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Kyoto University
An application of Uno correspondences in $p$-solvable groups

K. Uno extended the Glauberman-Isaacs correspondences between $\text{Irr}(G)^A$ and $\text{Irr}(C)$ to a correspondence between $\text{IBr}(G)^A$ and $\text{IBr}(C)$ when $G$ is $p$-solvable ([5], Theorem). We prove the following by using the Uno correspondence.

**Theorem 1** Assume Hypothesis 1 and that $G$ is $p$-solvable. Let $Q \leq C$ be a $p$-subgroup. There exists a bijection

\[ \pi(G, A; Q) : \text{Ind}(\mathcal{R}G|Q)^A \rightarrow \text{Ind}(\mathcal{R}C|Q) \]

which satisfies the following (i) and (ii). For $[V] \in \text{Ind}(\mathcal{R}G|Q)^A$, set $[V'] = \pi(G, A; Q)([V])$.

(i) If $B \leq A$, then $\pi(G, A; Q) = \pi(C_G(B), A/B; Q)\pi(G, B; Q)$.

(ii) Assume $A$ is an $r$-group where $r$ is a prime. Then $V'$ is a unique $A$-invariant indecomposable component of $V' \uparrow^G$ with vertex $Q$ and with the multiplicity prime to $r$. Moreover $V'$ is a unique indecomposable component of $V \downarrow^C$ with the multiplicity prime to $r$ and with vertex $Q$, and we have also $m(V', V \downarrow^C) \equiv m(V, V' \uparrow^G) \pmod{r}$.

In particular, if $A$ is solvable then $\pi(G, A; Q)$ is uniquely determined.

Let $G$ be an $\mathcal{R}$-algebra which is finitely generated as an $\mathcal{R}$-module. We denote by $\mathcal{P}(G)$ the set of points of $G$. For $\epsilon \in \mathcal{P}(G)$, we denote by $P_\epsilon$ a corresponding projective indecomposable $G$-module. If a group $A$ acts on $G$ via $\mathcal{R}$-algebra automorphisms, then $A$ acts on $\mathcal{P}(G)$.

1 Correspondences for principal indecomposable modules

Assume Hypothesis 1. Then $A$ acts on $\mathcal{R}G$ via $\mathcal{R}$-algebra automorphisms. Let $H \leq G$ and $L$ be an $\mathcal{R}H$-module. For $a \in A$, $L^a = \{ l^a \mid l \in L \}$ can be regarded as an $\mathcal{R}$-module isomorphic to $L$ by the map $l \mapsto l^a$. Moreover $L^a$ becomes an $\mathcal{R}H^a$-module by the action

\[ l^ah^a = (lh)^a \quad (l \in L, \ h \in H). \]
For $a, b \in A$, we have

$$(L^a)^b \cong L^{ab} \quad (a, b \in A).$$

Thus if $H$ is $A$-invariant, then $A$ acts on the $RH$-modules.

**Hypothesis 2** With Hypothesis 1, $G$ is $p$-solvable.

**Theorem 2** (Uno [5]) Assume Hypothesis 2. There exists a bijection

$$\rho(G, A) : \text{IBr}(G)^A \to \text{IBr}(C)$$

which satisfies the following (i) and (ii). For $\beta \in \text{IBr}(G)^A$, set $\beta' = \rho(G, A)(\beta)$.

(i) If $B \triangleleft A$, then $\rho(G, A) = \rho(C_G(B), A/B)\rho(G, B)$.

(ii) If $A$ is an $r$-group for a prime $r$, then $\beta'$ is a unique irreducible constituent of $\beta \downarrow_C$ with the multiplicity prime to $r$. Moreover $\beta$ is a unique $A$-invariant irreducible constituent of $\beta' \uparrow^G$ with the multiplicity prime to $r$, and we have also $m(\beta, \beta' \uparrow^G) \equiv m(\beta', \beta \downarrow_C) \pmod{r}$.

**Proof.** (i) is already shown. (ii) Assume $A$ is an $r$-group. In general if $\chi$ is a character of $G$, then the restriction of $\chi$ to the $p$-regular elements is denoted by $\chi^*$. By the arguments in [5], there is an element $\chi \in \text{Irr}(G)^A$ such that $\beta = \chi^*$, $\beta' = (\chi')^*$, where $\chi'$ is the Glauberman correspondent of $\chi$. Here note $\beta' \uparrow^G = (\chi' \uparrow^G)^*$. Now we can set

$$\chi' \uparrow^G = m\chi + \sum_{i=1}^{s} m_i \chi_i + \sum_{j=s+1}^{t} m_j \chi_j$$

where $\chi_i$ $(1 \leq i \leq s)$ are $A$-invariant irreducible characters of $G$ different from $\chi$ and $\chi_j$ $(s + 1 \leq j \leq t)$ are not $A$-invariant. Then $r \nmid m$ and $r | m_i$. We note if $\chi_j$ and $\chi_{j'}$ $(j, j' \geq s + 1)$ are $A$-conjugate, then $m_j = m_{j'}$. Then, for any $\gamma \in \text{IBr}(G)^A$, the decomposition numbers $d_{\chi_j \gamma}$ and $d_{\chi_{j'} \gamma}$ are equal. Hence, since $A$ is an $r$-group, $\beta$ is a unique $A$-invariant irreducible constituent of $\beta' \uparrow^G$ with the multiplicity prime to $r$, and $m(\beta, \beta' \uparrow^G) \equiv m (\beta', \beta \downarrow_C) \pmod{r}$. On the other hand $m = m(\chi', \chi \downarrow_C) \equiv m(\beta', \beta \downarrow_C) \pmod{r}$ because $\chi \downarrow_C = m\chi' + r\zeta$ where $\zeta = 0$ or $\zeta$ is a character of $C$. This completes the proof. 

Let $M_\epsilon$ be an irreducible $kG$-module corresponding to $\epsilon \in \mathcal{P}(kG)$. We have $(P_\epsilon)^a \cong P_\epsilon$ and $(M_\epsilon)^a \cong M_\epsilon$ for any $a \in A$. Hence by the above theorem and the Frobenius-Nakayama’s reciprocity theorem we have the following.

**Proposition 1** Assume Hypothesis 2. There exists a bijection

$$\tilde{\pi}(G, A) : \mathcal{P}(kG)^A \to \mathcal{P}(kC)$$

which satisfies the following (i) and (ii). For $\epsilon \in \mathcal{P}(kG)^A$, set $\epsilon' = \tilde{\pi}(G, A)(\epsilon)$.

(i) If $B \triangleleft A$, then $\tilde{\pi}(G, A) = \tilde{\pi}(C_G(B), A/B)\tilde{\pi}(G, B)$.

(ii) Assume $A$ is $r$-group for a prime $r$. Then $\epsilon$ is a unique element of $\mathcal{P}(kG)^A$ such that $r \nmid m(P_\epsilon, P_\epsilon \uparrow^G)$. Moreover $\epsilon'$ is a unique element of $\mathcal{P}(kC)$ such that $r \nmid m(P_\epsilon, P_\epsilon \downarrow_C)$, and we have also $m(P_\epsilon, P_\epsilon \uparrow^G) \equiv m(P_\epsilon, P_\epsilon \downarrow_C) \pmod{r}$.

**Remark 1** In the above proposition,

$$m(P_\epsilon, P_\epsilon \uparrow^G) \neq 0.$$
Proof. Since $\chi'$ is a constituent of $\chi \downarrow C$, $\beta'$ is a constituent of $\beta \downarrow C$ where $\beta \in \text{IBr}(G)^A$.

Hypothesis 3 With Hypothesis 1, $k_\ast G$ is a twisted group algebra of $G$ over $k$ with a basis $\{u_x \mid x \in G\}$ and with a factor set $\alpha$. Moreover $A$ acts on $k_\ast G$ via $k$-algebra automorphisms and the following holds:

$$(ku_x)^a = ku_{x^a} \quad (\forall x \in G, \forall a \in A).$$

Proposition 2 Assume Hypothesis 3. There is a central extension of $G$ which satisfies (i) - (iii).

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1$$

(i) $|Z| = (|G|_{p'})^2$,

(ii) The action of $A$ on $G$ is extended to $\hat{G}$, that is, $f(y^a) = f(y)^a$ ($\forall y \in \hat{G}, \forall a \in A$). Moreover $f^{-1}(C) = C_{\hat{G}}(A)$,

(iii) There are an idempotent $e$ of $kZ$ and a $k$-algebra isomorphism $\tilde{f} : e(k\hat{G}) \to k_\ast G$ compatible with the action of $A$.

Proof. We may assume $\alpha$ satisfies

$$\alpha(x, x')^{|G|} = 1, \quad \alpha(x, 1) = \alpha(1, x) = 1 \quad (\forall x, x' \in G).$$

Then $u_1$ is the identity element of $k_\ast G$. Set $V = \{\beta \in k^\times \mid \beta^{[G]} = 1\} = \{\beta \in k^\times \mid \beta^{[G]_{p'}} = 1\}$. By the assumption, for each $a \in A$ and $x \in G$, we can write

$$(u_x)^a = c(a, x)u_{x^a} \quad (c(a, x) \in k^\times).$$

From $(u_x u_{x'})^a = (u_x)^a (u_{x'})^a$

$$(1) \quad \alpha(x, x')c(a, xx') = c(a, x)c(a, x')\alpha(x^a, x'^a).$$

Therefore $c(a, x)c(a, x')c(a, xx')^{-1} \in V$ for each $a \in A$ and $x, x' \in G$. Hence $c(a, x)^{|G|^2} = 1$, and hence $c(a, x)^{(|G|_{p'})^2} = 1$. Since $(u_x)^{ab} = ((u_x)^a)^b$

$$(2) \quad c(ab, x) = c(a, x)c(b, x^a).$$

Moreover we have $c(a, 1) = 1$. Set $H = \{\beta \in k^\times \mid \beta^{[G]_{p'}} = 1\}$. (We will construct a central extension of $G$ using the method in [3], 3.5.15, and [6], (10.4)) We define the multiplication in $\hat{G} = H \times G$ as follows:

$$(h, x)(h', x') = (hh'\alpha(x, x'), xx').$$

Then $\hat{G}$ forms a group with the identity element $(1, 1)$. Also $Z = H \times 1$ is a central subgroup of $\hat{G}$. The map $\iota : Z \to H((h, 1) \mapsto h)$ is an isomorphism, in particular $Z$ is a $p'$-group. Moreover $(|Z|, |A|) = 1$. Therefore if

$$f : \hat{G} \to G ((h, x) \mapsto x)$$

then

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1$$
is a central extension of $G$ which satisfies (i).

By using (1) and (2)

$$(h, x)^a = (hc(a, x), x^a) \quad (\forall(h, x) \in \hat{G}, \forall a \in A)$$

defines an action of $A$ on $\hat{G}$ via group automorphisms. We note $A$ centralizes $Z$ and $f(y^a) = f(y)^a \quad (y \in \hat{G})$. Moreover $C_G(A) \subseteq f^{-1}(C)$. Let $c \in f^{-1}(C)$ and $a \in A$. We have $c^a = zc$ for some $z \in Z$. Since $z^{o(a)} = 1$, $z = 1$, so $c \in C_G(A)$. Thus (iii) holds.

For any $x \in G$, set $\hat{x} = (1, x)$. We have $\hat{x} \hat{x}' = (\alpha(x, x'), 1)\hat{xx}'$, $(h, x) = (h, 1)\hat{x}$ $(\forall h \in H)$. Moreover

$$e = \frac{1}{|Z|} \sum_{z \in Z} \iota(z^{-1})z$$

is an idempotent of $kZ$ and for any $z \in Z$, and we have $ze = \iota(z)e$. Therefore $e(k\hat{G}) = \bigoplus_{x \in G} k(e\hat{x})$, $(e\hat{x})(e\hat{x}') = \alpha(x, x')(e\hat{xx}')$. This implies

$$\tilde{f} : e(k\hat{G}) \to k_*G \left( \sum_{x \in G} c_x(e\hat{x}) \mapsto \sum_{x \in G} c_x u_x \right)$$
is an isomorphism of $k$-algebras. Moreover if $a \in A$, then

$$\tilde{f}(e\hat{x}^a) = c(a, x)u_{x^a} = \tilde{f}(e\hat{x})^a \quad (\forall x \in G).$$

Thus (iii) holds. \(\blacksquare\)

**Remark 2** With Hypothesis 3, $A$ centralizes $k_*C$.

**Proof.** Our proof is the same as the proof of [2], 7.6. From (1) and (2)

$$c(a, xy) = c(a, x)c(a, y),$$

$$c(ab, x) = c(a, x)c(b, x) \quad (\forall a, b \in A, \forall x, y \in C).$$

The fact that $(|A|, |C|) = 1$ implies $c(a, x) = 1$.

**Proposition 3** Assume Hypotheses 2 and 3. There exists a bijection

$$\pi_*(G, A) : \mathcal{P}(k_*G)^A \to \mathcal{P}(k_*C)$$

which satisfies the following (i) and (ii). For $\epsilon \in \mathcal{P}(k_*G)^A$, set $\epsilon' = \pi_*(G, A)(\epsilon)$.

(i) If $B \trianglelefteq A$, then $\pi_*(G, A) = \pi_*(C_G(B), A/B)\pi_*(G, B)$.

(ii) Assume $A$ is an $r$-group for a prime $r$. Then $\epsilon$ is a unique element of $\mathcal{P}(k_*G)^A$ such that $r \nmid m(P_\epsilon, P_\epsilon \otimes_{k_*C} k_*G)$. Moreover $\epsilon'$ is a unique element of $\mathcal{P}(k_*C)$ such that $r \nmid m(P_\epsilon', P_\epsilon' \downarrow_{k_*C})$, and we have also $m(P_\epsilon, P_\epsilon \downarrow_{k_*C}) \equiv m(P_\epsilon, P_\epsilon \otimes_{k_*C} k_*G) \mod r)$.

**Proof.** We will use Proposition 1. At first we note that $\hat{G}$ is $p$-solvable. For a subgroup $U$ of $G$, set $\hat{U} = f^{-1}(U)$. Then the $k$-algebras $e(k\hat{U})$ and $k_*U$ are isomorphic by the isomorphism $\tilde{f}|_{e(k\hat{U})}$. For $\delta \in \mathcal{P}(k_*U)$, we set $\delta = \tilde{f}(\delta) \in \mathcal{P}(e(k\hat{U}))$. Note that when
$U$ is $A$-invariant, $\hat{\delta}$ is $A$-invariant if and only if $\delta$ is $A$-invariant. For $\delta \in \mathcal{P}(k_*U)$ and $\epsilon \in \mathcal{P}(k_*G)$, we have

\[ m(P_{\epsilon}, P_\delta \otimes_{k_*U} k_*G) = m(P_{\epsilon}, P_\delta \uparrow^G) \]

\[ m(P_\delta, P_\epsilon \downarrow_{k_*G}) = m(P_\delta, P_\epsilon \downarrow_{k_*U}) \]

Since $Z$ is a central subgroup of $\hat{G}$, a point of $k\hat{G}$ is a point of the $k$-algebra $e(k\hat{G})$ or a point of $(1-e)(k\hat{G})$. If $\hat{\mu}$ is a point of $e(k\hat{G})$, then $(P_{\hat{\mu}} \uparrow^G)(1-e) = 0$. Hence the bijection $\tilde{\pi}(\hat{G}, A)$ in Proposition 1 induces a bijection from $\mathcal{P}(e(k\hat{G}))^A$ onto $\mathcal{P}(e(k\hat{C}))$ by Remark 1. Here we can define the bijection $\pi_*(G, A) : \mathcal{P}(k_*G)^A \rightarrow \mathcal{P}(k_*C)$ as follows

\[ \tilde{f}^{-1}(\pi_*(G, A)(\epsilon)) = \tilde{\pi}(\hat{G}, A)(\hat{\epsilon}) \]

From Proposition 1, (i), $\pi_*(G, A)$ satisfies (i). (ii) follows from Proposition 1, (ii), the definition of $\pi_*(G, A)$ and (3).

2 The endomorphism ring of an induced module

Let $Q \triangleleft G$ and let $S$ be an $RQ$-module. Let $H$ be a subgroup of $G$ containing $Q$. The $RH$-module $S \uparrow^H$ can be embedded in $S \uparrow^G$. Set $\hat{H} = H/Q$ and

\[ E_{\hat{H}} = \text{End}_{RH}(S \uparrow^H). \]

We can regard the $R$-algebra $E_{\hat{H}}$ as a subalgebra of $E_{\hat{C}}$. For $\delta \in \mathcal{P}(E_{\hat{H}})$, $V_\delta$ be an indecomposable component of the $RH$-module $S \uparrow^H$ corresponding to $\delta$ ([3], Theorem 1.5.4). We may assume $P_\delta = dE_{\hat{H}}$ and $V_\delta = d(S \uparrow^H)$ for some $d \in \delta$. We have

\[ V_\delta = P_\delta(S). \]

Proposition 4 Suppose that $Q \leq H \leq G$, and let $\delta \in \mathcal{P}(E_{\hat{H}})$. For $d \in \delta$, we have an isomorphism of $RG$-modules

\[ (d(S \uparrow^H)) \uparrow^G \cong d(S \uparrow^G). \]

In particular

\[ V_\delta \uparrow^G \cong \bigoplus_{\epsilon \in \mathcal{P}(E_{\hat{C}})} m(P_\epsilon, P_\delta \uparrow^G)V_\epsilon \]

where $P_\delta \uparrow^G = P_\delta \otimes_{E_{\hat{H}}} E_{\hat{G}}$.

Proof. This is clear. In fact, suppose that $G = \bigcup_{i=1}^{\lvert G:H \rvert} H x_i$. We have

\[ (d(S \uparrow^H)) \uparrow^G = \bigoplus_{i=1}^{\lvert G:H \rvert} d(S \uparrow^H) \otimes_H x_i, \]

\[ d(S \uparrow^G) = \bigoplus_{i=1}^{\lvert G:H \rvert} d(S \uparrow^H)x_i. \]

Therefore

\[ \sum_{i=1}^{\lvert G:H \rvert} u_i \otimes x_i \in (d(S \uparrow^H)) \uparrow^G \implies \sum_{i=1}^{\lvert G:H \rvert} u_i x_i \in d(S \uparrow^G) \]
is an isomorphism. ■

\[
\begin{array}{ccc}
\text{Pid}(E_{\overline{H}}) & \xrightarrow{\text{induction}} & \text{Pid}(E_{\overline{G}}) \\
1:1 & \downarrow & 1:1 \\
\text{Comp}(S \uparrow^{H}) & \xrightarrow{\text{induction}} & \text{Comp}(S \uparrow^{G})
\end{array}
\]

From now, assume that $S$ is $G$-invariant, that is, for any $x \in G$, $S \otimes x \cong S$ as $\mathcal{R}Q$-modules. For $\sigma \in \overline{G}$, let $x_\sigma$ be an element of $\sigma$. We have

$$S \uparrow^{G} = \bigoplus_{\sigma \in \overline{G}} S \otimes x_\sigma.$$  

We set

$$E_\sigma = \{ \psi \in E_{\overline{G}} \mid \psi(S \otimes 1) \subseteq S \otimes x_\sigma \}.$$  

By the assumption $E_\sigma$ contains an invertible element $\psi_\sigma$. We have

$$E_\sigma E_\tau = E_{\sigma \tau} \quad (\forall \sigma, \tau \in \overline{G}), \quad E_{\overline{H}} = \bigoplus_{\sigma \in \overline{H}} E_\sigma \quad (Q \leq H \leq G).$$

That is, $E_{\overline{H}}$ is a crossed product of $\overline{H}$ over $E_{\overline{1}}$.

Let a subgroup $H$ be fixed. Set

$$l_\delta = m(V_\delta, S \uparrow^{H}) = m(P_\delta, E_{\overline{H}}) \quad (\forall \delta \in \mathcal{P}(E_{\overline{H}})).$$

We also set $G = \bigcup_{i=1}^{\left|G:H\right|} y_i H$ and $\psi_i = \psi_{y_i Q}$. Since $\psi_i$ is invertible, we have the following

$$E_{\overline{G}} = \bigoplus_{i=1}^{\left|G:H\right|} \psi_i E_{\overline{H}} \cong \bigoplus_{\delta \in \mathcal{P}(E_{\overline{H}})} l_\delta \left( \bigoplus_{i=1}^{\left|G:H\right|} \psi_i P_\delta \right)$$

as $E_{\overline{H}}$-modules. Hence

$$\Psi : E_{\overline{G}} \otimes_{E_{\overline{H}}} (S \uparrow^{H}) \rightarrow S \uparrow^{G} \quad (\psi \otimes (s \otimes h) \mapsto \psi(s \otimes h))$$

is an isomorphism of $\mathcal{R}H$-modules (cf. Theorem A in [1]). Let

$$E_{\overline{G}} = \bigoplus_{s=1}^{v} P_s$$

be a decomposition of $E_{\overline{G}}$ into indecomposable $E_{\overline{H}}$-modules, where $v = |G:H| \sum_{\delta \in \mathcal{P}(E_{\overline{H}})} l_\delta$.

The isomorphism $\Psi$ induces a decomposition of $S \uparrow^{G}$ into $\mathcal{R}H$-modules:

$$S \uparrow^{G} = \bigoplus_{s=1}^{v} P_s(S \uparrow^{H}), \quad P_s \otimes_{E_{\overline{H}}} S \uparrow^{H} \cong P_s(S \uparrow^{H}).$$

We note that if $P_s$ and $P_t$ are isomorphic, then it is clear that $P_s \otimes_{E_{\overline{H}}} S \uparrow^{H} \cong P_t \otimes_{E_{\overline{H}}} S \uparrow^{H}$, and hence $P_s(S \uparrow^{H}) \cong P_t(S \uparrow^{H})$. Moreover, if $P_s \cong \psi_i P_\delta$, then $P_s(S \uparrow^{H}) \cong V_\delta$. Hence we have the following.
Proposition 5 For any \( \epsilon \in \mathcal{P}(E_{\overline{G}}) \),

\[
V_{\epsilon} \downarrow H \cong \bigoplus_{\delta \in \mathcal{P}(E_{\overline{H}})} m(P_{\delta}, P_{\epsilon} \downarrow E_{\overline{H}}) V_{\delta}.
\]

\[
\text{Pid}(E_{\overline{G}}) \xrightarrow{\text{restriction}} \text{Pid}(E_{\overline{H}})
\]

\[
\text{Comp}(S \uparrow^{G}) \xrightarrow{\text{restriction}} \text{Comp}(S \uparrow^{H})
\]

3 A correspondence between \( \text{Comp}(S \uparrow^{G})^{A} \) and \( \text{Comp}(S \uparrow^{C}) \)

In this section we assume Hypothesis 1 and let \( Q \) and \( S \) be as in the previous section. Moreover we assume \( Q \subseteq C \). Then \( A \) acts on \( \bar{G} \). Since \( (|A|, |Q|) = 1 \), \( C_{\overline{G}}(A) = \bar{C} \). Since \( Q \subseteq C \), the induced module \( S \uparrow^{G} \) is \( A \)-invariant, in fact, \( S \uparrow^{G} \) becomes an \( \mathcal{R}(G \times A) \)-module by the following action of \( A \) on \( S \uparrow^{G} \):

\[
(s \otimes x)a = s \otimes x^{a} \quad (s \in S, \ x \in G, \ a \in A).
\]

And we have

\[
(mx)a = (ma)x^{a} \quad (m \in S \uparrow^{G}, \ x \in G, \ a \in A).
\]

Moreover, \( A \) acts on \( E_{\overline{G}} \) via \( \mathcal{R} \)-algebra automorphisms as follows:

\[
\psi^{a}(m) = \psi(ma^{-1})a \quad (\psi \in E_{\overline{G}}, \ m \in S \uparrow^{G}, \ a \in A).
\]

If \( \psi \in E_{\sigma} \), then

\[
\psi^{a}(s \otimes 1) = \psi(s \otimes 1)a \in S \otimes (x_{\sigma})^{a},
\]

where \( x_{\sigma} \in \sigma \). Therefore

\[
(E_{\sigma})^{a} = E_{\sigma^{a}} \quad (\sigma \in \bar{G}, \ a \in A).
\]

Lemma 1 For \( \epsilon \in \mathcal{P}(E_{\overline{G}}) \) and \( a \in A \), we have

\[
(V_{\epsilon})^{a} \cong V_{\epsilon^{a}}.
\]

In particular \( \epsilon \) is \( A \)-invariant if and only if \( V_{\epsilon} \) is \( A \)-invariant.

Proof. We can set \( V_{\epsilon} = e(S \uparrow^{G}) \) ( \( e \in \epsilon \) ). From the action of \( A \) on \( E_{\overline{G}} \), for \( a \in A \),

\[
e^{a}(S \uparrow^{G}) = e(S \uparrow^{G})a = (V_{\epsilon})a.
\]

Therefore

\[
v^{a} \in (V_{\epsilon})^{a} \rightarrow va \in (V_{\epsilon})a
\]

is an isomorphism of \( \mathcal{R}G \)-modules (see (8)). ■

From now on we assume \( S \) is indecomposable. Let \( Q \leq H \leq G \). Then \( J(E_{(1)})E_{H} = E_{\overline{H}}J(E_{(1)}) \) is an ideal of \( E_{\overline{H}} \). We set

\[
\tilde{E}_{\overline{H}} = E_{\overline{H}}/J(E_{(1)})E_{\overline{H}}.
\]
\[ \mathcal{E}_\sigma = (E_\sigma + J(E_{(i)})E_G)/J(E_{(i)})E_G \quad (\forall \sigma \in \overline{G}). \]

We can regard $\mathcal{E}_H$ as a $k$-subalgebra of $\mathcal{E}_G$. Since $k$ is algebraically closed, $\mathcal{E}_G$ is a twisted group algebra of $\overline{G}$ over $k$. As $E_{(i)}$ is $A$-invariant, $A$ acts on $\mathcal{E}_G$ via $k$-algebra automorphisms. Moreover we have

\[(\mathcal{E}_\sigma)^a = \mathcal{E}_{\sigma^a} \quad (\sigma \in \overline{G}, \ a \in A).\]

Therefore $\overline{G}$, $A$ and $\mathcal{E}_G$ satisfies Hypothesis 2.

**Lemma 2** With the above notations, assume $\overline{G}$ is $p$-solvable. There exists a bijection

\[ \pi(E_{\overline{G}}, A) \cdot \mathcal{P}(E_{\overline{G}})^A \rightarrow \mathcal{P}(E_{\overline{C}}) \]

which satisfies the following (i) and (ii).

(i) If $B \subseteq A$, then $\pi(E_{\overline{G}}, A) = \pi(E_{\overline{G}}(B), A/B)\pi(E_{\overline{G}}, B)$.

(ii) Assume $A$ is an $r$-group for a prime $r$. Then $\epsilon'$ is a unique element of $\mathcal{P}(E_{\overline{G}})$ such that $r \nmid m(P_\epsilon, P_\epsilon \downarrow_{E_{\overline{G}}})$. Moreover $\epsilon$ is a unique element of $\mathcal{P}(E_{\overline{G}})^A$ such that $r \nmid m(P_\epsilon, P_{\epsilon'} \otimes_{E_{\overline{G}}} E_{\overline{G}})$, and we have also $m(P_\epsilon, P_{\epsilon'} \downarrow_{E_{\overline{G}}}) \equiv m(P_\epsilon, P_{\epsilon'} \otimes_{E_{\overline{G}}} E_{\overline{G}}) \pmod{r}$.

**Proof.** In our proof we will use lifting of idempotents ([6], Theorem 3.2) repeatedly. Let $Q \leq U \leq G$. Since $J(E_{(1)})E_0$ is contained in $J(E_{\overline{G}})$, the canonical homomorphism from $E_0$ onto $E_{\overline{G}}$ induces a bijection between $\mathcal{P}(E_0)$ and $\mathcal{P}(E_{\overline{G}})$. For $\delta \in \mathcal{P}(E_0)$, we denote by $\overline{\delta}$ the corresponding point of $E_{\overline{G}}$. When $U$ is $A$-invariant, $\delta$ is $A$-invariant if and only if $\overline{\delta}$ is $A$-invariant. Therefore by using the bijection $\pi_*(\mathcal{E}_{\overline{G}}, A)$ obtained in Proposition 3 for the twisted group algebra $\mathcal{E}_{\overline{G}}$, we can define the bijection $\pi(E_{\overline{G}}, A) : \mathcal{P}(E_{\overline{G}})^A \rightarrow \mathcal{P}(E_{\overline{C}})$ as follows

\[ \pi(E_{\overline{G}}, A)(\epsilon) = \pi_*(E_{\overline{G}}, A)(\overline{\epsilon}). \]

From Proposition 3, (i), $\pi(E_{\overline{G}}, A)$ satisfies (i). Now it is easy to see that

\[ m(P_\delta, P_\epsilon \downarrow_{E_0}) = m(P_\delta, P_\epsilon \downarrow_{E_{\overline{G}}}), \]

\[ m(P_\epsilon, P_\delta \otimes_{E_0} E_{\overline{G}}) = m(P_\epsilon, P_\delta \otimes_{E_0} E_{\overline{G}}) \]

because $P_\delta \otimes_{E_0} E_{\overline{G}} \cong (P_\delta \otimes_{E_0} E_{\overline{G}})/(P_\delta \otimes_{E_0} E_{\overline{G}})$ $m(P_\delta \otimes_{E_0} E_{\overline{G}})$. Hence from Proposition 3, (ii), (ii) holds.

Let $Q \leq U \leq G$ with $U$ $A$-invariant. We denote by $\text{Comp}(S \uparrow U)$ the isomorphism classes of indecomposable components of $S \uparrow U$. From (4), (6), Lemmas 1 and 2, the following holds.

**Proposition 6** With the above notations, assume $\overline{G}$ is $p$-solvable. There exists a bijection

\[ \pi(\overline{G}, A; S) : \text{Comp}(S \uparrow G)^A \rightarrow \text{Comp}(S \uparrow C) \]

which satisfies the following (i) and (ii).

(i) If $B \subseteq A$, then $\pi(\overline{G}, Q; S) = \pi(C_G(B), A/B; S)\pi(\overline{G}, B; S)$.

(ii) Assume $A$ is an $r$-group for a prime $r$. Then $V'$ is a unique indecomposable component of $V \downarrow C$ with the multiplicity prime to $r$. Moreover $V$ is a unique $A$-invariant indecomposable component of $V' \uparrow G$ with the multiplicity prime to $r$, and we have also $m(V', V \downarrow C) \equiv m(V, V' \uparrow G) \pmod{r}$. 


4 Proof of Theorem 1

We assume Hypothesis 2. Let $K \leq G$ and $X$ be an $\mathcal{R}K$-module. We have

$$X^a \uparrow^G \cong (X \uparrow^G)^a \quad (l^a \otimes_{H^a} x \mapsto (l \otimes_L x^{a^{-1}})^a).$$

Therefore if an indecomposable $\mathcal{R}G$-module $X$ has a vertex $D$, then $X^a$ has a vertex $D^a$.

Let $Q \leq C$. If an indecomposable $\mathcal{R}G$-module $V$ has a vertex $Q$, then $V^a$ has a vertex $Q$. We denote by $g_{NC(Q)}$ the Green correspondence from $\text{Ind} (\mathcal{R}N_G(Q)|Q)$ onto $\text{Ind} (\mathcal{R}G|Q)$. If $V'$ is the Green correspondent of an indecomposable $\mathcal{R}G$-module $V$, $V'^a$ is the Green correspondent of $V^a$. In particular $V$ is $A$-invariant if and only if $V'$ is $A$-invariant. Let $S$ be an indecomposable $\mathcal{R}Q$-module and set $T = N_G(Q,S)$, the stabilizer of $S$ in $N_G(Q)$. Then there is a natural bijection compatible with the action of $A$ between $\text{Comp}(S \uparrow^T)$ and $\text{Ind}(S \uparrow^{N_G(Q)})$ ([3], Corollary 4.6.8). Assume $Q$ is a vertex of $S$. We denote by $\text{Ind}(\mathcal{R}G||S)$ the set of isomorphism classes of indecomposable $\mathcal{R}G$-modules with a $Q$-source $S$. Hence there is a natural bijection compatible with the action of $A$ between $\text{Ind}(\mathcal{R}G||S)$ and $\text{Comp}(\mathcal{R}T||S)$ by Green correspondence. Now suppose that $Q \leq H \leq G$. For $M \in \text{Comp}(S \uparrow^T)$ and $L \in \text{Comp}(S \uparrow^{T\cap H})$, set $V = g_{NC(Q)}(M \uparrow^{N_G(Q)})$ and $W = g_{NH(Q)}(L \uparrow^{NH(Q)})$. By a property of Green correspondence we can see

$$m(V, W \uparrow^G) = m(M, L \uparrow^T), \quad (9)$$

$$m(W, V \downarrow^H) = m(L, M \downarrow_{H\cap T}). \quad (10)$$

**Proof.** At first we give a remark. By Hypothesis 1, if $x \in N_G(Q)$, then $x = cy$ ($c \in N_C(Q), y \in C_G(Q)$) by a theorem of Schur-Zassenhaus. Therefore if $\mathcal{R}Q$-modules $S_1$ and $S_2$ are $N_C(Q)$-conjugate, then those are $N_C(Q)$-conjugate.

Now let $[V] \in \text{Ind}(\mathcal{R}G|Q)^A$ and $S$ be a $Q$-source of $V$. Set $T = N_G(Q,S)$. By a property of Green correspondence there is a unique $M \in \text{Comp}(S \uparrow^T)^A$ such that $V$ is a component of $M \uparrow^G$, that is, $V$ is the Green correspondent of $M \uparrow^{N_G(Q)}$. Let $[M'] = \pi(T, A : S)([M])$ and $V' = g_{NC(Q)}(M \uparrow^{N_C(Q)})$ where $T = T/Q$. By the above remark, the map

$$[V] \in \text{Ind}(\mathcal{R}G|Q)^A \mapsto [V'] \in \text{Ind}(\mathcal{R}C|Q)$$

is a bijection. We denote it by $\pi(G, A; Q)$. By Proposition, (i), (i) holds.
\[
\begin{align*}
\text{Ind}(\mathcal{R}G||S)^A & \quad \longrightarrow \quad \text{Ind}(\mathcal{R}C||S) \\
\text{Green cor} \uparrow & \quad \quad \quad \longleftarrow \quad \text{Green cor} \\
\text{Ind}(\mathcal{R}N_G(Q)||S)^A & \quad \longrightarrow \quad \text{Ind}(\mathcal{R}N_C(Q)||S) \\
\text{induction} \uparrow & \quad \quad \quad \longleftarrow \quad \text{induction} \\
\text{Comp}(S^{1:T})^A & \quad \longrightarrow \quad \text{Comp}(S^{1:T\cap C}) \\
\pi(T,A;S) & \\
\text{Comp}(S^{1:T\cap C}) & \quad \longrightarrow \quad \text{Comp}(S^{1:T\cap C}) 
\end{align*}
\]

Now assume \(A\) is an \(r\)-group. Then the above \(M\) is an \(A\)-invariant unique indecomposable component of \(M' \uparrow^T\) with the multiplicity prime to \(r\) by Proposition 6, (ii). Therefore, from (9), we see \(V\) is a unique \(A\)-invariant indecomposable component of \(V' \uparrow^G\) with the multiplicity prime to \(r\) and with vertex \(Q\). (In fact we have \(m(V,V' \uparrow^G) = m(M,M' \uparrow^T)\))

On the other hand \(m(V',V \downarrow_C) = m(M',M \downarrow_{C\cap T})\) from (10). Hence \(m(V,V' \uparrow^G) \equiv m(V',V \downarrow_C) \pmod r\) by Proposition 6, (ii). Now suppose that

\[
vx(\tilde{V}) = C, \quad r \not|\ m(\tilde{V},V \downarrow_C)
\]

for an indecomposable \(\mathcal{R}C\)-module \(\tilde{V}\). Moreover let \(\tilde{S}\) be a \(C\)-source of \(\tilde{V}\). Then by Mackey decomposition, \(\tilde{S}\) and \(S\) are \(N_G(Q)\)-conjugate, and hence we may assume \(\tilde{S} = S\).

By Proposition 6, (ii) again, we see \([\tilde{V}] = [V']\). This completes the proof. \(\blacksquare\)

**Question:** Assume \(A\) is solvable. Let \(\beta \in \text{IBr}(G)^A\) and \(\beta'\) be the Uno correspondent of \(\beta\). Suppose \(\vx(\beta) = Q \leq C\). Then \(\vx(\beta') = Q\) by [4], Theorem. Let \(V_\beta\) be a \(kG\)-module with Brauer character \(\beta\). When is \(\pi(G,A;Q)(V_\beta) = V_{\beta'}\)?

**References**


