= -p + \left(2 - \frac{2}{3}\varepsilon\frac{d\xi}{dx}\right) - \frac{2\varepsilon}{3}\left(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = -p + 2\varepsilon\left\{\frac{2}{3}\frac{d\xi}{dx} - \frac{1}{3}\left(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)\right\} \\ &= -p + 2\varepsilon\left\{\frac{d\xi}{dx} - \frac{1}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)\right\} = -p + 2\varepsilon\left(\frac{d\xi}{dx} - \delta\right) \quad \Rightarrow \quad P_1 \text{ of Stokes' (20).} \end{aligned}$$

The other tensor components are likewise demonstrated but we omit the proof here for brevity. Moreover, Saint-Venant proposes that putting $\pi = \varpi - \varepsilon\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = \varpi - \varepsilon v_{k,k}$ then $t_{ij} = (\varpi - \varepsilon v_{k,k} + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma = (\varpi - \varepsilon v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$. This form of his tensor plays the key role in common with Navier's, Cauchy's and Poisson's constants.

5.5 Stokes' equations and tensor

In expressing the fluid equations in the following form

$$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases} \tag{18}$$

Stokes points out the coincidence with Poisson with the correspondence:

$\varpi = p + \frac{\alpha}{3}(K + k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) \quad \Rightarrow \quad \nabla\varpi = \nabla p + \frac{\beta}{3}\nabla \cdot (\nabla \cdot \mathbf{u})$.

Stokes also makes the comment:

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389)[18], but his principles differ from mine still more than do Poisson's. [21, p.77, footnote]

Stokes says : observing that $\alpha(K + k) \equiv \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ (=(15) in our renumbering) to the equation $(12)_S$ of this paper. Stokes proposes the Stokes' approximate equations in [21, p.93]:

$$(13)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0, \end{cases} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{19}$$

which are identical to $(7\text{-}9)_{Pf}$ (=(15), adding that: "these equations are applicable to the determination of the motion of water in pipes and canala, to the calculation of the effect of friction on the motions of tides and waves, and such questions." ([21, p.93]). Here we shall trace his deduction with the Stokes tensor in the form:

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p - 2\mu\left(\frac{du}{dx} - \delta\right) & -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - 2\mu\left(\frac{dv}{dy} - \delta\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - 2\mu\left(\frac{dw}{dz} - \delta\right) \end{bmatrix}, \text{ where } 3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \tag{20}$$

He remarks: "it may also be very easily provided directly that the value of 3δ, the rate of cubical dilatation". We find that Stokes' tensor can be described compactly as follows:

$$\begin{aligned} -t_{ij} &= \{p - 2\mu(v_{i,j} - \delta) + \gamma\}\delta_{ij} - \gamma, & &\Leftarrow \text{where,} \quad \gamma = \mu(v_{i,j} + v_{j,i}), \\ &= \{p - 2\mu v_{i,j}\}\delta_{ij} + \gamma(-\delta_{ij} + \delta_{ij} - 1) & &\Leftarrow \text{where,} \quad 2\mu v_{i,j}\delta_{ij} = \mu(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij}, \\ &= (p + 2\mu\gamma)\delta_{ij} - \gamma = (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i}) \end{aligned} \tag{21}$$

Therefore, the sign of $-t_{ij}$ depends on the location of the tensor in the equation.[19] Now, in considering the coincidence of (16) with (19), we see Stokes' tensor may have originated with Saint-Venant's tensor. The article by J.J.O'Connor and E.F.Robertson[14] point out this resemblance. Moreover, in 1846, Stokes has reported on the then academic activities within hydromechanics [22], in which he cites Saint-Venant[19]. It reads that, "the

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[17] cf.I.Imai [7, p.185].

[18] Navier[13].

[19] Schlichting writes Stokes' tensor with the minus sign as follows : $\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$ [20, p.58, in footnote].

same subject has been considered in a quite different point of view by Barré de Saint-Venant, in a communication to the French Academy in 1843, an abstruct of which is contained in the *Comtes Rendus*." Therefore, Stokes says : "I shall therefore suppose that for water, and by analogy for other incompressible fluids." ([21, p.93]). At any rate, we get $(13)_S$ (=(19)) with (20) and the following (22) :

$$\begin{cases} \rho\left(\frac{Du}{Dt}-X\right)+\frac{dP_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}=\rho\left(\frac{Du}{Dt}-X\right)+P=0, \\ \rho\left(\frac{Dv}{Dt}-Y\right)+\frac{dT_3}{dx}+\frac{dP_2}{dy}+\frac{dT_1}{dz}=\rho\left(\frac{Dv}{Dt}-Y\right)+Q=0, \\ \rho\left(\frac{Dw}{Dt}-Z\right)+\frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dP_3}{dz}=\rho\left(\frac{Dw}{Dt}-Z\right)+R=0, \end{cases} \quad \text{where,} \quad \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \tag{22}$$

6 Conclusions

It is called that "the two-constants theory" is the one now accepted for isotropic, homogeneous, linear elasticity. (Darrigol[4, p.121]). We showed in our report :

- the original mathematical evidence to clarify the genealogy of tensor; of which,
- tensors and the corresponding equations as developed historically by Navier(1822), Cauchy(1828), Poisson(1829), Saint-Venant(1843) and Stokes(1849) (sic. in order) ; and
- the appearance of the notion of tensors especially in the work of Saint-Venant. It is our contention that his was an epock-making contribution, by simplifying and identifying the concordance between these pioneers of MDNS equations, for using only tensor without the microscopically descriptions, and providing context for the contribution of Stokes.

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Remark : we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our papaer, the underscoring are by ours.

Table 4: Concurrences and variations of tensors

1	name	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elasticity	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(5\text{-}4)_{N^e}$ $-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	We define the coefficient matrix in elasticity : C_T^e as follows : C_T^e : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x\partial y} & \frac{\partial^2 w}{\partial z\partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y\partial z} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 v}{\partial y\partial z} \end{bmatrix},$ then $(6\text{-}1)_{N^e} \quad \Rightarrow \quad C_T^e = \; -\varepsilon\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \quad \Rightarrow \quad (3)$
1-2	Navier fluid	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ (3) $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & \epsilon' - 2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) & \epsilon' - 2\varepsilon\frac{dw}{dz}) \end{bmatrix},$ where $\epsilon' = p - \varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$	Samely, we define the coefficient matrix in fluid : C_T^f , which contains p in (1,1)-, (2,2)- and (3,3)-element. $(3) \quad \Rightarrow \quad C_T^f = \begin{bmatrix} p-3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p-3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p-3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2	Cauchy system (contains both elasticity and fluid)	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a} + K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b} + K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c} + K\nu \end{bmatrix},$ where $\nu = \frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$	$(46)_C \Rightarrow C_T^e = \begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix}$ $\Rightarrow \; R\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix},$ where $P = Q = R, \quad L = M = N, \quad L = 3R.$
3-1	Poisson elasticity	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(6)_{P^e}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	$(6)_{P^e}$ $\begin{cases} X = \frac{d^2u}{dt^2} - a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right), \\ Y = \frac{d^2v}{dt^2} - a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right), \\ Z = \frac{d^2w}{dt^2} - a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right), \end{cases}$ $\Rightarrow C_T^e = \; -\frac{a^2}{3}\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
3-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \pi+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \pi+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ \pi+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix},$ where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi^t}\frac{d\chi t}{dt}$	$(7\text{-}9)_{Pf} \quad \Rightarrow \quad C_T^f = \begin{bmatrix} \varpi+\beta & \beta & \beta & 0 & 0 \\ \beta & \varpi+\beta & \beta & 0 & 0 \\ \beta & \beta & \varpi+\beta & 0 & 0 \end{bmatrix}.$ According to Stokes : if we put $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$ $\Rightarrow \quad C_T^f = \begin{bmatrix} p+\frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p+\frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p+\frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix}$ $\Rightarrow (12)_S \; (= (18)).$ Remark : $\alpha(K+k) = \beta.$
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $\quad = (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $\begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix},$ where $\pi = \frac{1}{3}\left(P_{xx} + P_{yy} + P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ $\qquad \equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right) \quad (16)$	non description in [19].
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$ tensor $= -1 \times$ $\begin{bmatrix} p-2\mu\left(\frac{du}{dx}-\delta\right) & -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\mu\left(\frac{dv}{dy}-\delta\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\mu\left(\frac{dw}{dz}-\delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \quad (20)$	$(12)_S \quad \Rightarrow \quad C_T^f = \begin{bmatrix} -p+\frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p+\frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p+\frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix}$ $\Rightarrow (20).$ Remark : $\frac{4}{3}\mu = 2\mu(1-\frac{1}{3})$