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Hiai-Petzの幾何構造について
On a structure of the Hiai-Petz geometry

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0. Introduction

Let $\mathcal{M}$ (resp. $\mathcal{M}^+$) be the $n \times n$ (complex) matrices (resp. positive definite matrices). Throughout this paper, a *path* $\gamma(t)$ in $\mathcal{M}^+$ means a smooth curve for $t \in [0,1]$ and $|||\ |||$ stands for any unitarily invariant norm for $\mathcal{M}$. For $A, B \in \mathcal{M}^+$, the path of the geometric operator means in the sense of Kubo-Ando [14] is defined as

$$A \#_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

The geodesic in the CPR (Corach-Porta-Recht) geometry is $A \#_t B$ and the induced distance by their Finsler metric (which is the length of this geodesic) is related to the relative operator entropy [3, 5, 6]:

$$S(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

As I rephrase their result in [8], the distance is now called the Thompson (part) metric for a unitarily invariant norm $|||\ |||$:

$$d(A, B) = |||\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}|||.$$

Recently Hiai and Petz [11] introduced a new geometry parametrized by each real number r with a pull-back metric for a diffeomorphism $A \mapsto \ln_r A$ to the Euclidian space where $\ln_r$ is an extended logarithm

$$\ln_r(x) = \begin{cases} \frac{x^r - 1}{r} & (r \neq 0) \\ \log x & (r = 0). \end{cases}$$

In this geometry, the geodesic is a chaotic quasi-arithmetic mean [7]

$$A\, \mathfrak{m}_{r,t} B = \ln_r^{-1}((1-t)\ln_r(A) + t\ln_r(B)) = ((1-t)A^r + tB^r)^{\frac{1}{r}}$$

and the distance with respect to their metric is

$$\ell(A\,\mathfrak{m}_{r,t}B) = |||\ln_r B - \ln_r A||| = d(A,B).$$

Here a *chaotic mean* in [7] means the binary operation $A\,\mathfrak{m}B$ on positive (invertible) operators A and B satisfying the following conditions:

monotonicity: $A \le C$ and $B \le D$ imply $A\,\mathfrak{m}B \ll C\,\mathfrak{m}D$.

semi-continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\,\mathfrak{m}B_n \Downarrow A\,\mathfrak{m}B$.

normalization: $A\,\mathfrak{m}A = A$,

where $A \ll B$ is the chaotic order $\log A \le \log B$ and $A_n \Downarrow A$ is the monotone convergence in the chaotic order. In fact, if $r \in [-1,1]$, then $A\,\mathfrak{m}_{r,t}B$ is a chaotic mean. Though the above means do not have monotonicity any longer for $|r| > 1$, we use the same symbols for the sake of convenience in this paper.

Hiai-Petz [11, Theorem 3.3] also introduced another parametrized geometry for $\alpha > 0$ whose geodesic is $(A^\alpha \#_t B^\alpha)^{\frac{1}{\alpha}}$, which is an extension of CPR geometry and the distance is

$$d(A,B) = \left|\left|\left|\frac{1}{\alpha}\log A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}}\right|\right|\right|.$$

In these geometry, their interests mainly in metrics and distances for the geodesics.

As in [8], like the CPR geometry, we discuss an upper structure of their geometry and obtain the geodesic as the autoparallel curve, that is, a unique solution γ of the geodesic differential equation $\nabla_{\dot\gamma}\dot\gamma = O$, which does not depend on metrics. After this, we confirm that the Hiai-Petz geometry has the Finsler metric induced by each unitarily invariant norm and real number r (positive number α).

1. Hiai-Kosaki-Petz linear transform

To see a structure for the Hiai-Petz geometry, we need a certain linear transform Φ_A on $\mathcal{M}^h$ assigned to each $A \in \mathcal{M}^+$, which is introduced below. First we note the following key lemma in the Hiai-Petz geometry which is expressed by the Hadamard product $\circ$. This is closely related to the Hiai-Kosaki mean [10]: Let $\mathbf{L}_A$(resp. $\mathbf{R}_A$) be the multiplication operator from the left (resp. right) for a selfadjoint matrix A. Then the *Hiai-Kosaki mean* on X for a mean function ϕ is a kind of meta-operator mean defined by

$$\phi(\mathbf{L}_A, \mathbf{R}_B)X = U\Big(\big(\phi(d_i, e_j)\big) \circ U^*XV\Big)V^*$$

for any diagonalization diag $(d_i) = U^*AU$ and diag $(e_i) = V^*BV$ for some unitaries U and V where $\circ$ means the Hadamard product. Here we use such a formula for the case $A = B$. Though it is also known in the theory of quantum information geometry, we give a proof for the reader's convenience (see [10] for the infinite-dimensional version):

Basic Lemma. *For a continuous function* $\phi(x, y)$,

$$\phi(\mathbf{L}_A, \mathbf{R}_A)X = U\Big((\phi(d_i, d_j)) \circ U^*XU\Big)U^*.$$

Proof. In the case of a monomial $\phi(x, y) = x^m y^n$, we have

$$\begin{aligned}\phi(\mathbf{L}_A, \mathbf{R}_A)X &= A^m X A^n = UD^mU^*XUD^nU^* = U\Big(\mathbf{L}_D^m\mathbf{R}_D^n U^*XU\Big)U^* \\ &= U\Big((d_i^m d_j^n) \circ U^*XU\Big)U^* = U\Big((\phi(d_i, d_j)) \circ U^*XU\Big)U^*.\end{aligned}$$

Approximating a general ϕ by polynomials, we have the required result. □

This lemma shows that the linear map on the tangent vector space $T_A(\mathcal{M}^+)$

$$\Phi_A(X) = U\left((\phi(d_i, d_j)) \circ U^*XU\right)U^*$$

is well-defined for any diagonalization U^*AU and the inverse map is

$$\Phi_A^{-1}(X) = U\left(\left(\frac{1}{\phi(d_i, d_j)}\right) \circ U^*XU\right)U^*$$

if $\phi(d_i, d_j) \neq 0$ for all i, j. Let $\gamma(t)$ be a path of selfadjoint matrices. From now on, U (resp. U_t) is assumed to be any unitary such that U^*AU (resp. $U_t^*\gamma(t)U_t$) is a diagonal matrix D (resp. D_t) with entries d_j (resp. $d_j(t)$). For a continuously differentiable function f, define

$$f^{[1]}(x, y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & (x \neq y) \\ f'(x) & (x = y). \end{cases}$$

Then, putting $f_n(x) = x^n$, we have $f_n^{[1]}(x, y) = \frac{x^n - y^n}{x-y} = \sum_{k=1}^n x^{k-1}y^{n-k}$ (nx^{n-1} if $x = y$) and then

$$U_t\left((f_n^{[1]}(d_i(t), d_j(t))) \circ U_t^*\dot{\gamma}(t)U_t\right)U_t^* = f_n^{[1]}(\mathbf{L}_\gamma, \mathbf{R}_\gamma)\dot{\gamma}(t) = (\gamma(t)^n)'.$$

So, for general f, we have a well-known derivative formula, see [1, p.124] and [12, 6.6.30] (it is also called **the Deletskii-Krein formula**):

$$\frac{df(\gamma(t))}{dt} = U_t\left((f^{[1]}(d_i(t), d_j(t))) \circ U_t^*\dot{\gamma}(t)U_t\right)U_t^*$$

(Note that $f(\gamma(t))$ is differentiable though each U_t is not always so).

Now we define the Hiai-Petz action by the extended logarithmic function $\ln_r$. Here we mention that $\ln_0^{[1]}(x, y) = 1/\ell(x, y)$ where ℓ is the *logarithmic mean*. Since $\ln_r^{[1]}(x, y) > 0$ for all $x, y > 0$, we define the invertible linear map

$$\Phi_{A,r}(X) \equiv \Phi_{A,\ln_r}(X) = U\left(\left(\ln_r^{[1]}(d_i, d_j)\right) \circ U^*XU\right)U^*.$$

In this case, note that

$$\Phi_{\gamma(t),r}(\dot\gamma(t)) = \begin{cases} \frac{1}{r}\big(\gamma(t)^r - 1\big)' = \frac{1}{r}\big(\gamma(t)^r\big)' & (r \neq 0) \\ \big(\log(\gamma(t))\big)' & (r = 0). \end{cases}$$

For $r \in [-1, 1]$, the function $\ln_r$ is operator monotone and the Löwner matrix $\big(\ln_r^{[1]}(d_i, d_j)\big)$ is positive semidefinite. In general, though it is not always positive semidefinite, it is selfadjoint and so is $\Phi_{A,r}(B)$ for $B \in \mathcal{M}^+$. Moreover the map $\Phi_{A,r}$ leaves $\mathcal{M}^h$ invariant.

This map is well-behaved under unitary conjugation:

Lemma 1.1. *If V is a unitary matrix, then*

$$\Phi_{VAV^*,r}(VXV^*) = V\Phi_{A,r}(X)V^* \quad \textit{and} \quad \Phi^{-1}_{VAV^*,r}(VXV^*) = V\Phi^{-1}_{A,r}(X)V^*.$$

Proof. Under diagonalization $D = U^*AU = U^*V^*(VAV^*)VU$, we have

$$\begin{aligned} \Phi_{VAV^*,r}(VXV^*) &= VU\left(\left(\ln_r^{[1]}(d_i, d_j)\right) \circ U^*V^*(VXV^*)VU\right)U^*V^* \\ &= VU\left(\left(\ln_r^{[1]}(d_i, d_j)\right) \circ U^*XU\right)U^*V^* = V\Phi_{A,r}(X)V^*. \end{aligned}$$

The latter formula follows immediately from this. □

2. Chaotic mean type geometry

Now we observe the upper structure of one of the Hiai-Petz geometries whose geodesic is $A\,\mathrm{m}_{r,t}B$. Here it is called **the chatic mean type geometry**. For each real number r, consider the trivial principal bundle $\mathcal{P}_r = \mathcal{M}^+ \times \mathcal{U}$ for $\mathcal{M}^+$ with the trivial projection $\pi((A, V)) = A$. We may define the parametrized action $\Psi_r((A, V))X = \Phi^{-1}_{A,r}(VXV^*)$ of $\mathcal{P}_r$ on $T_A\mathcal{M}^+ = \mathcal{M}^h$. Here we observe the associated tangent vector bundle

$$\mathcal{P}_r \times \mathcal{M}^h / \,\mathcal{U} = \mathcal{P}_r \underset{\rho}{\times} \mathcal{M}^h$$

with the fiber $\mathcal{M}^h$ with the right action $(A, V)W = (A, VW)$ of $W \in \mathcal{U}$ on $\pi^{-1}(A) \subset \mathcal{P}_r$ and the left action $\rho(W)X = WXW^*$ on the tangent space $T_A\mathcal{M}^+ = \mathcal{M}^h$. We remark that it can be identified with $\mathcal{M}^h$ by $((A, V), X) \mapsto \Psi_r((A, V))(X)$ since

$$\begin{aligned} \Psi_r((A, V)W)\rho^{-1}(W)X = \Psi_r((A, VW))W^*XW &= \Phi^{-1}_{A,r}(VWW^*XWW^*V^*) \\ &= \Phi^{-1}_{A,r}(VXV^*) = \Psi_r((A, V))(X). \end{aligned}$$

This identification shows that we can determine the parallel displacement of tangent vectors along the curve γ by the connection of $\mathcal{P}_r$ and a horizontal lift of γ as in the below, see also [13]. The horizontality (hence connection) in the tangent space of $\mathcal{P}_r$ is naturally given by a common unitary entry (it is called the *canonical* *flat connection*). So the horizontal lift Γ of a path γ is $\Gamma(t) = (\gamma(t), V)$ for any fixed $V \in \mathcal{U}$. Recall that the notion of the connection of the principal bundle is equivalent to that of covariant derivative (hence parallel displacement) of the associated vector bundle. So we give the latter to obtain the geodesic for this connection. Since a tangent vector $Y \in \mathcal{M}^h$ also belongs to the associated bundle $\mathcal{M}^h$ of $\mathcal{P}_r$ and

$$\Psi_r((A,V))^{-1}Y = V^*\Phi_{A,r}(Y)V,$$

we have that the *parallel displacement* $P_t = P_t{}^0$ *from* 0 *to* t *along a path* γ of a tangent vector X on $\gamma(0)$ is obtained by

$$\begin{aligned} P_tX &= \Psi_r((\gamma(t),V))\left(\Psi_r((\gamma(0),V))^{-1}X\right) \\ &= \Phi^{-1}_{\gamma(t),r}(VV^*\Phi_{\gamma(0),r}(X)V^*V) = \Phi^{-1}_{\gamma(t),r}(\Phi_{\gamma(0),r}(X)). \end{aligned}$$

Then the *covariant derivative* for a vector field $X(t)$ is

$$\begin{aligned} \nabla_{\dot\gamma}X &= \lim_{\varepsilon\to 0}\frac{P_t{}^{t+\varepsilon}X(t+\varepsilon)-X(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{\Phi^{-1}_{\gamma(t),r}\left(\Phi_{\gamma(t+\varepsilon),r}(X(t+\varepsilon))\right)-X(t)}{\varepsilon} = \Phi^{-1}_{\gamma(t),r}\left(\left(\Phi_{\gamma(t),r}(X(t))\right)'\right). \end{aligned}$$

Let $\mathcal{M}^+_r$ be the manifold $\mathcal{M}^+$ with the principal bundle $\mathcal{P}_r$ and the actions above. Then we have geodesics in $\mathcal{M}^+_r$:

Theorem 2.1. *The geodesic* γ *from* A *to* B *in* $\mathcal{M}^+_r$ *is* $A\,\mathfrak{m}_{r,t}B$.

Proof. Suppose $r \neq 0$. Then the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = O$ implies

$$O = \Phi_{\gamma(t),r}(\nabla_{\dot\gamma}\dot\gamma) = \left(\Phi_{\gamma(t),r}(\dot\gamma(t))\right)' = \frac{1}{r}\left(\gamma(t)^r\right)''.$$

So there exist a selfadjoint C_1 and $C_2 \in \mathcal{M}^+$ with $\gamma(t)^r = tC_1 + C_2$. Since

$$A^r = \gamma(0)^r = C_2 \quad \text{and} \quad B^r = \gamma(1)^r = C_1 + C_2,$$

we have $C_2 = A^r$ and $C_1 = B^r - A^r$, so that $\gamma(t) = A\,\mathfrak{m}_{r,t}B$. For $r = 0$, we also have $\gamma(t) = \exp\left((1-t)\log A + t\log B\right)$ considering $\left(\log\gamma(t)\right)'' = O$. $\square$

Thus the *Hiai-Petz geometry* $\mathcal{M}^+_r$ has the above structure induced by $\mathcal{P}_r$.

Now we show the Hiai-Petz metric defines a Finsler one in the sense of Cartan [15, 16]:

Theorem 2.2. *For any unitarily invariant norm* $|||\ |||$, *the norm of* $X \in \mathcal{M}^h$ *defined as*

$$L_r(X;A) \equiv L_{r,|||\ |||}(X;A) \equiv \left|\left|\left|\Phi_{A,r}(X)\right|\right|\right| = \left|\left|\left|\left(\ln_r^{[1]}(d_i,d_j)\right)\circ U^*XU\right|\right|\right|.$$

is a Finsler metric, that is, it is equivalent to the original norm and satisfies the Finsler condition $L_r(X;\gamma(0)) = L_r(P_tX;\gamma(t))$ *for all path* γ.

Here we observe that this Finsler metric is not homogeneous in the preceding sense, but it is invariant under unitary conjugation.

Theorem 2.3. *For any unitarily invariant norm* $|||\ |||$, *if* V *is a unitary, then*

$$L_r(VXV^*; VAV^*) = L_r(X; A).$$

3. CPR type geometry

Next, we discuss structure of another Hiai-Petz parametrized geometry for $\alpha > 0$ in [11, Theorem 3.3] whose geodesic is $(A^\alpha \#_t B^\alpha)^{\frac{1}{\alpha}}$, which is a generalization of the CPR geometry and the Bhatia-Holbrook one [2].

Let $\mathcal{P}_{[\alpha]} = \{\mathcal{G}, \mathcal{M}^+, \mathcal{U}, \pi_{[\alpha]}\}$ be a principal bundle where $\pi_{[\alpha]}(G) = (GG^*)^{\frac{1}{\alpha}}$ with a natural right action of $V \in \mathcal{U}$: $G \mapsto GV$. Like the CPR geometry, the connection is defined by the horizontal subspace $\{GY | Y = Y^*\}$ of the tangent space $T_G\mathcal{G}$. Let Γ be a horizontal lift of a path γ. Then the horizontality shows $\Gamma^{-1}\dot{\Gamma} = (\Gamma^{-1}\dot{\Gamma})^* = \dot{\Gamma}^*(\Gamma^*)^{-1}$. Since $\gamma = \pi_{[\alpha]}(\Gamma) = (\Gamma\Gamma^*)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}(\gamma^\alpha)'\gamma^{-\alpha} &= (\dot{\Gamma}\Gamma^* + \Gamma\dot{\Gamma}^*)(\Gamma\Gamma^*)^{-1} = \dot{\Gamma}\Gamma^{-1} + \Gamma\dot{\Gamma}^*(\Gamma^*)^{-1}\Gamma^{-1} \\ &= \dot{\Gamma}\Gamma^{-1} + \Gamma\Gamma^{-1}\dot{\Gamma}\Gamma^{-1} = 2\dot{\Gamma}\Gamma^{-1},\end{aligned}$$

so that we have the *transport equation* which defines Γ: $\dot{\Gamma} = \frac{1}{2}(\gamma^\alpha)'\gamma^{-\alpha}\Gamma$. Based on an action by each function $f_\alpha(x) = x^\alpha$

$$\Phi_A(X) \equiv \Phi_A^{[\alpha]}(X) = U\left[\left(f_\alpha^{[1]}(d_i, d_j)\right) \circ U^*XU\right]U^*$$

for a diagonalization $U^*AU = D = \operatorname{diag}(d_j)$, we define an action of G on the tangent vector X at A by

$$\Theta(G)X \equiv \Theta_\alpha(G)X = \Phi_A^{-1}(GXG^*),$$

and consequently the inverse action is

$$\Theta(G)^{-1}X = G^{-1}\Phi_A(X)(G^*)^{-1}.$$

Consider the associated bundle $\mathcal{P}_{[\alpha]} \times \mathcal{M}^h/\mathcal{U}$ with the natural left action $\rho(V)X = VXV^*$ of $V \in \mathcal{U}$ on the tangent vector X at A. As in the former case, we can identify it with the tangent bundle $\mathcal{M}^h$ by the map $(G, X) \mapsto \Theta(G)X$ since

$$\Theta(GV)V^*XV = \Phi_A^{-1}(GV(V^*XV)V^*G^*) = \Phi_A^{-1}(GXG^*) = \Theta(G)X.$$

So we can discuss the parallel displacement of X from the structure of $\mathcal{P}_{[\alpha]}$.

Then the *parallel displacement* (from 0 to t) of the tangent vector field X along γ is

$$\begin{aligned} P_t X &\equiv P_t{}^0 X(t) = \Theta(\Gamma(t))\left(\Theta(\Gamma(0))^{-1}X(0)\right) \\ &= \Phi_{\gamma(t)}^{-1}\left(\Gamma(t)\Gamma(0)^{-1}\Phi_{\gamma(0)}(X(0))(\Gamma(0)^*)^{-1}\Gamma(t)^*\right) \end{aligned}$$

and hence the *covariant derivative* is obtained by

$$\begin{aligned} \nabla_{\dot\gamma}X &= \lim_{\varepsilon\to 0}\frac{P_t^{t+\varepsilon}X(t+\varepsilon)-X(t)}{\varepsilon} \\ &= \Theta(\Gamma(t))\left(\left[\Theta(\Gamma(t))^{-1}(X(t))\right]'\right) \\ &= \Phi_\gamma^{-1}\left(\Gamma(t)\left[\Gamma(t)^{-1}\Phi_\gamma(X(t))(\Gamma(t)^*)^{-1}\right]'\Gamma(t)^*\right) \\ &= \Phi_\gamma^{-1}\left((\Phi_\gamma(X))' - \dot\Gamma\Gamma^{-1}\Phi_\gamma(X) - \Phi_\gamma(X)(\Gamma^*)^{-1}\dot\Gamma^*\right) \\ &= \Phi_\gamma^{-1}\left((\Phi_\gamma(X))' - \frac{(\gamma^\alpha)'\gamma^{-\alpha}\Phi_\gamma(X) + \Phi_\gamma(X)\gamma^{-\alpha}(\gamma^\alpha)'}{2}\right). \end{aligned}$$

Therefore we have the *geodesic equation*

$$(\gamma^\alpha)'' = (\gamma^\alpha)'\gamma^{-\alpha}(\gamma^\alpha)'$$

because $\Phi_\gamma(\dot\gamma) = (\gamma^\alpha)'$ and $\nabla_{\dot\gamma}\dot\gamma = O$. Putting

$$f(t) = \gamma(0)^{-\alpha/2}\gamma(t)^\alpha\gamma(0)^{-\alpha/2}$$

for a path γ from A to B, we have

$$f(0) = I, \quad f(1) = A^{-\alpha/2}B^\alpha A^{-\alpha/2} \quad \text{and} \quad f'' = f'f^{-1}f'.$$

The CPR theory shows that $f(t) = (A^{-\alpha/2}B^\alpha A^{-\alpha/2})^t$ and consequently the geodesic is given by

$$\gamma(t)^\alpha = A^{\alpha/2}(A^{-\alpha/2}B^\alpha A^{-\alpha/2})^t A^{\alpha/2} = A^\alpha \#_t B^\alpha.$$

For each unitarily invariant norm $|||\ |||$, define a metric

$$L(X;A) \equiv L_{[\alpha]}(X;A) = \frac{1}{\alpha}|||A^{-\frac{\alpha}{2}}\Phi_A(X)A^{-\frac{\alpha}{2}}|||.$$

Then the unitary invariance shows that

$$\begin{aligned} L(X;A) &= \frac{1}{\alpha}\left|\left|\left|U^*A^{-\frac{\alpha}{2}}U\left[(f_\alpha^{[1]}(d_i,d_j))\circ U^*XU\right]U^*A^{-\frac{\alpha}{2}}U\right|\right|\right| \\ &= \frac{1}{\alpha}\left|\left|\left|D^{-\frac{\alpha}{2}}\left[(f_\alpha^{[1]}(d_i,d_j))\circ U^*XU\right]D^{-\frac{\alpha}{2}}\right|\right|\right| \\ &= \frac{1}{\alpha}\left|\left|\left|\left(\frac{f_\alpha^{[1]}(d_i,d_j)}{d_i^{\frac{\alpha}{2}}d_j^{\frac{\alpha}{2}}}\right)\circ U^*XU\right|\right|\right|, \end{aligned}$$

which is the Hiai-Petz metric in [11, Theorem 3.3]. Noting the matrix

$$V = \gamma(t)^{-\frac{\alpha}{2}}\Gamma(t)\Gamma(0)^{-1}\gamma(0)^{\frac{\alpha}{2}}$$

being unitary and the relation

$$\Phi_{\gamma(t)}(P_tX) = \Gamma(t)\Gamma(0)^{-1}\Phi_{\gamma(0)}(X)(\Gamma(0)^*)^{-1}\Gamma(t)^*,$$

we have it is a Finsler one:

$$\begin{aligned}\alpha L(P_tX;\gamma(t)) &= |||\gamma(t)^{-\frac{\alpha}{2}}\Phi_{\gamma(t)}(P_tX)\gamma(t)^{-\frac{\alpha}{2}}||| \\ &= |||\gamma(t)^{-\frac{\alpha}{2}}\Gamma(t)\Gamma(0)^{-1}\Phi_{\gamma(0)}(X)(\Gamma(0)^*)^{-1}\Gamma(t)^*\gamma(t)^{-\frac{\alpha}{2}}||| \\ &= |||V\gamma(0)^{-\frac{\alpha}{2}}\Phi_{\gamma(0)}(X)\gamma(0)^{-\frac{\alpha}{2}}V^*||| = \alpha L(X;\gamma(0)).\end{aligned}$$

Thus we summarize the above facts:

Theorem 3.1. *In the above setting, the principal bundle $\mathcal{P}_{[\alpha]} = \{\mathcal{G}, \mathcal{M}^+, \mathcal{U}, \pi_{[\alpha]}\}$ for $\alpha > 0$ defines a Finsler structure of $\mathcal{M}^+$ where the geodesic from A to B is $\gamma(t) = (A^\alpha \#_t B^\alpha)^{\frac{1}{\alpha}}$ and each metric*

$$L(X;A) = \frac{1}{\alpha}|||A^{-\frac{\alpha}{2}}\Phi_A(X)A^{-\frac{\alpha}{2}}|||$$

is a Finsler metric for each unitarily invariant norm $|||\ |||$.

4. Shortest path

Finally, we discuss whether the geodesic is the unique shortest path between two matrices. The length $\ell(\gamma)$ of a curve γ from A to B under a Finsler metric L is obtained by

$$\ell(\gamma) = \int_0^1 L(\dot\gamma(t);\gamma(t))dt.$$

The inavariant property under the parallel displacement shows if γ is a geodesic, then

$$L(\dot\gamma(t);\gamma(t)) = L(\dot\gamma(0);\gamma(0))$$

holds, so that the length of the geodesic is

$$\ell(\gamma) = L(\dot\gamma(0);\gamma(0)).$$

Thereby, in the chaotic mean type geometry, the Finsler metric is

$$L_r(X;A) = \left|\left|\left| \left(\ln_r^{[1]}(d_i,d_j)\right) \circ U^*XU \right|\right|\right|,$$

and then the length is

$$\ell(A\,\mathfrak{m}_{r,t}B) = |||\ln_r B - \ln_r A|||.$$

Also in the CPR type geometry, the Finsler metric is

$$L_{\alpha]}(X;A) = \frac{1}{\alpha}\left|\left|\left|A^{-\frac{\alpha}{2}}U\left[\left(f_{\alpha}^{[1]}(d_i,d_j)\right)\circ U^*XU\right]U^*A^{-\frac{\alpha}{2}}\right|\right|\right|$$

and the length is

$$\ell((A^{\alpha}\#_t B^{\alpha})^{\frac{1}{\alpha}}) = \frac{1}{\alpha}|||\log A^{-\frac{\alpha}{2}}B^{\alpha}A^{-\frac{\alpha}{2}}|||.$$

It is easy to see that these length are the shortest ones respectively.

Now, recall that a norm $|||\ |||$ is **strictly convex** if

$$|||(1-t)x+ty||| < 1$$

holds for $t \in (0,1)$ and distinct unit vectors x and y. Then we have ([9]):

Theorem 4.1. *If a unitarily invariant norm is strictly convex, the geodesic $A\,\mathfrak{m}_{r,t}B$ (resp. $(A^{\alpha}\#_t B^{\alpha})^{\frac{1}{\alpha}}$) is the unique shortest path under the Finsler metric $L_r(X;A)$ (resp. $L_{[\alpha]}(X;A)$).*

Typical unitarily invariant norms which are not strongly convex are Ky Fan's, that is $\|X\|_{(k)}$ means the sum of singular values for X from the largest to the k-th. In this case, the shortest paths are not uniquely determined for the Hiai-Petz geometries as in the following example: Let $B=(b_j)$ be a diagonal positive-definite matrix greater than I with b_j is (strictly) monotone decreasing. For a path from I to B, the shortest length is $\|\ln_r B\|_{(k)}$. Then, for two path of distinct means $\mathrm{m}_t \neq \mathrm{n}_t$, we have $I\mathrm{m}_t B$ is different from $I\mathrm{n}_t B$ as paths by the strict monotonicity for b_j.

First we give examples in the chaotic mean type geometry. In the case $r=1$, let $\delta(t) = B^t$ which differs from the geodesic $(1-t)I+tB$. Then $\dot{\delta}(t) = B^t \log B \geq O$ and $x^t \log x$ is monotone increasing for $x > 1$. Since $\ln_1'(x) = 1$, we can verify that δ also attains the shortest length. In the case $r \geqq 0$ and $r \neq 1$, let $\gamma(t) = (1-t)I+tB = I+t(B-I)$. Then γ attains the shortest length. In the case $r<0$, suppose $1 < b_k < 1-\frac{1}{r}$. Then γ attains the shortest.

In the CPR type geometry, a path defined by

$$\delta(t) = (1-t+tB^{\alpha})^{1/\alpha},$$

also attains the shortest.

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