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Positive Definite Kernels and Majorization

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1 Introduction

Definition 1.1 Let \( f(t) \) be a real continuous function defined on \( I \), and consider the functional calculus \( f(X) \) for a Hermitian matrix \( X \) with eigenvalues in \( I \).

- \( f \) is called an operator monotone function on \( I \) if \( f(A) \leqq f(B) \) whenever \( A \leqq B \) (of any order \( n \)).

- \( f \) is said to be operator decreasing if \(-f\) is operator monotone.

- \( f \) is called an operator convex function on \( I \) if \( f(sA + (1 - s)B) \leqq sf(A) + (1 - s)f(B) \) \((0 < s < 1)\) for every pair of bounded Hermitian operators \( A \) and \( B \) whose spectra are both in \( I \).

- An operator concave function is likewise defined.

Definition 1.2 Let \( K(t, s) \) be a real, continuous and symmetric function defined on \( I \times I \).
• $K(t, s)$ is called a positive semi-definite kernel on $I$ if

$$\iint_{I \times I} K(t, s) \phi(t) \phi(s) dt \, ds \geq 0$$

(1)

for all real continuous functions $\phi$ with compact support in $I$.

Remark It is evident that $K(t, s)$ is positive semi-definite on $I$ if and only if for each $n$ and for all $n$ points $t_i \in I$ the $n \times n$ matrices

$$(K(t_i, t_j))_{i,j=1}^n$$

are positive semi-definite.

• Suppose $K(t, s) \geq 0$ for every $t, s$ in $I$. Then the kernel $K(t, s)$ is said to be infinitely divisible on $I$ if $K(t, s)^r$ is a positive semi-definite kernel for every $r > 0$, i.e.,

$$\iint_{I \times I} K(t, s)^r \phi(t) \phi(s) dt \, ds \geq 0$$

• A kernel $K(t, s)$ is said to be conditionally positive semi-definite on $I$ if $\iint_{I \times I} K(t, s) \phi(t) \phi(s) dt \, ds \geq 0$ for $\phi$ such that the support of $\phi$ is compact and $\int_I \phi(t) dt = 0$.

• A kernel $K(t, s)$ is said to be conditionally negative semi-definite on $I$ if $-K(t, s)$ is conditionally positive semi-definite on $I$.

(Löwner) $C^1$ function $f$ is operator monotone on $I$ if and only if the Löwner kernel $K_f(t, s)$ defined by

$$K_f(t, s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t, t) = f'(t),$$
is positive semi-definite on $I$. (F. Krauss, J. Bendat- S. Sherman)

$g(t)$ is an operator convex function on $I$ if and only if $g(t)$ is of class $C^2(I)$
and for each $t_0 \in I$, the function $f(t)$ defined by

$$f(t) = \frac{g(t) - g(t_0)}{t - t_0} \quad (t \neq t_0), \quad f(t_0) = g'(t_0)$$

is operator monotone on $I$.

## 2 Operator convex functions

**Proposition 2.1** Let $f(t)$ be an operator monotone (or decreasing) function on $I$. Then the indefinite integral $\int f(t)dt$ is an operator convex (or concave) function on $I$.

**Example 2.1** $\int \log t dt = t \log t - t$, hence $t \log t$ and $\log \Gamma(t) = \int \frac{\Gamma'(t)}{\Gamma(t)} dt$
are both operator convex on $(0, \infty)$

But the converse is not true; $\frac{1}{t}$ on $(0, \infty)$ is a counter example.

**Proposition 2.2** Let $g(t)$ be an operator convex function on $(0, \infty)$. Then $g'(\sqrt{t})$ is operator monotone there.

(Well-known) Let $f(t) \geq 0$ be defined on $[0, \infty)$. Then $f$ is operator monotone $\iff f(t)$ is operator concave.

**Theorem 2.3** Let $f(t)$ be defined on $(a, \infty)$ with $a \geq -\infty$. Then

(i) $f(t)$ is operator decreasing $\iff f(t)$ is operator convex and $f(\infty) = \lim_{t \to \infty} f(t) < \infty$;
(ii) $f(t)$ is operator monotone $\iff f(t)$ is operator concave and $f(\infty) > -\infty$.

In (ii) the condition "$f(\infty) > -\infty$" is indispensable; for instance, $f(t) = -t^2$ is operator concave on $(0, \infty)$ but not operator monotone there.

**Corollary 2.4** Let $f(t)$ be defined on $(-\infty, b)$, where $b \leq \infty$. Then

(i) $f(t)$ is operator monotone on $(-\infty, b) \iff f(t)$ is operator convex on $(-\infty, b)$ and $f(-\infty) < \infty$

(ii) $f(t)$ is operator decreasing on $(-\infty, b) \iff f(t)$ is operator concave on $(-\infty, b)$ and $f(-\infty) > -\infty$.

**Corollary 2.5** *(Well-known)* Let $f(t)$ be defined on $(-\infty, \infty)$. Then

$f(t)$ is operator monotone on $(-\infty, \infty)$

$\iff f(t) = at + b \ (a \geq 0)$.

How about the case of finite intervals? $\tan t$ is operator monotone on $(-\pi/2, \pi/2)$.

**Proposition 2.6** Let $f(t)$ be an operator monotone function on a finite interval $(a, b)$. Then there is a decomposition of $f(t)$ such that

$$f(t) = f_+(t) + f_-(t) \quad (a < t < b)$$

where $f_+(t)$ and $f_-(t)$ are operator monotone on $(a, \infty)$ and $(-\infty, b)$ respectively.
3 Löwner kernels

(Bhatia and Sano) Let $f(t)$ be a $C^2$ function on $[0, \infty)$ such that $f(t) \geq 0$ and $f(0) = f'(0) = 0$. Then $f$ is operator convex on $[0, \infty)$ ⇔ the Löwner kernel $K_f(t, s)$ is conditionally negative semi-definite on $[0, \infty)$, where

$$K_f(t, s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t, t) = f'(t),$$

Proposition 3.1 Let $f(t)$ be a $C^1$ function on $(a, \infty)$. Then

(i) $f(t)$ is operator convex on $(a, \infty)$ ⇔

the Löwner kernel $K_f(t, s)$ is conditionally negative semi-definite and $$\lim_{t \to \infty} \frac{f(t)}{t} > -\infty;$$

(ii) $f(t)$ is operator concave on $(a, \infty)$ ⇔ the Löwner kernel $K_f(t, s)$ is conditionally positive semi-definite and $$\lim_{t \to \infty} \frac{f(t)}{t} < \infty.$$

In (i) the condition "$\lim_{t \to \infty} \frac{f(t)}{t} > -\infty$" is indispensible: in fact, the Löwner kernel $K_f(t, s) = -(t^2 + st + s^2)$ of $f(t) = -t^3$ is conditionally negative on $(0, \infty)$, but $f(t)$ is not operator convex there.

Theorem 3.2 Let $f(t)$ be $C^1$ function on $(a, \infty)$. Then the following hold:

(i) the Löwner kernel $K_f(t, s)$ is positive semi-definite on $(a, \infty)$ if and only if $K_f(t, s)$ is conditionally positive semi-definite on $(a, \infty)$, $$\lim_{t \to \infty} \frac{f(t)}{t} < \infty,$$ and $f(\infty) > -\infty;$
(ii) \( K_f(t, s) \) is negative semi-definite on \((a, \infty)\) if and only if \( K_f(t, s) \) is conditionally negative semi-definite on \((a, \infty)\), \( \lim_{t \to \infty} \frac{f(t)}{t} > -\infty \), and \( f(\infty) < \infty \).

**Corollary 3.3** Let \( f(t) \) be a \( C^1 \) function on \((-\infty, b)\). Then

(i) \( f(t) \) is operator convex on \((-\infty, b)\) if and only if the Löwner kernel \( K_f(t, s) \) is conditionally positive semi-definite; \( \lim_{t \to -\infty} \frac{f(t)}{t} < \infty \).

(ii) \( f(t) \) is operator concave on \((-\infty, b)\) if and only if the Löwner kernel \( K_f(t, s) \) is conditionally negative semi-definite, and \( \lim_{t \to -\infty} \frac{f(t)}{t} > -\infty \).

**Corollary 3.4** Let \( f(t) \) be \( C^1 \) function on \((-\infty, b)\). Then the following hold:

(i) the Löwner kernel \( K_f(t, s) \) is positive semi-definite on \((-\infty, b)\) if and only if \( K_f(t, s) \) is conditionally positive semi-definite on \((-\infty, b)\),
\[
\lim_{t \to -\infty} \frac{f(t)}{t} < \infty, \text{ and } f(-\infty) < \infty;
\]

(ii) the Löwner kernel \( K_f(t, s) \) is negative semi-definite on \((-\infty, b)\) if and only if \( K_f(t, s) \) is conditionally negative semi-definite on \((-\infty, b)\),
\[
\lim_{t \to -\infty} \frac{f(t)}{t} > -\infty, \text{ and } f(-\infty) > -\infty.
\]

4 Majorization and kernel functions

**Definition 4.1** Let \( h(t) \) and \( g(t) \) be \( C^1 \) functions on \( I \), and suppose that \( g(t) \) is increasing. Then \( h \) is said to be majorized by \( g \) and denoted by
$h \preceq g$ on $I$ if

$h(A) \leq h(B)$ whenever $g(A) \leq g(B)$ for $A, B$ whose spectra are both in $I$.

• $f(t) \preceq t$ on $I \iff f(t)$ is operator monotone on $I$.

**Definition 4.2** Let $h(t)$ and $g(t)$ be $C^1$ functions on $I$, and suppose that $g(t)$ is increasing. Then the kernel $K_{h,g}(t,s)$ defined by

$$K_{h,g}(t,s) = \frac{h(t) - h(s)}{g(t) - g(s)} \quad (s \neq t), \quad K_{h,g}(t,t) = \frac{h'(t)}{g'(t)}.$$

is continuous and symmetric.

• A Löwner kernel $K_f(t,s)$ can be written as $K_{f,t}(t,s)$.

**Proposition 4.1** The following statements are equivalent:

(i) The kernel $K_{h,g}(t,s)$ is positive semi-definite on $I$.

(ii) There is an operator monotone function $\varphi$ defined on $g(I)$ such that

$$h(t) = (\varphi \circ g)(t) \quad (t \in I).$$

(iii) $h \preceq g$ on $I$.

**Lemma 4.2** Let $h(t)$ and $g(t)$ be positive $C^1$ functions on an open interval $I$. Suppose $h(t)g(t)$ is increasing and its range is $(0, \infty)$. Then the kernel $K_{h,hg}$ is positive semi-definite on $I$ if and only if so is the kernel $K_{g,hg}$. 
Theorem 4.3 Let $h(t)$ and $g(t)$ be positive $C^1$ functions defined on $I$. Suppose $g$ is increasing and its range is $(0, \infty)$. If the kernel $K_{h,g}$ is positive semi-definite on $I$, then for $0 \leq i \leq n$, $0 \leq j \leq m$, $1 \leq m$, $i + j + 1 \leq n + m$

$$K_{h^i,g^j,h^n,g^m}(t, s) = \frac{h^i(t)g^j(t) - h^i(s)g^j(s)}{h^n(t)g^m(t) - h^n(s)g^m(s)}$$

is infinitely divisible.

Moreover, if $f$ is a (not necessarily positive) $C^1$ function such that the kernel $K_{f,g}(t, s)$ is positive semi-definite, then the kernel

$$K_{g,f,g}(t, s)$$

is infinitely divisible.

Example 4.1 (1). For $f(t) \preceq t$ on $(0, \infty)$

$$\frac{f(t)^i t^j - f(s)^i s^j}{f(t)^n t^m - f(s)^n s^m}$$

where $0 \leq i \leq n$, $0 \leq j \leq m$, $1 \leq m$, $i + j + 1 \leq n + m$, $1 \leq n + 1 \leq m$,

$$\frac{1}{t+s}$$

(Cauchy kernel),

$$\frac{t - s}{te^{-1/t} - se^{-1/s}}$$

are all infinitely divisible kernels on $(0, \infty)$.

(2). Consider a polynomial

$p(t) := \prod_{i=1}^{n}(t - a_i)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then the kernel

$$K_{t,p(t)}(t, s) = \frac{t - s}{p(t) - p(s)}$$

is infinitely divisible on $(a_1, \infty)$.
Theorem 4.4 Let $h(t)$ and $g(t)$ be positive $C^1$ functions defined on an open interval $(a, b)$, where $-\infty \leq a < b \leq \infty$. Suppose the range of $g$ is $(0, \infty)$. Then the following are equivalent:

(i) the kernel $K_{h,g}$ is conditionally negative;

(ii) there is an operator convex function $\varphi$ defined on $(0, \infty)$ such that $\varphi(g(t)) = h(t)$ for $t \in (a, b)$.

(iii) $\frac{h(t) - h(a + 0)}{g(t)} \preceq g(t)$ (a < t < b)