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<th>Title</th>
<th>Mean theoretic approach to a further extension of grand Furuta inequality (Prospects of non-commutative analysis in operator theory)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2010), 1678: 84-91</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141300">http://hdl.handle.net/2433/141300</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Mean theoretic approach to a further extension of grand Furuta inequality

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This report is based on “M. Ito and E. Kamei, Mean theoretic approach to a further extension of grand Furuta inequality, to appear in J. Math. Inequal.”

Abstract

Very recently, Furuta has shown a further extension of grand Furuta inequality. In this report, we obtain a more precise and clear expression of Furuta’s extension by considering a mean theoretic proof of grand Furuta inequality.

1 Introduction

In what follows, $A$ and $B$ are positive operators on a complex Hilbert space, and we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. strictly positive) operator.

Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$” is very famous as an order preserving operator inequality. As an extension of Löwner-Heinz theorem, Furuta [8] established the following result called Furuta inequality (see also [2, 3, 9, 12, 18, 20]).

**Theorem 1.A** (Furuta inequality [8]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) \[ (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}} \]

and

(ii) \[ (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{r}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{r}} \]

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

**Theorem 1.B** ([3]). Let $A \geq B \geq 0$ with $A > 0$. Then

\[
 f(p,r) = A^{\frac{r}{2}}(A^{\frac{p}{2}}A^{\frac{r}{2}})^{\frac{p+r}{p+r}} A^{\frac{r}{2}}
\]

is decreasing for $p \geq 1$ and $r \geq 0$. 

![Figure 1](image-url)
In [10], Furuta has shown an extension of Furuta inequality, which is called grand Furuta inequality (see also [5, 7, 11, 12, 13, 16, 21, 22, 23]). We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization, and we are also discussing Furuta inequality and Ando-Hiai inequality in [4, 6, 17].

**Theorem 1.C** (Grand Furuta inequality [10]). If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

$$F(r, s) = A^\frac{r}{2} \{A^\frac{r}{2} (A^\frac{-t}{2} B^p A^\frac{-t}{2})^s A^\frac{r}{2}\}^{\frac{1-t+r}{p-0 \cdot r}} A^\frac{r}{2}$$

is decreasing for $r \geq t$ and $s \geq 1$, and

$$A^{1-t+r} \geq A^\frac{r}{2} (A^\frac{-t}{2} B^p A^\frac{-t}{2})^s A^\frac{r}{2} \frac{1-t+r}{(p-t)s+r}$$

holds for $r \geq t$ and $s \geq 1$.

For $A > 0$ and $B \geq 0$, $\alpha$-power mean $\#_{\alpha}$ for $\alpha \in [0, 1]$ is defined by $A \#_{\alpha} B = A^\frac{1}{2} (A^\frac{-1}{2} BA^\frac{-1}{2})^\alpha A^\frac{1}{2}$. In this report, we use this operator mean as our main tool. We remark that the operator mean theory was established by Kubo-Ando [19].

It is known that $\alpha$-power mean is very useful for investigating Furuta inequality. As stated in [18], when $A > 0$ and $B \geq 0$, Theorem 1.A can be arranged in terms of $\alpha$-power mean as follows: If $A \geq B \geq 0$ with $A > 0$, then

$$A \geq B \geq A^{-r} \#_{\frac{1}{p+t}} B^p$$

for $p \geq 1$ and $r \geq 0$.

We can also rewrite (1.1) in Theorem 1.B by

$$f(p, r) = A^{-r} \#_{\frac{1}{p+t}} B^p . \quad (1.1')$$

Similarly, by putting $\beta = (p-t)s + t$ and $\gamma = r - t$, we can arrange Theorem 1.C in terms of $\alpha$-power mean as follows [5]: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$ with $p \neq t$,

$$F(\beta, \gamma) = A^{-\gamma} \#_{\frac{1}{p+t}} (A^t \#_{\frac{-t}{p-t}} B^p)$$

is decreasing for $\beta \geq p$ and $\gamma \geq 0$, and

$$A \geq B \geq A^{-\gamma} \#_{\frac{1}{p+t}} (A^t \#_{\frac{-t}{p-t}} B^p) \quad \text{for } \beta \geq p \text{ and } \gamma \geq 0 , \quad (1.2)$$

where $A \#_s B = A^\frac{1}{2} (A^\frac{-1}{2} BA^\frac{-1}{2})^s A^\frac{1}{2}$ for a real number $s$. (If $s \in [0, 1]$, then $\#_s = \#_s$.)

Very recently, Furuta [14, 15] has dug for a further extension of grand Furuta inequality, which is the following Theorem 1.D. We call this "FGF inequality" here.
Theorem 1. D (FGF inequality [14, 15]). Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n-1} \geq 1$ for natural number $n$. Then

$$G(r, p_{2n}) = A^\frac{-r}{2} \left[A^\frac{-r}{2} (A^\frac{-t}{2} \{A^\frac{t}{2} \ldots \{A^\frac{-t}{2} \{A^\frac{t}{2} \ldots} \}^{p_{2n-2} A^\frac{t}{2} \ldots A^\frac{-t}{2} \}^{p_{2n-1} A^\frac{t}{2}} \right]^{\frac{1-t+r}{q[2n]-t+r}} A^\frac{r}{2} \ldots (1.3)$$

is decreasing for $r \geq t$ and $p_{2n} \geq 1$, and

$$A^{1-t+r} \geq [A^\frac{t}{2} \{A^\frac{t}{2} \ldots \{A^\frac{-t}{2} \{A^\frac{t}{2} \ldots} \}^{p_{2n-2} A^\frac{t}{2} \ldots A^\frac{-t}{2} \}^{p_{2n-1} A^\frac{t}{2}} \right]^{\frac{1-t+r}{q[2n]-t+r}} \ldots (1.4)$$

holds for $r \geq t$ and $p_{2n} \geq 1$, where

$$q[2n] = \{ \cdots \{ (p_1 - t)p_2 + t \} p_3 - t \} p_4 + \cdots + t \} p_{2n-1} - t \} p_{2n} + t.$$

In this report, we obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of FGF inequality by scrutinizing the former argument.

2 FGF inequality

Firstly, we show that a sequence $\{B_i\}$ such that $B_i = (A^{t \# \alpha_i \rightarrow -t \beta - t} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$ is decreasing. Theorem 2.1 is a key result in the proof of FGF inequality.

Theorem 2.1. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^{t \# \alpha_i \rightarrow -t \beta - t} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Lemma 2.A ([5]). Let $A \geq B \geq 0$ with $A > 0$. Then

$$A \geq B \geq (A^{t \# \beta \rightarrow -t \alpha} B^p)^{\frac{1}{\beta}}$$

holds for $t \in [0,1]$, $\beta \geq p \geq 1$ and $p \neq t$. 
We remark that Lemma 2.A plays an important role in the proof of grand Furuta inequality (1.2).

**Proof of Theorem 2.1.** By applying Lemma 2.A to that $A \geq B \geq 0$ with $A > 0$, we have

$$A \geq B \geq (A^{t} \#_{\frac{\alpha_{i-1}}{\alpha_{i}}} B^{\alpha_{i}})^{\frac{1}{\beta_{i}}} = B_{1}$$

for $t \in [0, 1]$, $\beta_{1} \geq \alpha_{1} \geq 1$ and $\alpha_{1} \neq t$, and also by applying Lemma 2.A repeatedly to that $A \geq B_{i-1} \geq 0$ with $A > 0$ for $i = 1, 2, \ldots, n$, we have

$$B_{i-1} \geq (A^{t} \#_{\frac{\alpha_{i-1}}{\alpha_{i}}} B^{\alpha_{i}})^{\frac{1}{\beta_{i}}} = B_{i}$$

for $t \in [0, 1]$, $\beta_{i} \geq \alpha_{i} \geq 1$ and $\alpha_{i} \neq t$, so that

$$A \geq B \geq B_{1} \geq \cdots \geq B_{n-1} \geq B_{n}.$$

Hence the proof is complete. \(\square\)

Furuta [15] has given an extension of Lemma 2.A as an application of Theorem 1.D.

**Theorem 2.B** ([15]). Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_{1}, p_{2}, \ldots, p_{2n-1}, p_{2n} \geq 1$ for natural number $n$. Then

$$A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_{1}}A^{\frac{-t}{2}})^{p_{2}}A^{\frac{t}{2}}\}^{\frac{1}{q[2n]}} \geq \cdots \geq [A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_{1}}A^{\frac{-t}{2}})^{p_{2}}A^{\frac{t}{2}}]^{\frac{1}{q[2n]}}$$

where

$$q[2n] = (\{\cdots((p_{1} - t)p_{2} + t)p_{3} - t)p_{4} + \cdots + t\}p_{2n} - t)p_{2n} + t.$$  

We can rewrite Theorem 2.B by putting

$$\beta_{0} = 1, \alpha_{i} = \beta_{i-1}p_{2i-1}, \beta_{i} = (\alpha_{i} - t)p_{2i} + t \text{ and } \gamma = r - t$$

as follows:

**Theorem 2.B'.** Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0, 1]$, $\beta_{n} \geq \alpha_{n} \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_{1} \geq \alpha_{1} \geq 1$ and $\alpha_{i} \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq B_{1} \geq \cdots \geq B_{n-1} \geq B_{n},$$

where $B_{0} = B$ and $B_{i} = (A^{t} \#_{\frac{\alpha_{i}}{\alpha_{i}}- t} B^{\alpha_{i}})^{\frac{1}{\beta_{i}}}$.  

Therefore we recognize that Theorem 2.1 is a fine extension of Theorem 2.B. More precisely, $\beta_i \geq \alpha_i \geq 1$ in Theorem 2.1 is looser than $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$ in Theorem 2.B.

By using Theorem 2.1, we obtain an improvement of (1.4) in Theorem 1.D and Theorem 2.B. Theorem 2.2 is a satellite form of Theorem 1.D in our sense. Theorem 2.2 leads (1.4) in Theorem 1.D by the same replacement to (2.1).

Theorem 2.2. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0,1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$, $\gamma \geq 0$ and $\alpha_1 \neq t$,

$$A \geq B \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\beta_i \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\alpha_i \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\beta_i \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\alpha_i \geq \cdots \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\beta_i \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\alpha_i,$$

where $B_0 = B$ and $B_i = (A^t \#_{A^{-1/\beta_i}} B^\alpha_i)^{1/\beta_i}$.

Proof. Let $\beta_0 = 1$. By Theorem 2.1, $A \geq B_i^{-1}$ holds for $i = 1, 2, \ldots, n$, so that we have

$$A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\beta_i \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\alpha_i \geq \cdots \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\alpha_i \geq \cdots \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\beta_i \geq A^{-\gamma} \#_{A^{-1/\beta_i}}^t B^\alpha_i,$$

since $\beta_i \geq \alpha_i \geq \beta_i \geq \alpha_i \geq 1$. Hence the proof is complete. \qed

3 Variant of FGF inequality

In this section, we obtain a variant of FGF inequality by scrutinizing the argument in Section 2, and also we have a result on a FGF-type operator function. We omit their proofs here.

Theorem 3.1. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0,1]$, $\alpha_i \geq 1$, $1 \leq \frac{\beta_i - t}{\alpha_i - t} \leq 2$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$B_i^{\beta_i} \geq B_i^{\alpha_i},$$

where $B_0 = B$ and $B_i = (A^t \#_{A^{-1/\alpha_i}} B^\alpha_i)^{1/\beta_i}$. 

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**Theorem 3.2.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0,1] \), \( \alpha_i \geq 1 \), \( \beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq 1 \), \( 1 \leq \frac{\beta_i - t}{\alpha_i - t} \leq 2 \), \( \gamma \geq 0 \) and \( \alpha_i \neq t \) for \( i = 1, 2, \ldots, n \),

\[
A \geq B \geq A^{-\gamma} \| \frac{1}{\beta_i + \gamma} B_{\beta_i} \|_{\alpha_i} \geq A^{-\gamma} \| \frac{1}{\beta_{i+1} + \gamma} B_{\beta_{i+1}} \|_{\alpha_{i+1}} \geq \cdots \geq A^{-\gamma} \| \frac{1}{\beta_n + \gamma} B_{\beta_n} \|_{\alpha_n},
\]

where \( B_0 = B \) and \( B_i = (A^t \| \frac{1}{\alpha_i - t} B_{\alpha_i - t} \|_{\alpha_i} B_{i-1})^{\frac{1}{\beta_i}} \).

**Theorem 3.3.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0,1] \), \( \beta_i \geq \alpha_i \geq 1 \) for \( i = 1, 2, \ldots, n - 1 \), \( \alpha_n \geq 1 \), \( \gamma \geq 0 \) and \( \alpha_i \neq t \) for \( i = 1, 2, \ldots, n \),

\[
\hat{G}(\beta_n) = A^{-\gamma} \| \frac{1}{\beta_n + \gamma} (A^t \| \frac{1}{\alpha_n - t} B_{\alpha_n - t} \|_{\alpha_n} B_{n-1}) \|_{\alpha_n},
\]

is decreasing for \( \beta_n \geq \alpha_n \), where \( B_0 = B \) and \( B_i = (A^t \| \frac{1}{\alpha_i - t} B_{\alpha_i - t} \|_{\alpha_i} B_{i-1})^{\frac{1}{\beta_i}} \).

**Remark.** (3.1) is also decreasing for \( \gamma \geq 0 \) by Theorem 1.B since \( A \geq B \geq 0 \) with \( A > 0 \) ensures \( A \geq B_n = (A^t \| \frac{1}{\alpha_n - t} B_{\alpha_n - t} \|_{\alpha_n} B_{n-1})^{\frac{1}{\beta_n}} \) by Theorem 2.1. Therefore, similarly to Theorem 2.1, we recognize that Theorem 3.3 is a slight extension of (1.3) in Theorem 1.D.

**References**


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