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Positive Definite Kernels and Majorization

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1 Introduction

Definition 1.1 Let $f(t)$ be a real continuous function defined on $I$, and consider the functional calculus $f(X)$ for a Hermitian matrix $X$ with eigenvalues in $I$.

- $f$ is called an operator monotone function on $I$ if $f(A) \leqq f(B)$ whenever $A \leqq B$ (of any order $n$).

- $f$ is said to be operator decreasing if $-f$ is operator monotone.

- $f$ is called an operator convex function on $I$ if $f(sA + (1 - s)B) \leqq sf(A) + (1 - s)f(B)$ $(0 < s < 1)$ for every pair of bounded Hermitian operators $A$ and $B$ whose spectra are both in $I$.

- An operator concave function is likewise defined.

Definition 1.2 Let $K(t, s)$ be a real, continuous and symmetric function defined on $I \times I$. 
$ullet$ $K(t, s)$ is called a positive semi-definite kernel on $I$ if

$$\int_{I \times I} \int K(t, s) \phi(t) \phi(s) dt \, ds \geq 0$$

(1)

for all real continuous functions $\phi$ with compact support in $I$.

**Remark** It is evident that $K(t, s)$ is positive semi-definite on $I$ if and only if for each $n$ and for all $n$ points $t_i \in I$ the $n \times n$ matrices

$$(K(t_i, t_j))_{i,j=1}^{n}$$

are positive semi-definite.

$ullet$ Suppose $K(t, s) \geq 0$ for every $t, s$ in $I$. Then the kernel $K(t, s)$ is said to be infinitely divisible on $I$ if $K(t, s)^r$ is a positive semi-definite kernel for every $r > 0$, i.e.,

$$\int_{I \times I} \int K(t, s)^r \phi(t) \phi(s) dt \, ds \geq 0$$

$ullet$ A kernel $K(t, s)$ is said to be conditionally positive semi-definite on $I$ if $\int_{I \times I} \int K(t, s) \phi(t) \phi(s) dt \, ds \geq 0$ for $\phi$ such that the support of $\phi$ is compact and $\int_I \phi(t) dt = 0$.

$ullet$ A kernel $K(t, s)$ is said to be conditionally negative semi-definite on $I$ if $-K(t, s)$ is conditionally positive semi-definite on $I$.

(Łöwner) $C^1$ function $f$ is operator monotone on $I$ if and only if the Łöwner kernel $K_f(t, s)$ defined by

$$K_f(t, s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t, t) = f'(t),$$
is positive semi-definite on $I$. (F. Krauss, J. Bendat- S. Sherman)

$g(t)$ is an operator convex function on $I$ if and only if $g(t)$ is of class $C^2(I)$ and for each $t_0 \in I$, the function $f(t)$ defined by

$$f(t) = \frac{g(t) - g(t_0)}{t - t_0} \quad (t \neq t_0), \quad f(t_0) = g'(t_0)$$

is operator monotone on $I$.

2 Operator convex functions

Proposition 2.1 Let $f(t)$ be an operator monotone (or decreasing) function on $I$. Then the indefinite integral $\int f(t)dt$ is an operator convex (or concave) function on $I$.

Example 2.1 $\int \log t dt = t \log t - t$, hence $t \log t$ and $\log \Gamma(t) = \int \frac{\Gamma'(t)}{\Gamma(t)}dt$ are both operator convex on $(0, \infty)$

But the converse is not true; $\frac{1}{t}$ on $(0, \infty)$ is a counter example.

Proposition 2.2 Let $g(t)$ be an operator convex function on $(0, \infty)$. Then $g'(\sqrt{t})$ is operator monotone there.

(Well-known) Let $f(t) \geq 0$ be defined on $[0, \infty)$. Then $f$ is operator monotone $\iff f(t)$ is operator concave.

Theorem 2.3 Let $f(t)$ be defined on $(a, \infty)$ with $a \geq -\infty$. Then

(i) $f(t)$ is operator decreasing $\iff f(t)$ is operator convex and $f(\infty) = \lim_{t \to \infty} f(t) < \infty$;
(ii) $f(t)$ is operator monotone $\iff$ $f(t)$ is operator concave and $f(\infty) > -\infty$.

In (ii) the condition "$f(\infty) > -\infty$" is indispensable; for instance, $f(t) = -t^2$ is operator concave on $(0, \infty)$ but not operator monotone there.

**Corollary 2.4** Let $f(t)$ be defined on $(-\infty, b)$, where $b \leq \infty$. Then

(i) $f(t)$ is operator monotone on $(-\infty, b) \iff f(t)$ is operator convex on $(-\infty, b)$ and $f(-\infty) < \infty$

(ii) $f(t)$ is operator decreasing on $(-\infty, b) \iff f(t)$ is operator concave on $(-\infty, b)$ and $f(-\infty) > -\infty$.

**Corollary 2.5** *(Well-known)* Let $f(t)$ be defined on $(-\infty, \infty)$. Then $f(t)$ is operator monotone on $(-\infty, \infty)$ $\iff f(t) = at + b$ ($a \geq 0$).

How about the case of finite intervals? $\tan t$ is operator monotone on $(-\pi/2, \pi/2)$.

**Proposition 2.6** Let $f(t)$ be an operator monotone function on a finite interval $(a, b)$. Then there is a decomposition of $f(t)$ such that

$$f(t) = f_+(t) + f_-(t) \quad (a < t < b)$$

where $f_+(t)$ and $f_-(t)$ are operator monotone on $(a, \infty)$ and $(-\infty, b)$ respectively.
3 Löwner kernels

(Bhatia and Sano) Let \( f(t) \) be a \( C^2 \) function on \([0, \infty)\) such that \( f(t) \geq 0 \) and \( f(0) = f'(0) = 0 \). Then \( f \) is operator convex on \([0, \infty)\) ⇔ the Löwner kernel \( K_f(t, s) \) is conditionally negative semi-definite on \([0, \infty)\), where

\[
K_f(t, s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t, t) = f'(t),
\]

**Proposition 3.1** Let \( f(t) \) be a \( C^1 \) function on \((a, \infty)\). Then

(i) \( f(t) \) is operator convex on \((a, \infty)\) ⇔

the Löwner kernel \( K_f(t, s) \) is conditionally negative semi-definite
and \( \lim_{t \to \infty} \frac{f(t)}{t} > -\infty \);

(ii) \( f(t) \) is operator concave on \((a, \infty)\) ⇔ the Löwner kernel \( K_f(t, s) \) is

conditionally positive semi-definite and \( \lim_{t \to \infty} \frac{f(t)}{t} < \infty \).

In (i) the condition "\( \lim_{t \to \infty} \frac{f(t)}{t} > -\infty \)" is indispensible: in fact, the Löwner kernel \( K_f(t, s) = -(t^2 + st + s^2) \) of \( f(t) = -t^3 \) is conditionally negative on \((0, \infty)\), but \( f(t) \) is not operator convex there.

**Theorem 3.2** Let \( f(t) \) be \( C^1 \) function on \((a, \infty)\). Then the following hold:

(i) the Löwner kernel \( K_f(t, s) \) is positive semi-definite on \((a, \infty)\) if and

only if \( K_f(t, s) \) is conditionally positive semi-definite on \((a, \infty)\),

\[
\lim_{t \to \infty} \frac{f(t)}{t} < \infty, \text{ and } f(\infty) > -\infty;
\]
(ii) $K_f(t, s)$ is negative semi-definite on $(a, \infty)$ if and only if $K_f(t, s)$ is conditionally negative semi-definite on $(a, \infty)$, $\lim_{t \to \infty} \frac{f(t)}{t} > -\infty$, and $f(\infty) < \infty$.

**Corollary 3.3** Let $f(t)$ be a $C^1$ function on $(-\infty, b)$. Then

(i) $f(t)$ is operator convex on $(-\infty, b)$ if and only if the Löwner kernel $K_f(t, s)$ is conditionally positive semi-definite; $\lim_{t \to -\infty} \frac{f(t)}{t} < \infty$.

(ii) $f(t)$ is operator concave on $(-\infty, b)$ if and only if the Löwner kernel $K_f(t, s)$ is conditionally negative semi-definite, and $\lim_{t \to -\infty} \frac{f(t)}{t} > -\infty$.

**Corollary 3.4** Let $f(t)$ be $C^1$ function on $(-\infty, b)$. Then the following hold:

(i) the Löwner kernel $K_f(t, s)$ is positive semi-definite on $(-\infty, b)$ if and only if $K_f(t, s)$ is conditionally positive semi-definite on $(-\infty, b)$, $\lim_{t \to -\infty} \frac{f(t)}{t} < \infty$, and $f(-\infty) < \infty$;

(ii) the Löwner kernel $K_f(t, s)$ is negative semi-definite on $(-\infty, b)$ if and only if $K_f(t, s)$ is conditionally negative semi-definite on $(-\infty, b)$, $\lim_{t \to -\infty} \frac{f(t)}{t} > -\infty$, and $f(-\infty) > -\infty$.

**4 Majorization and kernel functions**

**Definition 4.1** Let $h(t)$ and $g(t)$ be $C^1$ functions on $I$, and suppose that $g(t)$ is increasing. Then $h$ is said to be majorized by $g$ and denoted by
$h \preceq g$ on $I$ if

$h(A) \leq h(B)$ whenever $g(A) \leq g(B)$ for $A, B$ whose spectra are both in $I$.

\[ f(t) \preceq t \text{ on } I \iff f(t) \text{ is operator monotone on } I. \]

**Definition 4.2** Let $h(t)$ and $g(t)$ be $C^1$ functions on $I$, and suppose that $g(t)$ is increasing. Then the kernel $K_{h,g}(t, s)$ defined by

\[
K_{h,g}(t, s) = \frac{h(t) - h(s)}{g(t) - g(s)} \quad (s \neq t), \quad K_{h,g}(t, t) = \frac{h'(t)}{g'(t)}.
\]

is continuous and symmetric.

- A Löwner kernel $K_f(t, s)$ can be written as $K_{f,t}(t, s)$.

**Proposition 4.1** The following statements are equivalent:

(i) The kernel $K_{h,g}(t, s)$ is positive semi-definite on $I$.

(ii) There is an operator monotone function $\varphi$ defined on $g(I)$ such that

\[
h(t) = (\varphi \circ g)(t) \quad (t \in I).
\]

(iii) $h \preceq g$ on $I$.

**Lemma 4.2** Let $h(t)$ and $g(t)$ be positive $C^1$ functions on an open interval $I$. Suppose $h(t)g(t)$ is increasing and its range is $(0, \infty)$. Then the kernel $K_{h,hg}$ is positive semi-definite on $I$ if and only if so is the kernel $K_{g,hg}$.
**Theorem 4.3** Let $h(t)$ and $g(t)$ be positive $C^1$ functions defined on $I$. Suppose $g$ is increasing and its range is $(0, \infty)$. If the kernel $K_{h,g}$ is positive semi-definite on $I$, then for $0 \leq i \leq n$, $0 \leq j \leq m$, $1 \leq m$, $i + j + 1 \leq n + m$

$$K_{h^i g^j, h^m g^n}(t, s) = \frac{h^i(t)g^j(t) - h^i(s)g^j(s)}{h^m(t)g^n(t) - h^n(s)g^m(s)}$$

is infinitely divisible.

Moreover, if $f$ is a (not necessarily positive) $C^1$ function such that the kernel $K_{f,g}(t, s)$ is positive semi-definite, then the kernel

$$K_{g, f^i g^j}(t, s)$$

is infinitely divisible.

**Example 4.1** (1). For $f(t) \leq t$ on $(0, \infty)$

$$\frac{f(t)^i t^j - f(s)^i s^j}{f(t)^n t^m - f(s)^n s^m}$$

where $0 \leq i \leq n$, $0 \leq j \leq m$, $1 \leq m$, $i + j + 1 \leq n + m$,

$$1 \leq n + 1 \leq n,$$

$$\frac{1}{t + s} (\text{Cauchy kernel}), \quad \frac{t - s}{te^{-1/t} - se^{-1/s}}$$

are all infinitely divisible kernels on $(0, \infty)$.

(2). Consider a polynomial

$p(t) := \prod_{i=1}^{n}(t - a_i)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then the kernel

$$K_{t, p(t)}(t, s) = \frac{t - s}{p(t) - p(s)}$$

is infinitely divisible on $(a_1, \infty)$.
**Theorem 4.4** Let $h(t)$ and $g(t)$ be positive $C^1$ functions defined on an open interval $(a, b)$, where $-\infty \leq a < b \leq \infty$. Suppose the range of $g$ is $(0, \infty)$. Then the following are equivalent:

(i) the kernel $K_{h,g}$ is conditionally negative;

(ii) there is an operator convex function $\varphi$ defined on $(0, \infty)$ such that $\varphi(g(t)) = h(t)$ for $t \in (a, b)$.

(iii) $\frac{h(t) - h(a+0)}{g(t)} \preceq g(t)$ \quad ($a < t < b$)