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An application of Uno correspondences in p-solvable groups

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Let $(\mathcal{K}, \mathcal{O}, k)$ be a sufficiently large p-modular system such that k is algebraically closed. We set $\mathcal{R}=\mathcal{O}$ or k. Let G be a finite group. For an indecomposable $\mathcal{R}G$-module M and an $\mathcal{R}G$-module N, $m(M, N)$ denotes the multiplicity of M as an indecomposable component of N. $\mathcal{R}G$-modules which we consider are finitely generated right modules. We use same notations for characters too. For a p-subgroup Q of G, we denote by Ind $(\mathcal{R}G|Q)$ the set of isomorphism classes of indecomposable $\mathcal{R}G$-modules with vertex Q. For an $\mathcal{R}G$-module V, $[V]$ denotes the isomorphism class containing V.

Hypothesis 1 *A finite group A acts on a finite group G via group automorphisms and $(|A|, |G|)=1$. Moreover $C=C_{G}(A)$.*

K. Uno extended the Glauberman-Isaacs correspondences between $\mathrm{Irr}(G)^{A}$ and $\mathrm{Irr}(C)$ to a correspondence between $\mathrm{IBr}(G)^{A}$ and $\mathrm{IBr}(C)$ when G is p-solvable ([5], Theorem). We prove the following by using the Uno correspondence.

Theorem 1 *Assume Hypothesis 1 and that G is p-solvable. Let $Q\leq C$ be a p-subgroup. There exists a bijection*

$$\pi(G, A;Q):\mathrm{Ind}(\mathcal{R}G|Q)^{A}\rightarrow \mathrm{Ind}(\mathcal{R}C|Q)$$

which satisfies the following (i) *and* (ii). *For $[V]\in \mathrm{Ind}(\mathcal{R}G|Q)^{A}$, set $[V']=\pi(G, A;Q)([V])$.*

(i) *If $B\trianglelefteq A$, then $\pi(G, A;Q)=\pi(C_{G}(B), A/B;Q)\pi(G, B;Q)$.*

(ii) *Assume A is an r-group where r is a prime. Then V is a unique A-invariant indecomposable component of $V'\uparrow^{G}$ with vertex Q and with the multiplicity prime to r. Moreover V' is a unique indecomposable component of $V\downarrow_{C}$ with the multiplicity prime to r and with vertex Q, and we have also $m(V', V\downarrow_{C})\equiv m(V, V'\uparrow^{G}) \pmod r$.*

In particular, if A is solvable then $\pi(G, A;Q)$ is uniquely determined.

Let $\mathbf{G}$ be an $\mathcal{R}$-algebra which is finitely generated as an $\mathcal{R}$-module. We denote by $\mathcal{P}(\mathbf{G})$ the set of points of $\mathbf{G}$. For $\epsilon\in \mathcal{P}(\mathbf{G})$, we denote by P_{ϵ} a corresponding projective indecomposable $\mathbf{G}$-module. If a group A acts on $\mathbf{G}$ via $\mathcal{R}$-algebra automorphisms, then A acts on $\mathcal{P}(\mathbf{G})$.

1 Correspondences for principal indecomposable modules

Assume Hypothesis 1. Then A acts on $\mathcal{R}G$ via $\mathcal{R}$-algebra automorphisms. Let $H\leq G$ and L be an $\mathcal{R}H$-module. For $a\in A$, $L^{a}=\{l^{a}\mid l\in L\}$ can be regarded as an $\mathcal{R}$-module isomorphic to L by the map $l\mapsto l^{a}$. Moreover L^{a} becomes an $\mathcal{R}H^{a}$-module by the action

$$l^{a}h^{a}=(lh)^{a}\quad (l\in L,\ h\in H).$$

For $a, b \in A$, we have

$$(L^a)^b \cong L^{ab} \quad (a, b \in A).$$

Thus if H is A-invariant, then A acts on the $\mathcal{R}H$-modules.

Hypothesis 2 *With Hypothesis 1, G is p-solvable.*

Theorem 2 (Uno [5]) *Assume Hypothesis 2. There exists a bijection*

$$\rho(G, A) : \mathrm{IBr}(G)^A \to \mathrm{IBr}(C)$$

which satisfies the following (i) *and* (ii)*. For $\beta \in \mathrm{IBr}(G)^A$, set $\beta' = \rho(G, A)(\beta)$.*

(i) *If $B \trianglelefteq A$, then $\rho(G, A) = \rho(C_G(B), A/B)\rho(G, B)$.*

(ii) *If A is an r-group for a prime r, β' is a unique irreducible constituent of $\beta \downarrow_C$ with the multiplicity prime to r. Moreover β is a unique A-invariant irreducible constituent of $\beta' \uparrow^G$ with the multiplicity prime to r, and we have also $m(\beta, \beta' \uparrow^G) \equiv m(\beta', \beta \downarrow C)$ (mod r).*

Proof. (i) is already shown. (ii) Assume A is an r-group. In general if χ is a character of G, then the restriction of χ to the p-regular elements is denoted by χ^*. By the arguments in [5], there is an element $\chi \in \mathrm{Irr}(G)^A$ such that $\beta = \chi^*$, $\beta' = (\chi')^*$, where χ' is the Glauberman correspondent of χ. Here note $\beta' \uparrow^G = (\chi' \uparrow^G)^*$. Now we can set

$$\chi' \uparrow^G = m\chi + \sum_{i=1}^{s} m_i \chi_i + \sum_{j=s+1}^{t} m_j \chi_j$$

where χ_i $(1 \leq i \leq s)$ are A-invariant irreducible characters of G different from χ and χ_j $(s+1 \leq j \leq t)$ are not A-invariant. Then $r \nmid m$ and $r \mid m_i$. We note if χ_j and $\chi_{j'}$ $(j, j' \geq s+1)$ are A-conjugate, then $m_j = m_{j'}$. Then, for any $\gamma \in \mathrm{IBr}(G)^A$, the decomposition numbers $d_{\chi_j \gamma}$ and $d_{\chi_{j'} \gamma}$ are equal. Hence, since A is an r-group, β is a unique A-invariant irreducible constituent of $\beta' \uparrow^G$ with the multiplicity prime to r, and $m(\beta, \beta' \uparrow^G) \equiv m$ (mod r). On the other hand $m = m(\chi', \chi \downarrow_C) \equiv m(\beta', \beta \downarrow_C)$ (mod r) because $\chi \downarrow_C = m\chi' + r\zeta$ where $\zeta = 0$ or ζ is a character of C. This completes the proof. ■

Let M_ϵ be an irreducible kG-module corresponding to $\epsilon \in \mathcal{P}(kG)$. We have $(P_\epsilon)^a \cong P_{\epsilon^a}$ and $(M_\epsilon)^a \cong M_{\epsilon^a}$ for any $a \in A$. Hence by the above theorem and the Frobenius-Nakayama's reciprocity theorem we have the following.

Proposition 1 *Assume Hypothesis 2. There exists a bijection*

$$\tilde{\pi}(G, A) : \mathcal{P}(kG)^A \to \mathcal{P}(kC)$$

which satisfies the following (i) *and* (ii)*. For $\epsilon \in \mathcal{P}(kG)^A$, set $\epsilon' = \tilde{\pi}(G, A)(\epsilon)$.*

(i) *If $B \trianglelefteq A$, then $\tilde{\pi}(G, A) = \tilde{\pi}(C_G(B), A/B)\tilde{\pi}(G, B)$.*

(ii) *Assume A is r-group for a prime r. Then ϵ is a unique element of $\mathcal{P}(kG)^A$ such that $r \nmid m(P_\epsilon, P_{\epsilon'} \uparrow^G)$. Moreover ϵ' is a unique element of $\mathcal{P}(kC)$ such that $r \nmid m(P_{\epsilon'}, P_\epsilon \downarrow_C)$, and we have also $m(P_{\epsilon'}, P_\epsilon \downarrow_C) \equiv m(P_\epsilon, P_{\epsilon'} \uparrow^G)$ (mod r).*

Remark 1 *In the above proposition,*

$$m(P_\epsilon, P_{\epsilon'} \uparrow^G) \neq 0.$$

Proof. Since χ' is a constituent of $\chi\downarrow_C$, β' is a constituent of $\beta\downarrow_C$ where $\beta \in \mathrm{IBr}(G)^A$.

Hypothesis 3 *With Hypothesis 1, k_*G is a twisted group algebra of G over k with a basis $\{u_x \mid x \in G\}$ and with a factor set α. Moreover A acts on k_*G via k-algebra automorphims and the following holds:*

$$(ku_x)^a = ku_{x^a} \ (\forall x \in G, \forall a \in A).$$

Proposition 2 *Assume Hypothesis 3. There is a central extension of G which satisfies* (i) - (iii).

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1$$

(i) $|Z| = (|G|_{p'})^2$,

(ii) *The action of A on G is extended to $\hat{G}$, that is, $f(y^a) = f(y)^a$ $(\forall y \in \hat{G}, \forall a \in A)$. Moreover $f^{-1}(C) = C_{\hat{G}}(A)$,*

(iii) *There are an idempotent e of kZ and a k-algebra isomorphism $\tilde{f} : e(k\hat{G}) \to k_*G$ compatible with the action of A.*

Proof. We may assume α satisfies

$$\alpha(x, x')^{|G|} = 1, \quad \alpha(x, 1) = \alpha(1, x) = 1 \quad (\forall x, x' \in G).$$

Then u_1 is the identity element of k_*G. Set $V = \{\beta \in k^\times \mid \beta^{|G|} = 1\} = \{\beta \in k^\times \mid \beta^{|G|_{p'}} = 1\}$. By the assumption, for each $a \in A$ and $x \in G$, we can write

$$(u_x)^a = c(a, x)u_{x^a} \ (c(a, x) \in k^\times).$$

From $(u_x u_{x'})^a = (u_x)^a (u_{x'})^a$

$$(1) \qquad \alpha(x, x')c(a, xx') = c(a, x)c(a, x')\alpha(x^a, x'^a).$$

Therefore $c(a, x)c(a, x')c(a, xx')^{-1} \in V$ for each $a \in A$ and $x, x' \in G$. Hence $c(a, x)^{|G|^2} = 1$, and hence $c(a, x)^{(|G|_{p'})^2} = 1$. Since $(u_x)^{ab} = ((u_x)^a)^b$

$$(2) \qquad c(ab, x) = c(a, x)c(b, x^a).$$

Moreover we have $c(a, 1) = 1$. Set $H = \{\beta \in k^\times \mid \beta^{(|G|_{p'})^2} = 1\}$. (We will construct a central extension of G using the method in [3], 3.5.15, and [6], (10.4)) We define the multiplication in $\hat{G} = H \times G$ as follows :

$$(h, x)(h', x') = (hh'\alpha(x, x'), xx').$$

Then $\hat{G}$ forms a group with the identity element $(1, 1)$. Also $Z = H \times 1$ is a central subgroup of $\hat{G}$. The map $\iota : Z \to H((h, 1) \mapsto h)$ is an isomorphism, in particular Z is a p'-group. Moreover $(|Z|, |A|) = 1$. Therefore if

$$f : \hat{G} \to G \ ((h, x) \mapsto x)$$

then

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1$$

is a central extension of G which satisfies (i).

By using (1) and (2)

$$(h, x)^a = (hc(a, x), x^a) \quad (\forall (h, x) \in \hat{G},\ \forall a \in A)$$

defines an action of A on $\hat{G}$ via group automorphisms. We note A centralizes Z and $f(y^a) = f(y)^a$ $(y \in \hat{G})$. Moreover $C_{\hat{G}}(A) \subseteq f^{-1}(C)$. Let $c \in f^{-1}(C)$ and $a \in A$. We have $c^a = zc$ for some $z \in Z$. Since $z^{o(a)} = 1$, $z = 1$, so $c \in C_{\hat{G}}(A)$. Thus (ii) holds.

For any $x \in G$, set $\hat{x} = (1, x)$. We have $\hat{x}\hat{x'} = (\alpha(x, x'), 1)\widehat{xx'}$, $(h, x) = (h, 1)\hat{x}$ $(\forall h \in H)$. Moreover

$$e = \frac{1}{|Z|}\sum_{z \in Z} \iota(z^{-1})z$$

is an idempotent of kZ and for any $z \in Z$, and we have $ze = \iota(z)e$. Therefore $e(k\hat{G}) = \bigoplus_{x \in G} k(e\hat{x})$, $(e\hat{x})(e\hat{x'}) = \alpha(x, x')(e\widehat{xx'})$. This implies

$$\tilde{f} : e(k\hat{G}) \to k_*G \ (\sum_{x \in G} c_x(e\hat{x}) \mapsto \sum_{x \in G} c_x u_x)$$

is an isomorphism of k-algebras. Moreover if $a \in A$, then

$$\tilde{f}((e\hat{x})^a) = c(a, x)u_{x^a} = \tilde{f}(e\hat{x})^a \ (\forall x \in G).$$

Thus (iii) holds. ■

Remark 2 *With Hypothesis 3, A centralizes k_*C.*

Proof. Our proof is the same as the proof of [2], 7.6. From (1) and (2)

$$c(a, xy) = c(a, x)c(a, y),$$

$$c(ab, x) = c(a, x)c(b, x) \quad (\forall a, b \in A,\ \forall x, y \in C).$$

The fact that $(|A|, |C|) = 1$ implies $c(a, x) = 1$.

Proposition 3 *Assume Hypotheses 2 and 3. There exists a bijection*

$$\pi_*(G, A) : \mathcal{P}(k_*G)^A \to \mathcal{P}(k_*C)$$

which satisfies the following (i) *and* (ii). *For $\epsilon \in \mathcal{P}(k_*G)^A$, set $\epsilon' = \pi_*(G, A)(\epsilon)$.*

(i) *If $B \trianglelefteq A$, then $\pi_*(G, A) = \pi_*(C_G(B), A/B)\pi_*(G, B)$.*

(ii) *Assume A is an r-group for a prime r. Then ϵ is a unique element of $\mathcal{P}(k_*G)^A$ such that $r \nmid m(P_\epsilon, P_{\epsilon'} \otimes_{k_*C} k_*G)$. Moreover ϵ' is a unique element of $\mathcal{P}(k_*C)$ such that $r \nmid m(P_{\epsilon'}, P_\epsilon \downarrow_{k_*C})$, and we have also $m(P_{\epsilon'}, P_\epsilon \downarrow_{k_*C}) \equiv m(P_\epsilon, P_{\epsilon'} \otimes_{k_*C} k_*G)$* (mod r).

Proof. We will use Proposition 1. At first we note that $\hat{G}$ is p-solvable. For a subgroup U of G, set $\hat{U} = f^{-1}(U)$. Then the k-algebras $e(k\hat{U})$ and k_*U are isomorphic by the isomorphism $\tilde{f}_{|e(k\hat{U})}$. For $\delta \in \mathcal{P}(k_*U)$, we set $\hat{\delta} = \tilde{f}^{-1}(\delta) \in \mathcal{P}(e(k\hat{U}))$. Note that when

U is A-invariant, $\hat{\delta}$ is A-invariant if and only if δ is A-invariant. For $\delta \in \mathcal{P}(k_*U)$ and $\epsilon \in \mathcal{P}(k_*G)$, we have

$$(3) \qquad m(P_\epsilon, P_\delta \otimes_{k_*U} k_*G) = m(P_{\hat{\epsilon}}, P_{\hat{\delta}} \uparrow^{\hat{G}}), \quad m(P_\delta, P_\epsilon \downarrow_{k_*U}) = m(P_{\hat{\delta}}, P_{\hat{\epsilon}} \downarrow_{\hat{U}}).$$

Since Z is a central subgroup of $\hat{G}$, a point of $k\hat{G}$ is a point of the k-algebra $e(k\hat{G})$ or a point of $(1-e)(k\hat{G})$. If $\hat{\mu}$ is a point of $e(k\hat{C})$, then $(P_{\hat{\mu}} \uparrow^{\hat{G}})(1-e) = 0$. Hence the bijection $\tilde{\pi}(\hat{G}, A)$ in Proposition 1 induces a bijection from $\mathcal{P}(e(k\hat{G}))^A$ onto $\mathcal{P}(e(k\hat{C}))$ by Remark 1. Here we can define the bijection $\pi_*(G,A) : \mathcal{P}(k_*G)^A \to \mathcal{P}(k_*C)$ as follows

$$\tilde{f}^{-1}(\pi_*(G,A)(\epsilon)) = \tilde{\pi}(\hat{G}, A)(\hat{\epsilon}).$$

From Proposition 1, (i), $\pi_*(G,A)$ satisfies (i). (ii) follows from Proposition 1, (ii), the definition of $\pi_*(G,A)$ and (3).

2 The endomorphism ring of an induced module

Let $Q \trianglelefteq G$ and let S be an $\mathcal{R}Q$-module. Let H be a subgroup of G containing Q. The $\mathcal{R}H$-module $S \uparrow^H$ can be embedded in $S \uparrow^G$. Set $\bar{H} = H/Q$ and

$$E_{\bar{H}} = \mathrm{End}_{\mathcal{R}H}(S \uparrow^H).$$

We can regard the $\mathcal{R}$-algebra $E_{\bar{H}}$ as a subalgebra of $E_{\bar{G}}$. For $\delta \in \mathcal{P}(E_{\bar{H}})$, V_δ be an an indecomposable component of the $\mathcal{R}H$-module $S \uparrow^H$ corresponding to δ ([3], Theorem 1.5.4). We may assume $P_\delta = dE_{\bar{H}}$ and $V_\delta = d(S \uparrow^H)$ for some $d \in \delta$. We have

$$V_\delta = P_\delta(S).$$

Proposition 4 *Suppose that $Q \le H \le G$, and let $\delta \in \mathcal{P}(E_{\bar{H}})$. For $d \in \delta$, we have an isomorphism of $\mathcal{R}G$-modules*

$$(d(S \uparrow^H)) \uparrow^G \cong d(S \uparrow^G).$$

In particular

$$(4) \qquad V_\delta \uparrow^G \cong \bigoplus_{\epsilon \in \mathcal{P}(E_{\bar{G}})} m(P_\epsilon, P_\delta \uparrow^G) V_\epsilon$$

where $P_\delta \uparrow^G = P_\delta \otimes_{E_{\bar{H}}} E_{\bar{G}}$.

Proof. This is clear. In fact, suppose that $G = \cup_{i=1}^{|G:H|} Hx_i$. We have

$$(d(S \uparrow^H)) \uparrow^G = \bigoplus_{i=1}^{|G:H|} d(S \uparrow^H) \otimes_H x_i,$$

$$d(S \uparrow^G) = \bigoplus_{i=1}^{|G:H|} d(S \uparrow^H) x_i.$$

Therefore

$$\sum_{i=1}^{|G:H|} u_i \otimes x_i \in (d(S \uparrow^H)) \uparrow^G \mapsto \sum_{i=1}^{|G:H|} u_i x_i \in d(S \uparrow^G)$$

is an isomorphism. ■

$$\begin{array}{ccc} \mathrm{Pid}(E_{\bar{H}}) & \xrightarrow[induction]{} & \mathrm{Pid}(E_{\bar{G}}) \\ {\scriptstyle 1:1}\uparrow & & {\scriptstyle 1:1}\uparrow \\ \mathrm{Comp}(S\uparrow^H) & \xrightarrow[induction]{} & \mathrm{Comp}(S\uparrow^G) \end{array}$$

From now, assume that S is G-invariant, that is, for any $x \in G$, $S \otimes x \cong S$ as $\mathcal{R}Q$-modules. For $\sigma \in \bar{G}$, let x_σ be an element of σ. We have

$$S\uparrow^G = \bigoplus_{\sigma\in\bar{G}} S \otimes x_\sigma.$$

We set

$$E_\sigma = \{\psi \in E_{\bar{G}} \mid \psi(S \otimes 1) \subseteq S \otimes x_\sigma\}.$$

By the assumption E_σ contains an invertible element ψ_σ. We have

$$E_\sigma E_\tau = E_{\sigma\tau}\ (\forall\sigma,\tau \in \bar{G}),\quad E_{\bar{H}} = \bigoplus_{\sigma\in\bar{H}} E_\sigma \quad (Q \le H \le G).$$

That is, $E_{\bar{H}}$ is a crossed product of $\bar{H}$ over $E_{\bar{1}}$.

Let a subgroup H be fixed. Set

$$l_\delta = m(V_\delta, S\uparrow^H) = m(P_\delta, E_{\bar{H}}) \quad (\forall\delta \in \mathcal{P}(E_{\bar{H}})).$$

We also set $G = \cup_{i=1}^{|G:H|} y_i H$ and $\psi_i = \psi_{y_i Q}$. Since ψ_i is invertible, we have the following

$$E_{\bar{G}} = \bigoplus_{i=1}^{|G:H|} \psi_i E_{\bar{H}} \cong \bigoplus_{\delta\in\mathcal{P}(E_{\bar{H}})} l_\delta\Big(\bigoplus_{i=1}^{|G:H|} \psi_i P_\delta\Big) \tag{5}$$

as $E_{\bar{H}}$-modules. Hence

$$\Psi : E_{\bar{G}} \otimes_{E_{\bar{H}}} (S\uparrow^H) \to S\uparrow^G \quad (\psi \otimes (s \otimes h) \mapsto \psi(s \otimes h))$$

is an isomorphism of $\mathcal{R}H$-modules (cf. Theorem A in [1]). Let

$$E_{\bar{G}} = \bigoplus_{s}^{v} P_s$$

be a decomposition of $E_{\bar{G}}$ into indecomposable $E_{\bar{H}}$-modules, where $v = |G : H| \sum_{\delta\in\mathcal{P}(E_{\bar{H}})} l_\delta$. The isomorphism Ψ induces a decomposition of $S\uparrow^G$ into $\mathcal{R}H$-modules:

$$S\uparrow^G = \bigoplus_{s=1}^{v} P_s(S\uparrow^H), \quad P_s \otimes_{E_{\bar{H}}} S\uparrow^H \cong P_s(S\uparrow^H).$$

We note that if P_s and P_t are isomorphic, then it is clear that $P_s \otimes_{E_{\bar{H}}} S\uparrow^H \cong P_t \otimes_{E_{\bar{H}}} S\uparrow^H$, and hence $P_s(S\uparrow^H) \cong P_t(S\uparrow^H)$. Moreover, if $P_s \cong \psi_i P_\delta$, then $P_s(S\uparrow^H) \cong V_\delta$. Hence we have the following.

Proposition 5 *For any $\epsilon \in \mathcal{P}(E_{\bar{G}})$,*

$$V_\epsilon \downarrow_H \cong \bigoplus_{\delta \in \mathcal{P}(E_{\bar{H}})} m(P_\delta, P_\epsilon \downarrow_{E_{\bar{H}}}) V_\delta. \tag{6}$$

$$\begin{array}{ccc} \mathrm{Pid}(E_{\bar{G}}) & \xrightarrow{\text{restriction}} & \mathrm{Pid}(E_{\bar{H}}) \\ {\scriptstyle 1:1}\Big\uparrow & & {\scriptstyle 1:1}\Big\uparrow \\ \mathrm{Comp}(S\uparrow^G) & \xrightarrow{\text{restriction}} & \mathrm{Comp}(S\uparrow^H) \end{array}$$

3 A correspondence between Comp($S\uparrow^G)^A$ and Comp($S\uparrow^C$)

In this section we assume Hypothesis 1 and let Q and S be as in the previous section. Moreover we assume $Q \leq C$. Then A acts on $\bar{G}$. Since $(|A|, |Q|) = 1$, $C_{\bar{G}}(A) = \bar{C}$. Since $Q \subseteq C$, the induced module $S\uparrow^G$ is A-invariant, in fact, $S\uparrow^G$ becomes an $\mathcal{R}(G \rtimes A)$-module by the following action of A on $S\uparrow^G$:

$$(s \otimes x)a = s \otimes x^a \ (s \in S,\ x \in G,\ a \in A). \tag{7}$$

And we have

$$(mx)a = (ma)x^a \ \ (m \in S\uparrow^G,\ x \in G, a \in A). \tag{8}$$

Moreover, A acts on $E_{\bar{G}}$ via $\mathcal{R}$-algebra automorphisms as follows :

$$\psi^a(m) = \psi(ma^{-1})a \ (\psi \in E_{\bar{G}},\ m \in S\uparrow^G,\ a \in A).$$

If $\psi \in E_\sigma$, then

$$\psi^a(s \otimes 1) = \psi(s \otimes 1)a \in S \otimes (x_\sigma)^a,$$

where $x_\sigma \in \sigma$. Therefore

$$(E_\sigma)^a = E_{\sigma^a} \ (\sigma \in \bar{G},\ a \in A).$$

Lemma 1 *For $\epsilon \in \mathcal{P}(E_{\bar{G}})$ and $a \in A$, we have*

$$(V_\epsilon)^a \cong V_{\epsilon^a}.$$

In particular ϵ is A-invariant if and only if V_ϵ is A-invariant.

Proof. We can set $V_\epsilon = e(S\uparrow^G)$ ($e \in \epsilon$). From the action of A on $E_{\bar{G}}$, for $a \in A$, $e^a(S\uparrow^G) = e(S\uparrow^G)a = (V_\epsilon)a$. Therefore

$$v^a \in (V_\epsilon)^a \to va \in (V_\epsilon)a$$

is an isomorphism of $\mathcal{R}G$-modules (see (8)). ■

From now on we assume S is indecomposable. Let $Q \leq H \leq G$. Then $J(E_{\langle\bar{1}\rangle})E_{\bar{H}} = E_{\bar{H}}J(E_{\langle\bar{1}\rangle})$ is an ideal of $E_{\bar{H}}$. We set

$$\bar{E}_{\bar{H}} = E_{\bar{H}}/J(E_{\langle\bar{1}\rangle})E_{\bar{H}},$$

$$\bar{E}_\sigma = (E_\sigma + J(E_{\langle \bar{1} \rangle})E_{\bar{G}})/J(E_{\langle \bar{1} \rangle})E_{\bar{G}} \quad (\forall \sigma \in \bar{G}).$$

We can regard $\bar{E}_H$ as a k-subalgebra of $\bar{E}_G$. Since k is algebraically closed, $\bar{E}_G$ is a twisted group algebra of $\bar{G}$ over k. As $E_{\langle \bar{1} \rangle}$ is A-invariant, A acts on $\bar{E}_{\bar{G}}$ via k-algebra automorphisms. Moreover we have

$$(\bar{E}_\sigma)^a = \bar{E}_{\sigma^a} \ (\sigma \in \bar{G}, \ a \in A).$$

Therefore $\bar{G}$, A and $\bar{E}_G$ satisfies Hypothesis 2.

Lemma 2 *With the above notations, assume $\bar{G}$ is p-solvable. There exists a bijection*

$$\pi(E_{\bar{G}}, A) : \mathcal{P}(E_{\bar{G}})^A \to \mathcal{P}(E_{\bar{C}})$$

which satisfies the following (i) *and* (ii). *For $\epsilon \in \mathcal{P}(E_{\bar{G}})^A$, set $\epsilon' = \pi(E_{\bar{G}}, A)(\epsilon)$.*

(i) *If $B \trianglelefteq A$, then $\pi(E_{\bar{G}}, A) = \pi(E_{C_{\bar{G}}(B)}, A/B)\pi(E_{\bar{G}}, B)$.*

(ii) *Assume A is an r-group for a prime r. Then ϵ' is a unique element of $\mathcal{P}(E_{\bar{C}})$ such that $r \nmid m(P_{\epsilon'}, P_\epsilon \downarrow_{E_{\bar{C}}})$. Moreover ϵ is a unique element of $\mathcal{P}(E_{\bar{G}})^A$ such that $r \nmid m(P_\epsilon, P_{\epsilon'} \otimes_{E_{\bar{C}}} E_{\bar{G}})$, and we have also $m(P_{\epsilon'}, P_\epsilon \downarrow_{E_{\bar{C}}}) \equiv m(P_\epsilon, P_{\epsilon'} \otimes_{E_{\bar{C}}} E_{\bar{G}})$* (mod r).

Proof. In our proof we will use lifting of idempotents ([6], Theorem 3.2) repeatedly. Let $Q \leq U \leq G$. Since $J(E_{\langle \bar{1} \rangle})E_{\bar{U}}$ is contained in $J(E_{\bar{U}})$, the canonical homomorphism from $E_{\bar{U}}$ onto $\bar{E}_{\bar{U}}$ induces a bijection between $\mathcal{P}(E_{\bar{U}})$ and $\mathcal{P}(\bar{E}_{\bar{U}})$. For $\delta \in \mathcal{P}(E_{\bar{U}})$, we denote by $\bar{\delta}$ the corresponding point of $\bar{E}_{\bar{U}}$. When U is A-invariant, δ is A-invariant if and only if $\bar{\delta}$ is A-invariant. Therefore by using the bijection $\pi_*(\bar{E}_{\bar{G}}, A)$ obtained in Proposition 3 for the twisted group algebra $\bar{E}_{\bar{G}}$, we can define the bijection $\pi(E_{\bar{G}}, A) : \mathcal{P}(E_{\bar{G}})^A \to \mathcal{P}(E_{\bar{C}})$ as follows

$$\overline{\pi(E_{\bar{G}}, A)(\epsilon)} = \pi_*(\bar{E}_{\bar{G}}, A)(\bar{\epsilon}).$$

From Proposition 3, (i), $\pi(E_{\bar{G}}, A)$ satisfies (i). Now it is easy to see that

$$m(P_\delta, P_\epsilon \downarrow_{E_{\bar{U}}}) = m(P_{\bar{\delta}}, P_{\bar{\epsilon}} \downarrow_{\bar{E}_{\bar{U}}}),$$

$$m(P_\epsilon, \ P_\delta \otimes_{E_{\bar{U}}} E_{\bar{G}}) = m(P_{\bar{\epsilon}}, \ P_{\bar{\delta}} \otimes_{\bar{E}_{\bar{U}}} \bar{E}_{\bar{G}})$$

because $P_{\bar{\delta}} \otimes_{\bar{E}_{\bar{U}}} \bar{E}_{\bar{G}} \cong (P_\delta \otimes_{E_{\bar{U}}} E_{\bar{G}})/((P_\delta \otimes_{E_{\bar{U}}} E_{\bar{G}})J(E_{\langle \bar{1} \rangle}))$. Hence from Proposition 3, (ii), (ii) holds. ■

Let $Q \leq U \leq G$ with U A-invariant. We denote by $\mathrm{Comp}(S \uparrow^U)$ the isomorphism classes of indecomposable components of $S \uparrow^U$. From (4), (6), Lemmas 1 and 2, the following holds.

Proposition 6 *With the above notations, assume $\bar{G}$ is p-solvable. There exists a bijection*

$$\pi(\bar{G}, A; S) : \mathrm{Comp}(S \uparrow^G)^A \to \mathrm{Comp}(S \uparrow^C)$$

which satisfies the following (i) *and* (ii). *For $[V] \in \mathrm{Comp}(S \uparrow^G)^A$, set $[V'] = \pi(\bar{G}, A; S)([V])$.*

(i) *If $B \trianglelefteq A$, then $\pi(\bar{G}, Q; S) = \pi(C_{\bar{G}}(B), A/B; S)\pi(\bar{G}, B; S)$.*

(ii) *Assume A is an r-group for a prime r. Then V' is a unique indecomposable component of $V \downarrow_C$ with the multiplicity prime to r. Moreover V is a unique A-invariant indecomposable component of $V' \uparrow^G$ with the multiplicity prime to r, and we have also $m(V', V \downarrow_C) \equiv m(V, V' \uparrow^G)$* (mod r).

$$\begin{array}{ccc}
\mathcal{P}(\bar{E}_{\bar{G}})^A & \xrightarrow{\pi(\bar{E}_{\bar{G}},A)} & \mathcal{P}(\bar{E}_{\bar{C}}) \\
{\scriptstyle 1:1}\uparrow & & \uparrow{\scriptstyle 1:1} \\
\mathcal{P}(E_{\bar{G}})^A & \xrightarrow{\pi(E_{\bar{G}},A)} & \mathcal{P}(E_{\bar{C}}) \\
{\scriptstyle 1:1}\uparrow & & \uparrow{\scriptstyle 1:1} \\
\mathrm{Comp}(S\uparrow^G)^A & \xrightarrow[\pi(\bar{G},A;S)]{} & \mathrm{Comp}(S\uparrow^C)
\end{array}$$

4 Proof of Theorem 1

We assume Hypothesis 2. Let $K \le G$ and X be an $\mathcal{R}K$-module. We have

$$X^a \uparrow^G \cong (X\uparrow^G)^a \quad (l^a \otimes_{H^a} x \mapsto (l \otimes_L x^{a^{-1}})^a).$$

Therefore if an indecomposable $\mathcal{R}G$-module X has a vertex D, then X^a has a vertex D^a.

Let $Q \le C$. If an indecomposable $\mathcal{R}G$-module V has a vertex Q, then V^a has a vertex Q. We denote by $g_{N_G(Q)}$ the Green correspondence from $\mathrm{Ind}(\mathcal{R}N_G(Q)|Q)$ onto $\mathrm{Ind}\ (\mathcal{R}G|Q)$. If V' is the Green correspondent of an indecomposable $\mathcal{R}G$-module V, V'^a is the Green correspondent of V^a. In particular V is A-invariant if and only if V' is A-invariant. Let S be an indecomposable $\mathcal{R}Q$-module and set $T = N_G(Q,S)$, the stabilizer of S in $N_G(Q)$. Then there is a natural bijection compatible with the action of A between $\mathrm{Comp}(S\uparrow^T)$ and $\mathrm{Ind}(S\uparrow^{N_G(Q)})$ ([3], Corollary 4.6.8). Assume Q is a vertex of S. We denote by $\mathrm{Ind}(\mathcal{R}G||S)$ the set of isomorphism classes of indecomposable $\mathcal{R}G$-modules with a Q-source S. Hence there is a natural bijection compatible with the action of A between $\mathrm{Ind}(\mathcal{R}G||S)$ and $\mathrm{Comp}(\mathcal{R}T||S)$ by Green correspondence. Now suppose that $Q \le H \le G$. For $M \in \mathrm{Comp}(S\uparrow^T)$ and $L \in \mathrm{Comp}(S\uparrow^{T\cap H})$, set $V = g_{N_G(Q)}(M\uparrow^{N_G(Q)})$ and $W = g_{N_H(Q)}(L\uparrow^{N_H(Q)})$. By a property of Green correspondence we can see

(9) $$m(V, W\uparrow^G) = m(M, L\uparrow^T),$$

(10) $$m(W, V\downarrow_H) = m(L, M\downarrow_{H\cap T}).$$

Proof. At first we give a remark. By Hypothesis 1, if $x \in N_G(Q)$, then $x = cy$ $(c \in N_C(Q), y \in C_G(Q))$ by a theorem of Schur-Zassenhaus. Therefore if $\mathcal{R}Q$-modules S_1 and S_2 are $N_G(Q)$-conjugate, then those are $N_C(Q)$-conjugate.

Now let $[V] \in \mathrm{Ind}(\mathcal{R}G|Q)^A$ and S be a Q-source of V. Set $T = N_G(Q,S)$. By a property of Green correspondence there is a unique $M \in \mathrm{Comp}(S\uparrow^T)^A$ such that V is a component of $M\uparrow^G$, that is, V is the Green correspondent of $M\uparrow^{N_G(Q)}$. Let $[M'] = \pi(\bar{T}, A : S)([M])$ and $V' = g_{N_C(Q)}(M\uparrow^{N_C(Q)})$ where $\bar{T} = T/Q$. By the above remark, the map

$$[V] \in \mathrm{Ind}(\mathcal{R}G|Q)^A \mapsto [V'] \in \mathrm{Ind}(\mathcal{R}C|Q)$$

is a bijection. We denote it by $\pi(G, A; Q)$. By Proposition, (i), (i) holds.

$$\begin{array}{ccc}
\mathrm{Ind}(\mathcal{R}G||S)^A & \xrightarrow[1:1]{} & \mathrm{Ind}(\mathcal{R}C||S) \\
{\scriptstyle Green\ cor}\uparrow & & \uparrow{\scriptstyle Green\ cor} \\
\mathrm{Ind}(\mathcal{R}N_G(Q)||S)^A & \xrightarrow[1:1]{} & \mathrm{Ind}(\mathcal{R}N_C(Q)||S) \\
{\scriptstyle induction}\uparrow & & \uparrow{\scriptstyle induction} \\
\mathrm{Comp}(S\uparrow^T)^A & \xrightarrow[\pi(\bar{T},A;S)]{} & \mathrm{Com}(S\uparrow^{T\cap C})
\end{array}$$

Now assume A is an r-group. Then the above M is an A-invariant unique indecomposable component of $M'\uparrow^T$ with the multiplicity prime to r by Proposition 6, (ii). Therefore, from (9), wee see V is a unique A-invariant indecomposable component of $V'\uparrow^G$ with the multiplicity prime to r and with vertex Q. (In fact we have $m(V, V'\uparrow^G) = m(M, M'\uparrow^T)$) On the other hand $m(V', V\downarrow_C) = m(M', M\downarrow_{C\cap T})$ from (10). Hence $m(V, V'\uparrow^G) \equiv m(V', V\downarrow_C)$ (mod r) by Proposition 6, (ii). Now suppose that

$$vx(\tilde{V}) =_C Q,\ r \not| \, m(\tilde{V}, V\downarrow_C)$$

for an indecomposable $\mathcal{R}C$-module $\tilde{V}$. Moreover let $\tilde{S}$ be a Q-source of $\tilde{V}$. Then by Mackey decomposition, $\tilde{S}$ and S are $N_G(Q)$-conjugate, and hence we may assume $\tilde{S} = S$. By Proposition 6, (ii) again, we see $[\tilde{V}] = [V']$. This completes the proof. ■

Question: Assume A is solvable. Let $\beta \in \mathrm{IBr}(G)^A$ and β' be the Uno correspondent of β. Suppose $\mathrm{vx}(\beta) = Q \leq C$. Then $\mathrm{vx}(\beta') = Q$ by [4], Theorem. Let V_β be a kG-module with Brauer character β. When is $\pi(G, A; Q)(V_\beta) = V_{\beta'}$?

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