Title: On circular operators (Prospects of non-commutative analysis in operator theory)

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Circular 作用素について
(On circular operators)
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1  Circularity

A densely defined operator $T$ in a Hilbert space $\mathcal{H}$ is said to be circular if $T$ is unitarily equivalent to $e^{it}T$ for all $t \in \mathbb{R}$. Clearly the spectrum of a circular operator is circularly symmetric at origin.

**Example 1** Let $S$ be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\} (n \in \mathbb{Z})$ and a sequence $\{w_n\} (w_n \neq 0, n \in \mathbb{Z})$ of complex numbers such that

$$\mathcal{D}(S) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \right\}$$

and

$$Se_n = w_n e_{n+1}$$

for all $n \in \mathbb{Z}$, then $S$ is called a bilateral (injective) weighted shift with weights $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift is defined by the replacement $\mathbb{Z}$ with $\mathbb{N}$ analogously.

Every bilateral or unilateral, weighted shift is circular.

Let us recall irreducibility for a possibly unbounded operator in $\mathcal{H}$. Let $T$ be a closed densely defined operator in $\mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to reduce $T$ if the following two conditions are satisfied:
1. $P_{\mathcal{M}}D(T) \subseteq D(T)$.

2. $T(\mathcal{M} \cap D(T)) \subseteq \mathcal{M}$ and $T(\mathcal{M}^\perp \cap D(T)) \subseteq \mathcal{M}^\perp$.

Here $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. If there is no non-trivial reducing subspace of $T$, then $T$ is said to be irreducible.

**Lemma 2** Let $T$ be an irreducible, closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If $T$ is circular, then there are a family $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators on $\mathcal{H}$ and a mapping $m(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{T}$ such that

1. $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$.

2. $U_s U_t = m(s, t) U_{s+t}$, $U_0 = I$ (identity operator) for all $s, t \in \mathbb{R}$.

3. the map $m(\cdot, \cdot)$ satisfies

$$m(s, 0) = m(0, s) = 1 \quad \text{and} \quad m(s + t, u) m(s, t) = m(s, t + u) m(t, u)$$

for $s, t \in \mathbb{R}$.

Here, $\mathbb{T}$ is the multiplicative group of complex numbers with modulus 1.

Moreover, if the above $\{U_t\}$ is so chosen that $t \mapsto U_t$ is measurable, there exists a strongly continuous one-parameter unitary group $\{V_t\}$ satisfying the above condition 1, that is, $V_t T = e^{it} T V_t$ for all $t \in \mathbb{R}$.

## 2 Strong circularity

Let $T$ be a closed densely defined operator in a Hilbert space $\mathcal{H}$. If there is a strongly continuous one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$ such that

$$U_t T = e^{it} T U_t \quad (t \in \mathbb{R}),$$

then $T$ is said to be **strongly circular** and $\{U_t\}_{t \in \mathbb{R}}$ is called a unitary group associated with $T$.

**Example 3** (Mlak) If $S$ is the creation operator in a separable Hilbert space, that is, the unilateral weighted shift with weights $\{w_n\}$ given by $w_n = \sqrt{n+1}$ $(n \in \mathbb{N})$, then $S$ is strongly circular.
Let $S$ be a unilateral or bilateral weighted shift in a separable Hilbert space $\mathcal{H}$. Then $S$ is strongly circular.

In fact, let $S$ be a bilateral weighted shift in $\mathcal{H}$ with weights $\{w_n\}$ with respect to $\{e_n\}$. Define a closed densely defined operator by

$$\mathcal{D}(N) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |n|^2 < \infty \right\}$$

and

$$Ne_n = ne_n \ (n \in \mathbb{Z}).$$

Then $N$ is self-adjoint, and

$$e^{itN} S e_n = e^{it} S e^{itN} e_n$$

for all $n \in \mathbb{Z}$. It follows that $S$ is a strongly circular operator with the associated unitary group $\{e^{itN}\}$.

For a bounded operator $B$ and a densely defined operator $T$, $BT \subseteq TB$ means that

$$BD(T) \subseteq D(T) \text{ and } BT \eta = TB \eta \ (\eta \in D(T)).$$

**Lemma 4** Let $S$ be a densely defined operator in a Hilbert space $\mathcal{H}$ and $T$ be a closed densely defined operator in $\mathcal{H}$. Let $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ be strongly continuous one-parameter unitary groups on $\mathcal{H}$ with infinitesimal generators $A$ and $B$ respectively, that is, $U_t = e^{itA}$, $V_t = e^{itB}$. Then the following conditions are equivalent:

1. For all $t \in \mathbb{R}$,
   $$U_t S \subseteq TV_t.$$

2. For all $\lambda \in \mathbb{C}$ with $\Im \lambda \neq 0$,
   $$(\lambda - A)^{-1} S \subseteq T(\lambda - B)^{-1}.$$

**Theorem 5** Let $T$ be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then $T$ is strongly circular if and only if there is a self-adjoint operator $A$ in $\mathcal{H}$ such that

$$(\lambda - A)^{-1} T \subseteq T(\lambda - I - A)^{-1}$$

for all $\lambda \in \mathbb{C}$ with $\Im \lambda \neq 0$. 

130
Proof. Suppose $T$ is strongly circular. Then there is a strongly continuous one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$ such that $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$. Set

$$V_t = e^{it} U_t$$

for each $t \in \mathbb{R}$.

Then, $\{V_t\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Let $A$ be the infinitesimal generator of $\{U_t\}_{t \in \mathbb{R}}$. Then it follows from the semigroup theory that the infinitesimal generator of $\{V_t\}_{t \in \mathbb{R}}$ is $I + A$. Putting $T = S$ in the above, $A$ satisfies relation (1).

Conversely, suppose $A$ is a self-adjoint operator satisfying relation (1). Put

$$U_t = e^{itA} \quad \text{and} \quad V_t = e^{it} e^{itA}$$

for $t \in \mathbb{R}$. Then we obtain $U_t T \subseteq e^{it} T U_t$ for all $t \in \mathbb{R}$. Since each $U_t$ is unitary, $U_t \mathcal{D}(T) = \mathcal{D}(T)$. Hence, $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$.

3 $q$-deformed circularity

Let $T$ be a densely defined operator in a Hilbert space $\mathcal{H}$. If there is a positive real number $q$ with $q \neq 1$ such that $T$ is unitarily equivalent to $qT$, then we say that $T$ has property $Q$.

Proposition 6. Suppose that a nontrivial closed densely defined operator $T$ has property $Q$. Then,

1. $T$ is unbounded.
2. The spectrum contains zero.
3. The absolute value $|T|$ has also property $Q$.

Example 7. Let $T$ be a closed densely defined operator in $\mathcal{H}$. If $T$ satisfies

$$TT^* = q T^* T \quad (q > 0, q \neq 1),$$

then $T$ is called a $q$-normal operator. It should be noticed that elements satisfying this relation in a formal algebraic sense appear at various circumstances in the theory of quantum group theory. A non-trivial $q$-normal operator $T$ is always unbounded and has sufficient large spectrum in the sense of the planar Lebesgue measure. Especially, every $q$-normal operator $T$ is unitarily equivalent to $qT$. Thus the class of operators possessing property $Q$ contains all $q$-normal operators.
**Definition 8** Let $T$ be a densely defined operator in $\mathcal{H}$. If there is a positive real number $q$ with $q \neq 1$ such that $T$ is unitarily equivalent to $qe^{it}T$ for all $t \in \mathbb{R}$, then $T$ is called a $q$-deformed circular (simply, $q$-circular) operator.

Circularity may be considered as $q$ tends to 1 in the above. Clearly, a $q$-circular operator has property $Q$.

**Example 9 (q-circular weighted shifts)** If a bilateral weighted shift has property $Q$, then it is $q$-circular. Hence, a $q$-normal bilateral weighted shift is $q$-circular. Moreover the spectrum of a $q$-circular weighted shift is equal to the whole complex plain.

**Theorem 10** Let $T$ be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then $T$ is $q$-circular if and only if $T$ is circular and has property $Q$.

**Proof.** Suppose $T$ is $q$-circular. Then there is a family $\{U_t\}_{t\in \mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that

$$U_t T = q e^{it} T U_t .$$  \hfill (2)

for all $t \in \mathbb{R}$. It is clear that $T$ has property $Q$. Put

$$V_t = U_t U_0^{-1} .$$  \hfill (3)

for all $t \in \mathbb{R}$. We have by above relation (2)

$$V_t T = q U_t U_0^{-1} T = U_t T U_0$$

$$= q e^{it} U_t U_0^{-1} = e^{it} T V_t .$$

Thus $T$ is circular. The converse is easily proved by a simple calculation.

**Theorem 11** Let $T$ be a closed densely defined operator in $\mathcal{H}$ with the polar decomposition $T = U |T|$. Then $T$ is $q$-circular if and only if the following statements hold:

1. There is a unitary operator $U_0$ on $\mathcal{H}$ that commutes with $U$ and satisfies

$$U_0 |T| = q |T| U_0 .$$  \hfill (4)

2. There is a family $\{V_t\}_{t\in \mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that

$$V_t U = e^{it} U V_t$$

and

$$V_t |T| = |T| V_t$$  \hfill (5)

for all $t \in \mathbb{R}$.

Especially, if this is the case, $U$ is circular.
参考文献


