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Circular作用素について
(On circular operators)

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稠密な定義域を持つHilbert空間上の作用素に対して、‘circular’なる概念を導入する。この概念は、有界作用素に対してW. Mlak等によって、作用素論および量子力学の観点から研究されている。本講演ではcircular作用素に付随して現れる強連続1径数ユニタリー群について考察し、q-変形作用素族のq-正規作用素がもつ‘性質Q’との関わりについて述べる。

1 Circularity

A densely defined operator T in a Hilbert space $\mathcal{H}$ is said to be **circular** if T is unitarily equivalent to $e^{it}T$ for all $t \in \mathbb{R}$. Clearly the spectrum of a circular operator is circularly symmetric at origin.

Example 1 *Let S be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\}$ $(n \in \mathbb{Z})$ and a sequence $\{w_n\}(w_n \neq 0\ , n \in \mathbb{Z})$ of complex numbers such that*

$$\mathcal{D}(S) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \right\}$$

and

$$Se_n = w_n e_{n+1}$$

for all $n \in \mathbb{Z}$, then S is called a bilateral (injective) weighted shift with weights $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift is defined by the replacement $\mathbb{Z}$ with $\mathbb{N}$ analogously.

Every bilateral or unilateral, weighted shift is circular.

Let us recall irreducibility for a possibly unbounded operator in $\mathcal{H}$. Let T be a closed densely defined operator in $\mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to reduce T if the following two conditions are satisfied:

1. $P_{\mathcal{M}}\mathcal{D}(T) \subseteq \mathcal{D}(T)$.

2. $T(\mathcal{M} \cap \mathcal{D}(T) \subseteq \mathcal{M}$ and $T(\mathcal{M}^{\perp} \cap \mathcal{D}(T) \subseteq \mathcal{M}^{\perp}$.

Here $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. If there is no non-trivial reducing subspace of T, then T is said to be *irreducible*.

Lemma 2 *Let T be an irreducible, closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If T is circular, then there are a family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ and a mapping $m(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{T}$ such that*

1. $U_tT = \mathrm{e}^{it}\,TU_t$ *for all* $t \in \mathbb{R}$.

2. $U_sU_t = m(s,t)\,U_{s+t}$, $U_0 = I$ *(identity operator) for all* $s,t \in \mathbb{R}$.

3. *the map* $m(\cdot,\cdot)$ *satisfies*

$$m(s,0) = m(0,s) = 1 \quad \text{and} \quad m(s+t,u)\,m(s,t) = m(s,t+u)\,m(t,u)$$

for s, $t \in \mathbb{R}$.

Here, $\mathbb{T}$ is the multiplicative group of complex numbers with modulus 1.

Moreover, if the above $\{U_t\}$ is so chosen that $t \mapsto U_t$ is measurable, there exists a strongly continuous one-parameter unitary group $\{V_t\}$ satisfying the above condition 1, *that is,* $V_tT = \mathrm{e}^{it}\,TV_t$ *for all* $t \in \mathbb{R}$.

2 Strong circularity

Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. If there is a strongly continuous one-prameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ such that

$$U_tT = \mathrm{e}^{it}\,TU_t \quad (t \in \mathbb{R})\,,$$

then T is said to be **strongly circular** and $\{U_t\}_{t\in\mathbb{R}}$ is called a unitary group associated with T.

Example 3 (Mlak) *If S is the creation operator in a separable Hilbert space, that is, the unilateral weighted shift with weights $\{w_n\}$ given by $w_n = \sqrt{n+1}$ ($n \in \mathbb{N}$), then S is strongly circular.*

Let S be a unilateral or bilateral weighted shift in a separable Hilbert space $\mathcal{H}$. Then S is strongly circular.
In fact, let S be a bilateral weighted shift in $\mathcal{H}$ with with weights $\{w_n\}$ with respect to $\{e_n\}$. Define a closed densely defined operator by

$$\mathcal{D}(N) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |n|^2 < \infty \right\}$$

and

$$N e_n = n e_n \quad (n \in \mathbb{Z}) .$$

Then N is self-adjoint, and

$$\mathrm{e}^{itN} S e_n = \mathrm{e}^{it} S \mathrm{e}^{itN} e_n$$

for all $n \in \mathbb{Z}$. It follows that S is a strongly circular operator with the associated unitary group $\{\mathrm{e}^{itN}\}$.

For a bounded operator B and a densely defined operator T, $BT \subseteq TB$ means that

$$B\mathcal{D}(T) \subseteq \mathcal{D}(T) \text{ and } BT\eta = TB\eta \ (\eta \in \mathcal{D}(T)).$$

Lemma 4 *Let S be a densely defined operator in a Hilbert space $\mathcal{H}$ and T be a closed densely defined operator in $\mathcal{H}$. Let $\{U_t\}_{t\in\mathbb{R}}$ and $\{V_t\}_{t\in\mathbb{R}}$ be strongly continuous one-prameter unitary groups on $\mathcal{H}$ with infinitesimal generators A and B respectively, that is, $U_t = \mathrm{e}^{itA}$, $V_t = \mathrm{e}^{itB}$. Then the following conditions are equivalent:*

1. *For all $t \in \mathbb{R}$,*

$$U_t\, S \subseteq T\, V_t .$$

2. *For all $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$,*

$$(\lambda - A)^{-1} S \subseteq T\, (\lambda - B)^{-1} .$$

Theorem 5 *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then T is strongly circular if and only if there is a self-adjoint operator A in $\mathcal{H}$ such that*

$$(\lambda - A)^{-1} T \subseteq T\, (\lambda - I - A)^{-1} \tag{1}$$

for all $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$.

Proof. Suppose T is strongly circular. Then there is a strongly continuous one-prameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ such that $U_tT = e^{it}TU_t$ for all $t \in \mathbb{R}$. Set

$$V_t = e^{it}U_t \quad \text{for each } t \in \mathbb{R}.$$

Then, $\{V_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-prameter unitary group on $\mathcal{H}$. Let A be the infinitesimal generator of $\{U_t\}_{t\in\mathbb{R}}$. Then it follows from the semigroup theory that the infinitesimal generator of $\{V_t\}_{t\in\mathbb{R}}$ is $I + A$. Putting $T = S$ in the above, A satisfies relation (1).

Conversely, suppose A is a self-adjoint operator satisfying relation (1). Put

$$U_t = e^{itA} \quad \text{and} \quad V_t = e^{it}e^{itA}$$

for $t \in \mathbb{R}$. Then we obtain $U_tT \subseteq e^{it}TU_t$ for all $t \in \mathbb{R}$. Since each U_t is unitary, $U_t\mathcal{D}(T) = \mathcal{D}(T)$. Hence, $U_tT = e^{it}TU_t$ for all $t \in \mathbb{R}$.

3 q-deformed circularity

Let T be a densely defined operator in a Hilbert space $\mathcal{H}$. If there is a positive real number q with $q \neq 1$ such that T is unitarily equivalent to qT, then we say that T **has property *Q***.

Proposition 6 . *Suppose that a nontrivial closed densely defined operator T has property* Q. *Then,*

1. *T is unbounded.*
2. *The spectrum contains zero.*
3. *The absolute value $|T|$ has also property* Q.

Example 7 *Let T be a closed densely defined operator in $\mathcal{H}$. If T satisfies*

$$TT^* = qT^*T \quad (q > 0,\, q \neq 1),$$

then T is called a q-normal operator. It should be noticed that elements satisfying this relation in a formal algebraic sense appear at various circumstances in the theory of quantum group theory. A non-trivial q-normal operator T is always unbounded and has sufficient large spectrum in the sense of the planar Lebesgue measure. Especially, every q-normal operator T is unitarily equivalent to qT. Thus the class of operators possessing property Q *contains all q-normal operators.*

Definition 8 Let T be a densely defined operator in $\mathcal{H}$. If there is a positive real number q with $q \neq 1$ such that T is unitarily equivalent to $qe^{it}T$ for all $t \in \mathbb{R}$, then T is called a *q-deformed circular(simply, q-circular) operator.*

Circularity may be considered as q tends to 1 in the above. Clearly, a q-circular operator has property Q.

Example 9 (q-circular weighted shifts) *If a bilateral weighted shift has property* Q, *then it is q-circular. Hence, a q-normal bilateral weighted shift is q-circular. Moreover the spectrum of a q-circular weighted shift is equal to the whole complex plain.*

Theorem 10 *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then T is q-circular if and only if T is circular and has property* Q.

Proof. Suppose T is q-circular. Then there is a family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that

$$U_t T = qe^{it}TU_t \ . \tag{2}$$

for all $t \in \mathbb{R}$. It is clear that T has property Q. Put

$$V_t = U_t U_0^{-1} \ . \tag{3}$$

for all $t \in \mathbb{R}$. We have by above relation (2)

$$\begin{aligned} V_t T &= qU_t U_0^{-1} T = U_t T U_0 \\ &= qe^{it} T U_t U_0^{-1} = e^{it} T V_t \ . \end{aligned}$$

Thus T is circular. The converse is easily proved by a simple calculation.

Theorem 11 *Let T be a closed densely defined operator in $\mathcal{H}$ with the polar decomposition $T = U\,|T|$. Then T is q-circular if and only if the following statements hold:*

1. *There is a unitary operator U_0 on $\mathcal{H}$ that commutes with U and satisfies*

$$U_0\,|T| = q\,|T|\,U_0 \, . \tag{4}$$

2. *There is a family $\{V_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that*

$$V_t U = e^{it} U V_t \quad \text{and} \quad V_t\,|T| = |T|\,V_t \tag{5}$$

   for all $t \in \mathbb{R}$.

Especially, if this is the case, U is circular.

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