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<td>Ito, Masatoshi; Kamei, Eizaburo</td>
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Kyoto University
Mean theoretic approach to a further extension of grand Furuta inequality

前橋工科大学　伊藤 公智 (Masatoshi Ito)
Maebashi Institute of Technology
前橋工科大学　亀井 榮三郎 (Eizaburo Kamei)
Maebashi Institute of Technology

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Abstract

Very recently, Furuta has shown a further extension of grand Furuta inequality. In this report, we obtain a more precise and clear expression of Furuta’s extension by considering a mean theoretic proof of grand Furuta inequality.

1 Introduction

In what follows, $A$ and $B$ are positive operators on a complex Hilbert space, and we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. strictly positive) operator.

Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$” is very famous as an order preserving operator inequality. As an extension of Löwner-Heinz theorem, Furuta [8] established the following result called Furuta inequality (see also [2, 3, 9, 12, 18, 20]).

Theorem 1.A (Furuta inequality [8]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.

Theorem 1.B ([3]). Let $A \geq B \geq 0$ with $A > 0$. Then

\[ f(p, r) = A^{\frac{r}{2}}(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{p+r}}A^{\frac{r}{2}} \]

is decreasing for $p \geq 1$ and $r \geq 0$. 

\[ (1+r)q = p + r \]
In [10], Furuta has shown an extension of Furuta inequality, which is called grand Furuta inequality (see also [5, 7, 11, 12, 13, 16, 21, 22, 23]). We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization, and we are also discussing Furuta inequality and Ando-Hiai inequality in [4, 6, 17].

**Theorem 1.C (Grand Furuta inequality [10]).** If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,

$$F(r, s) = A^{\frac{-r}{2}} \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{s} A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and

$$A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{s} A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

For $A > 0$ and $B \geq 0$, $\alpha$-power mean $\#_{\alpha}$ for $\alpha \in [0,1]$ is defined by $A \#_{\alpha} B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\alpha} A^{\frac{1}{2}}$. In this report, we use this operator mean as our main tool. We remark that the operator mean theory was established by Kubo-Ando [19].

It is known that $\alpha$-power mean is very useful for investigating Furuta inequality. As stated in [18], when $A > 0$ and $B \geq 0$, Theorem 1.A can be arranged in terms of $\alpha$-power mean as follows: If $A \geq B \geq 0$ with $A > 0$, then

$$A \geq B \geq A^{-r} \#_{p+1} B^p \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$  

We can also rewrite (1.1) in Theorem 1.B by

$$f(p,r) = A^{-r} \#_{p+1} B^p. \quad (1.1')$$

Similarly, by putting $\beta = (p-t)s + t$ and $\gamma = r - t$, we can arrange Theorem 1.C in terms of $\alpha$-power mean as follows [5]: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$ with $p \neq t$,

$$\tilde{F}(\beta, \gamma) = A^{-\gamma} \#_{\beta+1} (A^{t} \#_{p-1} B^p)$$

is decreasing for $\beta \geq p$ and $\gamma \geq 0$, and

$$A \geq B \geq A^{-\gamma} \#_{\beta+1} (A^{t} \#_{p-1} B^p) \quad \text{for } \beta \geq p \text{ and } \gamma \geq 0, \quad (1.2)$$

where $A \#_{s} B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{s} A^{\frac{1}{2}}$ for a real number $s$. (If $s \in [0,1]$, then $\#_{s} = \#_{s}$.)

Very recently, Furuta [14, 15] has dug for a further extension of grand Furuta inequality, which is the following Theorem 1.D. We call this “FGF inequality” here.
Theorem 1.1 (FGF inequality [14, 15]). Let \( A \geq B \geq 0 \) with \( A > 0 \), \( t \in [0, 1] \) and \( p_1, p_2, \ldots, p_{2n-1} \geq 1 \) for natural number \( n \). Then

\[
G(r, p_{2n}) = A^{\frac{-r}{2}} \left[ A^{\frac{r}{2}} \left( A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \cdots \{ A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \right)^{p_1} A^{\frac{-t}{2}} \}^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}} \}^{p_4} \cdots A^{\frac{t}{2}} \}^{p_{2n}} \right]^{\frac{1-t+r}{q[2n]-t+r}} A^{\frac{-r}{2}}
\]  

is decreasing for \( r \geq t \) and \( p_{2n} \geq 1 \), and

\[
A^{1-t+r} \geq \left[ A^{\frac{t}{2}} \left( A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \cdots \{ A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \right)^{p_1} A^{\frac{-t}{2}} \}^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}} \}^{p_4} \cdots A^{\frac{t}{2}} \}^{p_{2n-1}} A^{\frac{t}{2}} \}^{p_{2n}} A^{\frac{-r}{2}} \right]^{\frac{1-t+r}{q[2n]-t+r}}
\]  

holds for \( r \geq t \) and \( p_{2n} \geq 1 \), where

\[
q[2n] = \left( \cdots \{ (p_1-t)p_2 + t \}p_3 - t \}p_4 + \cdots + t \right) p_{2n-1} - t \right) p_{2n} + t.
\]

In this report, we obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of FGF inequality by scrutinizing the former argument.

2 FGF inequality

Firstly, we show that a sequence \( \{B_i\} \) such that \( B_i = (A^t \mathfrak{h}^\alpha_{\frac{\beta-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_k}} \) is decreasing. Theorem 2.1 is a key result in the proof of FGF inequality.

Theorem 2.1. Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0, 1] \), \( \beta_i \geq \alpha_i \geq 1 \) and \( \alpha_i \neq t \) for \( i = 1, 2, \ldots, n \),

\[
A \geq B \geq B_1 \cdots \geq B_{n-1} \geq B_n,
\]

where \( B_0 = B \) and \( B_i = (A^t \mathfrak{h}^\alpha_{\frac{\beta-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}} \).

Lemma 2.1 ([5]). Let \( A \geq B \geq 0 \) with \( A > 0 \). Then

\[
A \geq B \geq (A^t \mathfrak{h}^\beta_{\frac{\beta-t}{p-1}} B^p)^{\frac{1}{\beta}}
\]

holds for \( t \in [0, 1] \), \( \beta \geq p \geq 1 \) and \( p \neq t \).
We remark that Lemma 2.A plays an important role in the proof of grand Furuta inequality (1.2).

**Proof of Theorem 2.1.** By applying Lemma 2.A to that \( A \geq B \geq 0 \) with \( A > 0 \), we have

\[
A \geq B \geq (A^t \overline{\frac{\beta_{i-1} - t}{\alpha_{i-1}}} B^\alpha_{i-1})^\frac{1}{\beta_i} = B_1
\]

for \( t \in [0, 1] \), \( \beta_1 \geq \alpha_1 \geq 1 \) and \( \alpha_1 \neq t \), and also by applying Lemma 2.A repeatedly to that \( A \geq B_{i-1} \geq 0 \) with \( A > 0 \) for \( i = 1, 2, \ldots, n \), we have

\[
B_{i-1} \geq (A^t \overline{\frac{\beta_{i-1} - t}{\alpha_{i-1}}} B^\alpha_{i-1})^\frac{1}{\beta_i} = B_i
\]

for \( t \in [0, 1] \), \( \beta_i \geq \alpha_i \geq 1 \) and \( \alpha_i \neq t \), so that

\[
A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n.
\]

Hence the proof is complete. \( \square \)

Furuta [15] has given an extension of Lemma 2.A as an application of Theorem 1.D.

**Theorem 2.B** ([15]). Let \( A \geq B \geq 0 \) with \( A > 0 \), \( t \in [0, 1] \) and \( p_1, p_2, \ldots, p_{2n-1}, p_{2n} \geq 1 \) for natural number \( n \). Then

\[
A \geq B \geq (A^\frac{t}{2} B^{p_1} A^\frac{-t}{2} B^{p_2} A^\frac{t}{2})^\frac{1}{q[2n]} \geq \cdots \geq (A^\frac{t}{2} B^{p_{2n-1}} A^\frac{-t}{2} B^{p_{2n}} A^\frac{t}{2})^\frac{1}{q[2n]},
\]

where

\[
q[2n] = \{(p_1 - t)p_2 + t\}p_3 - t)p_4 + \cdots + t\}p_{2n-1} - t)p_{2n} + t.
\]

We can rewrite Theorem 2.B by putting

\[
\beta_0 = 1, \ \alpha_i = \beta_{i-1} p_{2i-1}, \ \beta_i = (\alpha_i - t)p_{2i} + t \text{ and } \gamma = r - t
\]

as follows:

**Theorem 2.B'.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0, 1] \), \( \beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1 \) and \( \alpha_i \neq t \) for \( i = 1, 2, \ldots, n \),

\[
A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,
\]

where \( B_0 = B \) and \( B_i = (A^t \overline{\frac{\beta_{i-1} - t}{\alpha_{i-1}}} B^\alpha_{i-1})^\frac{1}{\beta_i} \).
Therefore we recognize that Theorem 2.1 is a fine extension of Theorem 2.B. More precisely, \( \beta_i \geq \alpha_i \geq 1 \) in Theorem 2.1 is looser than \( \beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1 \) in Theorem 2.B.

By using Theorem 2.1, we obtain an improvement of (1.4) in Theorem 1.D and Theorem 2.B. Theorem 2.2 is a satellite form of Theorem 1.D in our sense. Theorem 2.2 leads (1.4) in Theorem 1.D by the same replacement to (2.1).

**Theorem 2.2.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0,1] \), \( \beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1 \), \( \gamma \geq 0 \) and \( \alpha_i \neq t \),

\[
A \geq B \geq A^{-\gamma} \#_{\frac{1+\gamma}{\alpha_1+t}} B^\alpha_1 \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_1+t}} B^\beta_1 \geq A^{-\gamma} \#_{\frac{1+\gamma}{\alpha_2+t}} B^\alpha_2 \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_2+t}} B^\beta_2 \\
\geq \cdots \geq A^{-\gamma} \#_{\frac{1+\gamma}{\alpha_{n-1}+t}} B^\alpha_{n-1} \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_{n-1}+t}} B^\beta_{n-1} \geq A^{-\gamma} \#_{\frac{1+\gamma}{\alpha_n+t}} B^\alpha_n \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_n+t}} B^\beta_n,
\]

where \( B_0 = B \) and \( B_i = (A^t \#_{\frac{\beta_i}{\alpha_i+t}} B^\alpha_i)_{i=1}^n \).

**Proof.** Let \( \beta_0 = 1 \). By Theorem 2.1, \( A \geq B_{i-1} \) holds for \( i = 1, 2, \ldots, n \), so that we have

\[
A^{-\gamma} \#_{\frac{1+\gamma}{\alpha_{i-1}+t}} B^\alpha_{i-1} \\
\geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_{i-1}+t}} B^\beta_{i-1} \quad \text{by Theorem 1.B}
\]

\[
\geq A^{-\gamma} \#_{\frac{1+\gamma}{\alpha_i+t}} B^\alpha_i \quad \text{by Theorem 1.C}
\]

since \( \beta_i \geq \alpha_i \geq \beta_{i-1} \geq 1 \). Hence the proof is complete. \( \square \)

3 Variant of FGF inequality

In this section, we obtain a variant of FGF inequality by scrutinizing the argument in Section 2, and also we have a result on a FGF-type operator function. We omit their proofs here.

**Theorem 3.1.** Let \( A \geq B \geq 0 \) with \( A > 0 \) and \( n \) be a natural number. Then for \( t \in [0,1] \), \( \alpha_i \geq 1 \), \( 1 \leq \frac{\beta_i-t}{\alpha_i-t} \leq 2 \) and \( \alpha_i \neq t \) for \( i = 1, 2, \ldots, n \),

\[
B^\beta_{i-1} \geq B^\beta_i,
\]

where \( B_0 = B \) and \( B_i = (A^t \#_{\frac{\beta_i}{\alpha_i+t}} B^\alpha_i)_{i=1}^n \).
Theorem 3.2. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0,1]$, $\alpha_i \geq 1$, $\beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq 1$, $1 \leq \frac{\beta_i - t}{\alpha_i - t} \leq 2$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq A^{-\gamma} \frac{\beta_1}{\beta_1 + \gamma} B_{\beta_1} \geq A^{-\gamma} \frac{\beta_2}{\beta_2 + \gamma} B_{\beta_2} \geq \cdots \geq A^{-\gamma} \frac{\beta_n}{\beta_n + \gamma} B_{\beta_n} \geq A^{-\gamma} \frac{1 + \gamma}{\beta_1 + \gamma} B_{\beta_1} \geq \cdots \geq A^{-\gamma} \frac{1 + \gamma}{\beta_n + \gamma} B_{\beta_n},$$

where $B_0 = B$ and $B_i = (A^t \frac{\beta_i - t}{\alpha_i - t} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Theorem 3.3. Let $A \geq B \geq 0$ with $A > 0$ and $n$ be a natural number. Then for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ for $i = 1, 2, \ldots, n-1$, $\alpha_n \geq 1$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$G'(\beta_n) = A^{-\gamma} \frac{1}{\beta_n + \gamma} (A^t \frac{\beta_n - t}{\alpha_n - t} B_{\beta_n}^{\alpha_n})$$

is decreasing for $\beta_n \geq \alpha_n$, where $B_0 = B$ and $B_i = (A^t \frac{\beta_i - t}{\alpha_i - t} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Remark. (3.1) is also decreasing for $\gamma \geq 0$ by Theorem 1.B since $A \geq B \geq 0$ with $A > 0$ ensures $A \geq B_n = (A^t \frac{\beta_n - t}{\alpha_n - t} B_{\beta_n}^{\alpha_n})^{\frac{1}{\beta_n}}$ by Theorem 2.1. Therefore, similarly to Theorem 2.1, we recognize that Theorem 3.3 is a slight extension of (1.3) in Theorem 1.D.

References


[8] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


(Masatoshi Ito) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN
E-mail address: m-ito@maebashi-it.ac.jp

(Eizaburo Kamei) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN
E-mail address: kamei@maebashi-it.ac.jp