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Kyoto University
Circular 作用素について
(On circular operators)
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1 Circularity

A densely defined operator $T$ in a Hilbert space $\mathcal{H}$ is said to be circular if $T$ is unitarily equivalent to $e^{it}T$ for all $t \in \mathbb{R}$. Clearly the spectrum of a circular operator is circularly symmetric at origin.

Example 1 Let $S$ be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\} (n \in \mathbb{Z})$ and a sequence $\{w_n\} (w_n \neq 0, n \in \mathbb{Z})$ of complex numbers such that

$$\mathcal{D}(S) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \right\}$$

and

$$Se_n = w_n e_{n+1}$$

for all $n \in \mathbb{Z}$, then $S$ is called a bilateral (injective) weighted shift with weights $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift is defined by the replacement $\mathbb{Z}$ with $\mathbb{N}$ analogously.

Every bilateral or unilateral, weighted shift is circular.

Let us recall irreducibility for a possibly unbounded operator in $\mathcal{H}$. Let $T$ be a closed densely defined operator in $\mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to reduce $T$ if the following two conditions are satisfied:
1. \( P_{\mathcal{M}} \mathcal{D}(T) \subseteq \mathcal{D}(T) \).

2. \( T(\mathcal{M} \cap \mathcal{D}(T)) \subseteq \mathcal{M} \) and \( T(\mathcal{M}^\perp \cap \mathcal{D}(T)) \subseteq \mathcal{M}^\perp \).

Here \( P_{\mathcal{M}} \) denotes the orthogonal projection onto \( \mathcal{M} \). If there is no non-trivial reducing subspace of \( T \), then \( T \) is said to be irreducible.

**Lemma 2** Let \( T \) be an irreducible, closed densely defined operator in a separable Hilbert space \( \mathcal{H} \). If \( T \) is circular, then there are a family \( \{U_t\}_{t \in \mathbb{R}} \) of unitary operators on \( \mathcal{H} \) and a mapping \( m(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{T} \) such that

1. \( U_t T = e^{it} T U_t \) for all \( t \in \mathbb{R} \).

2. \( U_s U_t = m(s, t) U_{s+t} \), \( U_0 = I \) (identity operator) for all \( s, t \in \mathbb{R} \).

3. the map \( m(\cdot, \cdot) \) satisfies

\[
m(s, 0) = m(0, s) = 1 \quad \text{and} \quad m(s + t, u) m(s, t) = m(s, t + u) m(t, u)
\]

for \( s, t \in \mathbb{R} \).

Here, \( \mathbb{T} \) is the multiplicative group of complex numbers with modulus 1.

Moreover, if the above \( \{U_t\} \) is so chosen that \( t \to U_t \) is measurable, there exists a strongly continuous one-parameter unitary group \( \{V_t\} \) satisfying the above condition 1, that is, \( V_t T = e^{it} TV_t \) for all \( t \in \mathbb{R} \).

### 2 Strong circularity

Let \( T \) be a closed densely defined operator in a Hilbert space \( \mathcal{H} \). If there is a strongly continuous one-prameter unitary group \( \{U_t\}_{t \in \mathbb{R}} \) such that

\[
U_t T = e^{it} T U_t \quad (t \in \mathbb{R})
\]

then \( T \) is said to be strongly circular and \( \{U_t\}_{t \in \mathbb{R}} \) is called a unitary group associated with \( T \).

**Example 3** (Mlak) If \( S \) is the creation operator in a separable Hilbert space, that is, the unilateral weighted shift with weights \( \{w_n\} \) given by \( w_n = \sqrt{n+1} \) \( (n \in \mathbb{N}) \), then \( S \) is strongly circular.
Let $S$ be a unilateral or bilateral weighted shift in a separable Hilbert space $\mathcal{H}$. Then $S$ is strongly circular. In fact, let $S$ be a bilateral weighted shift in $\mathcal{H}$ with weights $\{w_{n}\}$ with respect to $\{e_{n}\}$. Define a closed densely defined operator by

$$D(N) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_{n}e_{n} \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_{n}|^{2}|n|^{2} < \infty \right\}$$

and

$$Ne_{n} = ne_{n} \quad (n \in \mathbb{Z}).$$

Then $N$ is self-adjoint, and

$$e^{itN}Se_{n} = e^{it}Se^{itN}e_{n}$$

for all $n \in \mathbb{Z}$. It follows that $S$ is a strongly circular operator with the associated unitary group $\{e^{itN}\}$.

For a bounded operator $B$ and a densely defined operator $T$,

$$BT \subseteq TB$$

means that

$$BD(T) \subseteq D(T) \text{ and } BT\eta = TB\eta \quad (\eta \in D(T)).$$

**Lemma 4** Let $S$ be a densely defined operator in a Hilbert space $\mathcal{H}$ and $T$ be a closed densely defined operator in $\mathcal{H}$. Let $\{U_{t}\}_{t \in \mathbb{R}}$ and $\{V_{t}\}_{t \in \mathbb{R}}$ be strongly continuous one-parameter unitary groups on $\mathcal{H}$ with infinitesimal generators $A$ and $B$ respectively, that is, $U_{t} = e^{itA}$, $V_{t} = e^{itB}$. Then the following conditions are equivalent:

1. For all $t \in \mathbb{R}$,

   $$U_{t}S \subseteq TV_{t}.$$  

2. For all $\lambda \in \mathbb{C}$ with $\Re \lambda \neq 0$,

   $$(\lambda - A)^{-1}S \subseteq T(\lambda - B)^{-1}.$$  

**Theorem 5** Let $T$ be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then $T$ is strongly circular if and only if there is a self-adjoint operator $A$ in $\mathcal{H}$ such that

$$ (\lambda - A)^{-1}T \subseteq T(\lambda - I - A)^{-1}$$

(1)

for all $\lambda \in \mathbb{C}$ with $\Re \lambda \neq 0$. 

Proof. Suppose $T$ is strongly circular. Then there is a strongly continuous one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$ such that $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$. Set

$$V_t = e^{it} U_t$$

for each $t \in \mathbb{R}$. Then, $\{V_t\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Let $A$ be the infinitesimal generator of $\{U_t\}_{t \in \mathbb{R}}$. Then it follows from the semigroup theory that the infinitesimal generator of $\{V_t\}_{t \in \mathbb{R}}$ is $I + A$. Putting $T = S$ in the above, $A$ satisfies relation (1).

Conversely, suppose $A$ is a self-adjoint operator satisfying relation (1). Put

$$U_t = e^{it A} \quad \text{and} \quad V_t = e^{it} e^{it A}$$

for $t \in \mathbb{R}$. Then we obtain $U_t T \subseteq e^{it} T U_t$ for all $t \in \mathbb{R}$. Since each $U_t$ is unitary, $U_t \mathcal{D}(T) = \mathcal{D}(T)$. Hence, $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$.

### 3 q-deformed circularity

Let $T$ be a densely defined operator in a Hilbert space $\mathcal{H}$. If there is a positive real number $q$ with $q \neq 1$ such that $T$ is unitarily equivalent to $q T$, then we say that $T$ has property Q.

**Proposition 6.** Suppose that a nontrivial closed densely defined operator $T$ has property Q. Then,

1. $T$ is unbounded.
2. The spectrum contains zero.
3. The absolute value $|T|$ has also property Q.

**Example 7.** Let $T$ be a closed densely defined operator in $\mathcal{H}$. If $T$ satisfies

$$TT^* = q T^* T \quad (q > 0, \ q \neq 1),$$

then $T$ is called a q-normal operator. It should be noticed that elements satisfying this relation in a formal algebraic sense appear at various circumstances in the theory of quantum group theory. A non-trivial q-normal operator $T$ is always unbounded and has sufficient large spectrum in the sense of the planar Lebesgue measure. Especially, every q-normal operator $T$ is unitarily equivalent to $q T$. Thus the class of operators possessing property Q contains all q-normal operators.
Definition 8 Let $T$ be a densely defined operator in $\mathcal{H}$. If there is a positive real number $q$ with $q \neq 1$ such that $T$ is unitarily equivalent to $q e^{it} T$ for all $t \in \mathbb{R}$, then $T$ is called a $q$-deformed circular (simply, $q$-circular) operator.

Circularity may be considered as $q$ tends to 1 in the above. Clearly, a $q$-circular operator has property $Q$.

Example 9 (q-circular weighted shifts) If a bilateral weighted shift has property $Q$, then it is $q$-circular. Hence, a $q$-normal bilateral weighted shift is $q$-circular. Moreover the spectrum of a $q$-circular weighted shift is equal to the whole complex plane.

Theorem 10 Let $T$ be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then $T$ is $q$-circular if and only if $T$ is circular and has property $Q$.

Proof. Suppose $T$ is $q$-circular. Then there is a family $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that

$$ U_t T = q e^{it} T U_t \quad (2) $$

for all $t \in \mathbb{R}$. It is clear that $T$ has property $Q$. Put

$$ V_t = U_t U_0^{-1} \quad (3) $$

for all $t \in \mathbb{R}$. We have by above relation (2)

$$ V_t T = q U_t U_0^{-1} T = U_t T U_0 = q e^{it} T U_t U_0^{-1} = e^{it} T V_t. $$

Thus $T$ is circular. The converse is easily proved by a simple calculation.

Theorem 11 Let $T$ be a closed densely defined operator in $\mathcal{H}$ with the polar decomposition $T = U |T|$. Then $T$ is $q$-circular if and only if the following statements hold:

1. There is a unitary operator $U_0$ on $\mathcal{H}$ that commutes with $U$ and satisfies

$$ U_0 |T| = q |T| U_0 \quad (4) $$

2. There is a family $\{V_t\}_{t \in \mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that

$$ V_t U = e^{it} U V_t \quad \text{and} \quad V_t |T| = |T| V_t \quad (5) $$

for all $t \in \mathbb{R}$.

Especially, if this is the case, $U$ is circular.
参考文献


