Mean theoretic approach to a further extension of grand Furuta inequality

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Abstract

Very recently, Furuta has shown a further extension of grand Furuta inequality. In this report, we obtain a more precise and clear expression of Furuta’s extension by considering a mean theoretic proof of grand Furuta inequality.

1 Introduction

In what follows, $A$ and $B$ are positive operators on a complex Hilbert space, and we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. strictly positive) operator.

Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$” is very famous as an order preserving operator inequality. As an extension of Löwner-Heinz theorem, Furuta [8] established the following result called Furuta inequality (see also [2, 3, 9, 12, 18, 20]).

Theorem 1.A (Furuta inequality [8]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\left( B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^\frac{1}{q} \geq \left( B^{\frac{r}{2}} B^p B^{\frac{r}{2}} \right)^\frac{1}{q}$

and

(ii) $\left( A^{\frac{r}{2}} A^p A^{\frac{r}{2}} \right)^\frac{1}{q} \geq \left( A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^\frac{1}{q}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Theorem 1.B ([3]). Let $A \geq B \geq 0$ with $A > 0$. Then

$$f(p,r) = A^{\frac{r}{q}} \left( A^{\frac{r}{2}} A^p A^{\frac{r}{2}} \right)^{1+\frac{pr}{q+pr}} A^{\frac{r}{q}}$$

(1.1)

is decreasing for $p \geq 1$ and $r \geq 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
In [10], Furuta has shown an extension of Furuta inequality, which is called grand Furuta inequality (see also [5, 7, 11, 12, 13, 16, 21, 22, 23]). We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization, and we are also discussing Furuta inequality and Ando-Hiai inequality in [4, 6, 17].

**Theorem 1.C** (Grand Furuta inequality [10]). If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,\

$$F(r, s) = A^\frac{r}{2} \{ A^{\frac{r}{2}} (A^{\frac{f}{2}} B^p A^{\frac{f}{2}})^{s} A^{\frac{r}{2}} \}^\frac{1-\frac{r}{p}+t}{(p-t)s+r} A^\frac{r}{2}$$

is decreasing for $r \geq t$ and $s \geq 1$, and

$$A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{s} A^{\frac{r}{2}} \}^\frac{1-\frac{r}{p}+t}{(p-t)s+r}$$

holds for $r \geq t$ and $s \geq 1$.

For $A > 0$ and $B \geq 0$, $\alpha$-power mean $\#_{\alpha}$ for $\alpha \in [0,1]$ is defined by $A \#_{\alpha} B = A^\frac{1}{2} (A^{\frac{-1}{2} B A^{\frac{-1}{2}}} )^\alpha A^\frac{1}{2}$. In this report, we use this operator mean as our main tool. We remark that the operator mean theory was established by Kubo-Ando [19].

It is known that $\alpha$-power mean is very useful for investigating Furuta inequality. As stated in [18], when $A > 0$ and $B \geq 0$, Theorem 1.A can be arranged in terms of $\alpha$-power mean as follows: If $A \geq B \geq 0$ with $A > 0$, then

$$A \geq B \geq A^{-r} \#_{\frac{1+p}{1+r}} B^p \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$ 

We can also rewrite (1.1) in Theorem 1.B by

$$f(p, r) = A^{-r} \#_{\frac{1+p}{1+r}} B^p.$$  \hspace{1cm} (1.1')

Similarly, by putting $\beta = (p-t)s+t$ and $\gamma = r-t$, we can arrange Theorem 1.C in terms of $\alpha$-power mean as follows [5]: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$ with $p \neq t$,

$$\hat{F}(\beta, \gamma) = A^{-\gamma} \#_{\frac{1+p}{1+\gamma}} (A^t \#_{\frac{p-t}{p-t}} B^p) \quad \text{is decreasing for } \beta \geq p \text{ and } \gamma \geq 0,$$

and

$$A \geq B \geq A^{-\gamma} \#_{\frac{1+p}{1+\gamma}} (A^t \#_{\frac{p-t}{p-t}} B^p) \quad \text{for } \beta \geq p \text{ and } \gamma \geq 0,$$

(1.2)

where $A \#_{s} B = A^{\frac{1}{2}} (A^{\frac{-1}{2} B A^{\frac{-1}{2}}} )^s A^{\frac{1}{2}}$ for a real number $s$. (If $s \in [0,1]$, then $\#_{s} = \#_{s}$.)

Very recently, Furuta [14, 15] has dug for a further extension of grand Furuta inequality, which is the following Theorem 1.D. We call this “FGF inequality” here.
Theorem 1.D (FGF inequality [14, 15]). Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n-1} \geq 1$ for natural number $n$. Then

$$G(r, p_{2n}) = A^{\frac{-r}{2}}[A^{\frac{r}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\cdots(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(1.3)$$

$$(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}\cdots A^{\frac{t}{2}}\}^{p_{2n-1}}A^{\frac{t}{2}}\}^{p_{2n}}A^{\frac{r}{2}} \frac{1-t+r}{q[2n]-t+r}A^{\frac{-f}{2}}$$

is decreasing for $r \geq t$ and $p_{2n} \geq 1$, and

$$A^{1-t+r} \geq [A^{\frac{t}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\cdots(A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(1.4)$$

$$(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}\cdots A^{\frac{t}{2}}\}^{p_{2n-1}}A^{\frac{t}{2}}\}^{p_{2n}}A^{\frac{r}{2}} \frac{1-t+r}{q[2n]-t+r}$$

holds for $r \geq t$ and $p_{2n} \geq 1$, where

$$q[2n] = ((\cdots((p_1-t)p_2+t)p_3-t)p_4+\cdots+t)p_{2n-1}-t)p_{2n}+t.$$

In this report, we obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of FGF inequality by scrutinizing the former argument.

2 FGF inequality

Firstly, we show that a sequence $\{B_i\}$ such that $B_i = (A^{t} \#arrow_{\alpha_{i-1}}^{\beta-t} B_{i-1}^\alpha)^{\frac{1}{\beta}}$ is decreasing. Theorem 2.1 is a key result in the proof of FGF inequality.

Theorem 2.1. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^{t} \#arrow_{\alpha_{i-1}}^{\beta-t} B_{i-1}^\alpha)^{\frac{1}{\beta}}$.

Lemma 2.A ([5]). Let $A \geq B \geq 0$ with $A > 0$. Then

$$A \geq B \geq (A^{t} \#arrow_{p+1}^{\beta-t} B^p)^{\frac{1}{p}}$$

holds for $t \in [0,1]$, $\beta \geq p \geq 1$ and $p \neq t$. 
We remark that Lemma 2.A plays an important role in the proof of grand Furuta inequality (1.2).

**Proof of Theorem 2.1.** By applying Lemma 2.A to that $A \geq B \geq 0$ with $A > 0$, we have

$$A \geq B \geq (A^t \overline{\frac{\gamma_{a_1-t}}{a_1-t}} B^{\alpha_1})^{\frac{1}{\beta_1}} = B_1$$

for $t \in [0, 1]$, $\beta_1 \geq \alpha_1 \geq 1$ and $\alpha_1 \neq t$, and also by applying Lemma 2.A repeatedly to that $A \geq B_{i-1} \geq 0$ with $A > 0$ for $i = 1, 2, \ldots, n$, we have

$$B_{i-1} \geq (A^t \overline{\frac{\gamma_{a_{i-1}-t}}{a_{i-1}-t}} B_{i-1}^{\alpha_{i-1}})^{\frac{1}{\beta_{i-1}}} = B_i$$

for $t \in [0, 1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$, so that

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n.$$

Hence the proof is complete. $\square$

Furuta [15] has given an extension of Lemma 2.A as an application of Theorem 1.D.

**Theorem 2.B** ([15]). Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \ldots, p_{2n-1}, p_{2n} \geq 1$ for natural number $n$. Then

$$A \geq B \geq \{A^\frac{t}{2} (A^\frac{-t}{2} B^{p_1} A^\frac{-t}{2})^{p_2} A^\frac{t}{2}\}^{\overline{q[2n]}} \geq \cdots \geq [A^\frac{t}{2} \{A^\frac{-t}{2} \{A^\frac{t}{2} (A^\frac{-t}{2} B^{p_1} A^\frac{-t}{2})^{p_2} A^\frac{t}{2}\}^{p_3} A^\frac{-t}{2} \}^{p_4} \cdots A^\frac{t}{2}\}^{p_{2n-1}} A^\frac{-t}{2}]^{\overline{q[2n]}}$$

where

$$q[2n] = \{(\cdots((p_1-t)p_2+t)p_3-t)p_4+\cdots+t)p_{2n-1}-t)p_{2n}+t\}.$$  

We can rewrite Theorem 2.B by putting

$$\beta_0 = 1, \alpha_i = \beta_{i-1} p_{2i-1}, \beta_i = (\alpha_i - t)p_{2i} + t \text{ and } \gamma = r - t \quad (2.1)$$

as follows:

**Theorem 2.B'.** Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0, 1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^t \overline{\frac{\gamma_{a_{i-1}-t}}{a_{i-1}-t}} B_{i-1}^{\alpha_{i-1}})^{\frac{1}{\beta_i}}$. 
Therefore we recognize that Theorem 2.1 is a fine extension of Theorem 2.B. More precisely, \( \beta_i \geq \alpha_i \geq 1 \) in Theorem 2.1 is looser than \( \beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1 \) in Theorem 2.B.

By using Theorem 2.1, we obtain an improvement of (1.4) in Theorem 1.D and Theorem 2.B. Theorem 2.2 is a satellite form of Theorem 1.D in our sense. Theorem 2.2 leads (1.4) in Theorem 1.D by the same replacement to (2.1).

**Theorem 2.2.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0,1] \), \( \beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1, \gamma \geq 0 \) and \( \alpha_1 \neq t \),

\[
A \geq B \geq A^\gamma \frac{1+t}{\alpha_1+t} B^{\alpha_1} \geq A^{-\gamma} \frac{1+t}{\beta_1+t} B_1^{\beta_1} \geq A^{-\gamma} \frac{1+t}{\beta_2+t} B_2^{\beta_2} \geq \cdots \geq A^{-\gamma} \frac{1+t}{\beta_{n-1}+t} B_{n-1}^{\beta_{n-1}} \geq A^{-\gamma} \frac{1+t}{\beta_{n}+t} B_{n}^{\beta_{n}},
\]

where \( B_0 = B \) and \( B_i = (A^t \frac{\beta_{i-1}}{\alpha_{i-1}} B_{i-1}^{\alpha_{i-1}})^{\frac{1}{\beta_i}} \).

**Proof.** Let \( \beta_0 = 1 \). By Theorem 2.1, \( A \geq B_{i-1} \) holds for \( i = 1, 2, \ldots, n \), so that we have

\[
A^{-\gamma} \frac{1+t}{\beta_{i-1}+t} B_{i-1}^{\beta_{i-1}} \geq A^{-\gamma} \frac{1+t}{\alpha_i+t} B_i^{\alpha_i} \text{ by Theorem 1.B}
\]

\[
\geq A^{-\gamma} \frac{1+t}{\beta_i+t} (A^t \frac{\beta_{i-1}}{\alpha_{i-1}} B_{i-1}^{\alpha_{i-1}}) = A^{-\gamma} \frac{1+t}{\beta_i+t} B_i^{\beta_i} \text{ by Theorem 1.C}
\]

since \( \beta_i \geq \alpha_i \geq \beta_{i-1} \geq 1 \). Hence the proof is complete. \( \square \)

### 3 Variant of FGF Inequality

In this section, we obtain a variant of FGF inequality by scrutinizing the argument in Section 2, and also we have a result on a FGF-type operator function. We omit their proofs here.

**Theorem 3.1.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0,1] \), \( \alpha_i \geq 1, 1 \leq \frac{\beta_i-t}{\alpha_i-t} \leq 2 \) and \( \alpha_i \neq t \) for \( i = 1, 2, \ldots, n \),

\[
B_{i-1}^{\beta_i} \geq B_i^{\beta_i},
\]

where \( B_0 = B \) and \( B_i = (A^t \frac{\beta_{i-1}}{\alpha_{i-1}} B_{i-1}^{\alpha_{i-1}})^{\frac{1}{\beta_i}} \).
Theorem 3.2. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0, 1]$, $\alpha_i \geq 1$, $\beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq 1$, $1 \leq \frac{\beta_{i-t}}{\alpha_{i-t}} \leq 2$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq A^{-\gamma} \frac{\beta_1}{\alpha_1 + \gamma} B_{\beta_1} \geq A^{-\gamma} \frac{\beta_2}{\alpha_2 + \gamma} B_{\beta_2} \geq \cdots \geq A^{-\gamma} \frac{\beta_n}{\alpha_n + \gamma} B_{\beta_n},$$

where $B_0 = B$ and $B_i = (A^t \frac{\beta_i}{\alpha_i - t} B_{\beta_i-1})^{\frac{1}{\beta_i}}$.

Theorem 3.3. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0, 1]$, $\beta_i \geq \alpha_i \geq 1$ for $i = 1, 2, \ldots, n-1$, $\alpha_n \geq 1$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$, 

$$G'(\beta_n) = A^{-\gamma} \frac{\beta_n}{\alpha_n + \gamma} (A^t \frac{\beta_n}{\alpha_n - t} B_{\beta_n-1})$$

(3.1)

is decreasing for $\beta_n \geq \alpha_n$, where $B_0 = B$ and $B_i = (A^t \frac{\beta_i}{\alpha_i - t} B_{\beta_i-1})^{\frac{1}{\beta_i}}$.

Remark. (3.1) is also decreasing for $\gamma \geq 0$ by Theorem 1.B since $A \geq B \geq 0$ with $A > 0$ ensures $A \geq B_n = (A^t \frac{\beta_n}{\alpha_n - t} B_{\beta_n-1})^{\frac{1}{\beta_n}}$ by Theorem 2.1. Therefore, similarly to Theorem 2.1, we recognize that Theorem 3.3 is a slight extension of (1.3) in Theorem 1.D.

References


[8] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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