An application of Uno correspondences in $p$-solvable groups

(Cohomology Theory of Finite Groups and Related Topics)

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An application of Uno correspondences in $p$-solvable groups

K. Uno extended the Glauberman-Isaacs correspondences between $\text{Irr}(G)^A$ and $\text{Irr}(C)$ to a correspondence between $\text{IBr}(G)^A$ and $\text{IBr}(C)$ when $G$ is $p$-solvable ([5], Theorem). We prove the following by using the Uno correspondence.

**Theorem 1** Assume Hypothesis 1 and that $G$ is $p$-solvable. Let $Q \leq C$ be a $p$-subgroup. There exists a bijection

$$\pi(G, A; Q) : \text{Ind}(\mathcal{R}G|Q)^A \to \text{Ind}(\mathcal{R}C|Q)$$

which satisfies the following (i) and (ii). For $[V] \in \text{Ind}(\mathcal{R}G|Q)^A$, set $[V'] = \pi(G, A; Q)([V])$.

(i) If $B \leq A$, then $\pi(G, A; Q) = \pi(C_G(B), A/B; Q)\pi(G, B; Q)$.

(ii) Assume $A$ is an $r$-group where $r$ is a prime. Then $V$ is a unique $A$-invariant indecomposable component of $V' \uparrow^G$ with vertex $Q$ and with the multiplicity prime to $r$. Moreover $V'$ is a unique indecomposable component of $V \downarrow_C$ with the multiplicity prime to $r$ and with vertex $Q$, and we have also $m(V', V \downarrow_C) \equiv m(V, V' \uparrow^G) \pmod{r}$.

In particular, if $A$ is solvable then $\pi(G, A; Q)$ is uniquely determined.

Let $G$ be an $\mathcal{R}$-algebra which is finitely generated as an $\mathcal{R}$-module. We denote by $\mathcal{P}(G)$ the set of points of $G$. For $e \in \mathcal{P}(G)$, we denote by $P_e$ a corresponding projective indecomposable $G$-module. If a group $A$ acts on $G$ via $\mathcal{R}$-algebra automorphisms, then $A$ acts on $\mathcal{P}(G)$.

1 Correspondences for principal indecomposable modules

Assume Hypothesis 1. Then $A$ acts on $\mathcal{R}G$ via $\mathcal{R}$-algebra automorphisms. Let $H \leq G$ and $L$ be an $\mathcal{R}H$-module. For $a \in A$, $L^a = \{l^a \mid l \in L\}$ can be regarded as an $\mathcal{R}$-module isomorphic to $L$ by the map $l \mapsto l^a$. Moreover $L^a$ becomes an $\mathcal{R}H^a$-module by the action

$$l^a h^a = (lh)^a \quad (l \in L, \ h \in H).$$
For $a, b \in A$, we have

$$(L^a)^b \cong L^{ab} \quad (a, b \in A).$$

Thus if $H$ is $A$-invariant, then $A$ acts on the $RH$-modules.

**Hypothesis 2** With Hypothesis 1, $G$ is $p$-solvable.

**Theorem 2** (Uno [5]) Assume Hypothesis 2. There exists a bijection

$$\rho(G, A) : \text{IBr}(G)^A \to \text{IBr}(C)$$

which satisfies the following (i) and (ii). For $\beta \in \text{IBr}(G)^A$, set $\beta' = \rho(G, A)(\beta)$.

(i) If $B \triangleleft A$, then $\rho(G, A) = \rho(C_G(B), A/B)\rho(G, B)$.

(ii) If $A$ is an $r$-group for a prime $r$, $\beta'$ is a unique irreducible constituent of $\beta \downarrow_C$ with the multiplicity prime to $r$. Moreover $\beta$ is a unique $A$-invariant irreducible constituent of $\beta' \uparrow^G$ with the multiplicity prime to $r$, and we have also $m(\beta, \beta' \uparrow^G) \equiv m(\beta', \beta \downarrow_C) \pmod r$.

**Proof.** (i) is already shown. (ii) Assume $A$ is an $r$-group. In general if $\chi$ is a character of $G$, then the restriction of $\chi$ to the $p$-regular elements is denoted by $\chi^*$. By the arguments in [5], there is an element $\chi \in \text{Irr}(G)^A$ such that $\beta = \chi^*$, $\beta' = (\chi')^*$, where $\chi'$ is the Glauberman correspondent of $\chi$. Here note $\beta' \uparrow^G = (\chi' \uparrow^G)^*$. Now we can set

$$\chi' \uparrow^G = m\chi + \sum_{i=1}^s m_i \chi_i + \sum_{j=s+1}^t m_j \chi_j$$

where $\chi_i \ (1 \leq i \leq s)$ are $A$-invariant irreducible characters of $G$ different from $\chi$ and $\chi_j \ (s + 1 \leq j \leq t)$ are not $A$-invariant. Then $r \nmid m$ and $r \mid m_i$. We note if $\chi_j$ and $\chi_j' \ (j, j' \geq s + 1)$ are $A$-conjugate, then $m_j = m_j'$. Then, for any $\gamma \in \text{IBr}(G)^A$, the decomposition numbers $d_{\chi_j \gamma}$ and $d_{\chi_j' \gamma}$ are equal. Hence, since $A$ is an $r$-group, $\beta$ is a unique $A$-invariant irreducible constituent of $\beta' \uparrow^G$ with the multiplicity prime to $r$, and $m(\beta, \beta' \uparrow^G) \equiv m \pmod r$. On the other hand $m = m(\chi', \chi \downarrow_C) \equiv m(\beta', \beta \downarrow_C) \pmod r$ because $\chi \downarrow_C = m\chi' + r\zeta$ where $\zeta = 0$ or $\zeta$ is a character of $C$. This completes the proof. \[\blacksquare\]

Let $M_\epsilon$ be an irreducible $kG$-module corresponding to $\epsilon \in \mathcal{P}(kG)$. We have $(P_\epsilon)^a \cong P_\epsilon$ and $(M_\epsilon)^a \cong M_\epsilon$ for any $a \in A$. Hence by the above theorem and the Frobenius-Nakayama's reciprocity theorem we have the following.

**Proposition 1** Assume Hypothesis 2. There exists a bijection

$$\tilde{\pi}(G, A) : \mathcal{P}(kG)^A \to \mathcal{P}(kC)$$

which satisfies the following (i) and (ii). For $\epsilon \in \mathcal{P}(kG)^A$, set $\epsilon' = \tilde{\pi}(G, A)(\epsilon)$.

(i) If $B \triangleleft A$, then $\tilde{\pi}(G, A) = \tilde{\pi}(C_G(B), A/B)\tilde{\pi}(G, B)$.

(ii) Assume $A$ is an $r$-group for a prime $r$. Then $\epsilon$ is a unique element of $\mathcal{P}(kG)^A$ such that $r \nmid m(P_\epsilon, P_\epsilon \uparrow^G)$. Moreover $\epsilon'$ is a unique element of $\mathcal{P}(kC)$ such that $r \nmid m(P_\epsilon, P_\epsilon \downarrow_C)$, and we have also $m(P_\epsilon, P_\epsilon \uparrow^C) \equiv m(P_{\epsilon'}, P_\epsilon \downarrow_C) \pmod r$.

**Remark 1** In the above proposition,

$$m(P_\epsilon, P_\epsilon \uparrow^G) \neq 0.$$
Proof. Since $\chi'$ is a constituent of $\chi \downarrow C$, $\beta'$ is a constituent of $\beta \downarrow C$ where $\beta \in \mathrm{IBr}(G)^A$.

Hypothesis 3 With Hypothesis 1, $k_\ast G$ is a twisted group algebra of $G$ over $k$ with a basis \( \{u_x \mid x \in G\} \) and with a factor set $\alpha$. Moreover $A$ acts on $k_\ast G$ via $k$-algebra automorphisms and the following holds:

\[ (ku_x)^a = ku_{xa} \quad (\forall x \in G, \forall a \in A). \]

Proposition 2 Assume Hypothesis 3. There is a central extension of $G$ which satisfies (i) - (iii).

\[ 1 \to Z \to \hat{G} \xrightarrow{f} G \to 1 \]

(i) $|Z| = |G|_{p'}^2$,

(ii) The action of $A$ on $G$ is extended to $\hat{G}$, that is, $f(y^a) = f(y)^a$ (\forall y \in \hat{G}, \forall a \in A).$

Moreover $f^{-1}(C) = C_G(A)$,

(iii) There are an idempotent $e$ of $kZ$ and a $k$-algebra isomorphism $\tilde{f} : e(k\hat{G}) \to k_\ast G$ compatible with the action of $A$.

Proof. We may assume $\alpha$ satisfies

\[ \alpha(x, x')|G| = 1, \quad \alpha(x, 1) = \alpha(1, x) = 1 \quad (\forall x, x' \in G). \]

Then $u_1$ is the identity element of $k_\ast G$. Set $V = \{ \beta \in k^\times \mid \beta^{|G|} = 1 \} = \{ \beta \in k^\times \mid \beta^{|G|_{p'}} = 1 \}$. By the assumption, for each $a \in A$ and $x \in G$, we can write

\[ (u_x)^a = c(a, x)u_{xa} \quad (c(a, x) \in k^\times). \]

From $(u_xu_{x'})^a = (u_x)^a(u_{x'})^a$

(1)

\[ \alpha(x, x')c(a, xx') = c(a, x)c(a, x')\alpha(x^a, x'^a). \]

Therefore $c(a, x)c(a, x')c(a, xx')^{-1} \in V$ for each $a \in A$ and $x, x' \in G$. Hence $c(a, x)^{|G|^2} = 1$, and hence $c(a, x)^{|G|_{p'}^2} = 1$. Since $(u_x)^{ab} = ((u_x)^a)^b$

(2)

\[ c(ab, x) = c(a, x)c(b, x^a). \]

Moreover we have $c(a, 1) = 1$. Set $H = \{ \beta \in k^\times \mid \beta^{|G|^2} = 1 \}$. (We will construct a central extension of $G$ using the method in [3], 3.5.15, and [6], (10.4)) We define the multiplication in $\hat{G} = H \times G$ as follows:

\[ (h, x)(h', x') = (hh'\alpha(x, x'), xx'). \]

Then $\hat{G}$ forms a group with the identity element $(1, 1)$. Also $Z = H \times 1$ is a central subgroup of $\hat{G}$. The map $\iota : Z \to H((h, 1) \mapsto h)$ is an isomorphism, in particular $Z$ is a $p'$-group. Moreover $(|Z|, |A|) = 1$. Therefore if

\[ f : \hat{G} \to G ((h, x) \mapsto x) \]

then

\[ 1 \to Z \to \hat{G} \xrightarrow{f} G \to 1 \]
is a central extension of $G$ which satisfies (i).

By using (1) and (2)

$$(h, x)^a = (hc(a, x), x^a) \quad (\forall(h, x) \in \hat{G}, \forall a \in A)$$

defines an action of $A$ on $\hat{G}$ via group automorphisms. We note $A$ centralizes $Z$ and $f(y^a) = f(y)^a$ ($y \in \hat{G}$). Moreover $C_G(A) \subseteq f^{-1}(C)$. Let $c \in f^{-1}(C)$ and $a \in A$. We have $c^a = zc$ for some $z \in Z$. Since $z^{o(a)} = 1$, $z = 1$, so $c \in C_{\hat{G}}(A)$. Thus (iii) holds.

For any $x \in G$, set $\hat{x} = (1, x)$. We have $\hat{x} \hat{x}^l = (\alpha(x, x'), 1) \hat{xx}'$ ($\forall h \in H$).

Moreover $e = \frac{1}{|Z|} \sum_{z \in Z} \iota(z^{-1})z$ is an idempotent of $kZ$ and for any $z \in Z$, and we have $ze = \iota(z)e$. Therefore $e(k\hat{G}) = \bigoplus_{x \in G} k(e \hat{x})$, $(e \hat{x})(e \hat{x}') = \alpha(x, x') (e \hat{xx}')$. This implies

$$f : e(k\hat{G}) \to k_*G (\sum_{x \in G} c_x(e \hat{x}) \mapsto \sum_{x \in G} c_xu_x)$$

is an isomorphism of $k$-algebras. Moreover if $a \in A$, then

$$\tilde{f}((e \hat{x})^a) = c(a, x)u_x^a = \tilde{f}(e \hat{x})^a \quad (\forall x \in G).$$

Thus (iii) holds. $lacksquare$

**Remark 2** With Hypothesis 3, $A$ centralizes $k_*C$.

**Proof.** Our proof is the same as the proof of [2], 7.6. From (1) and (2)

$$c(a, xy) = c(a, x)c(a, y),$$

$$c(ab, x) = c(a, x)c(b, x) \quad (\forall a, b \in A, \forall x, y \in C).$$

The fact that $(|A|, |C|) = 1$ implies $c(a, x) = 1$.

**Proposition 3** Assume Hypotheses 2 and 3. There exists a bijection

$$\pi_*(G, A) : \mathcal{P}(k_*G)^A \to \mathcal{P}(k_*C)$$

which satisfies the following (i) and (ii). For $\epsilon \in \mathcal{P}(k_*G)^A$, set $\epsilon' = \pi_*(G, A)(\epsilon)$.

(i) If $B \triangleleft A$, then $\pi_*(G, A) = \pi_*(C_G(B), A/B)\pi_*(G, B)$.

(ii) Assume $A$ is an $r$-group for a prime $r$. Then $\epsilon$ is a unique element of $\mathcal{P}(k_*G)^A$ such that $r \mid m(P, P' \otimes_{k_*C} k_*G)$. Moreover $\epsilon'$ is a unique element of $\mathcal{P}(k_*C)$ such that $r \mid m(P, P' \downarrow_{k_*C})$, and we have also $m(P, P' \downarrow_{k_*C}) \equiv m(P, P' \otimes_{k_*C} k_*G) \pmod{r}$.

**Proof.** We will use Proposition 1. At first we note that $\hat{G}$ is $p$-solvable. For a subgroup $U$ of $G$, set $\hat{U} = f^{-1}(U)$. Then the $k$-algebras $e(k\hat{U})$ and $k_*U$ are isomorphic by the isomorphism $\tilde{f}|_{e(k\hat{U})}$. For $\delta \in \mathcal{P}(k_*U)$, we set $\delta = \tilde{f}^{-1}(\delta) \in \mathcal{P}(e(k\hat{U}))$. Note that when
$U$ is $A$-invariant, $\hat{\delta}$ is $A$-invariant if and only if $\delta$ is $A$-invariant. For $\delta \in \mathcal{P}(k_*U)$ and $\epsilon \in \mathcal{P}(k_*G)$, we have

\[ m(P_\epsilon, P_\delta \downarrow_{k_*U}) = m(P_\epsilon \uparrow_{k_*G}, P_\delta). \]

Since $Z$ is a central subgroup of $\hat{G}$, a point of $k\hat{G}$ is a point of the $k$-algebra $e(k\hat{G})$ or a point of $(1-e)(k\hat{G})$. If $\mu$ is a point of $e(k\hat{C})$, then \((P_\mu \uparrow_{k\hat{G}}) (1-e) = 0\). Hence the bijection $\tilde{\pi}(\hat{G}, A)$ in Proposition 1 induces a bijection from \(\mathcal{P}(e(k\hat{G}))^A\) onto \(\mathcal{P}(e(k\hat{C}))\) by Remark 1.

From Proposition 1, (i), $\pi_*(G, A)$ satisfies (i). (ii) follows from Proposition 1, (ii), the definition of $\pi_*(G, A)$ and (3).

2 The endomorphism ring of an induced module

Let $Q \triangleleft G$ and let $S$ be an $\mathcal{R}Q$-module. Let $H$ be a subgroup of $G$ containing $Q$. The $\mathcal{R}H$-module $S \uparrow^H$ can be embedded in $S \uparrow^G$. Set $\hat{H} = H/Q$ and

\[ E_{\hat{H}} = \text{End}_{\mathcal{R}H}(S \uparrow^H). \]

We can regard the $\mathcal{R}$-algebra $E_{\hat{H}}$ as a subalgebra of $E_{\hat{C}}$. For $\delta \in \mathcal{P}(E_{H})$, $V_\delta$ be an an indecomposable component of the $\mathcal{R}H$-module $S \uparrow^H$ corresponding to $\delta$ ([3], Theorem 1.5.4). We may assume $P_\delta = dE_{\hat{H}}$ and $V_\delta = d(S \uparrow^H)$ for some $d \in \delta$. We have

\[ V_\delta \downarrow^G = P_\delta \otimes_{E_{\hat{H}}} E_{\hat{G}}. \]

Proposition 4 Suppose that $Q \leq H \leq G$, and let $\delta \in \mathcal{P}(E_{H})$. For $d \in \delta$, we have an isomorphism of $\mathcal{R}G$-modules

\[ (d(S \uparrow^H)) \uparrow^G \cong d(S \uparrow^G). \]

In particular

\[ V_\delta \uparrow^G \cong \bigoplus_{\epsilon \in \mathcal{P}(E_{G})} m(P_\epsilon, P_\delta \uparrow^G) V_\epsilon \]

where $P_\delta \uparrow^G = P_\delta \otimes_{E_{\hat{H}}} E_{\hat{G}}$.

Proof. This is clear. In fact, suppose that $G = \bigcup_{i=1}^{[G:H]} H x_i$. We have

\[ (d(S \uparrow^H)) \uparrow^G = \bigoplus_{i=1}^{[G:H]} d(S \uparrow^H) \otimes_H x_i, \]

\[ d(S \uparrow^G) = \bigoplus_{i=1}^{[G:H]} d(S \uparrow^H) x_i. \]

Therefore

\[ \sum_{i=1}^{[G:H]} u_i \otimes x_i \in (d(S \uparrow^H)) \uparrow^G \implies \sum_{i=1}^{[G:H]} u_i x_i \in d(S \uparrow^G). \]
is an isomorphism. ■

\[
\begin{array}{ccc}
\text{Pid}(E_{\overline{H}}) & \xrightarrow{\text{induction}} & \text{Pid}(E_{\overline{G}}) \\
\uparrow^{1:1} & & \uparrow^{1:1} \\
\text{Comp}(S \uparrow^{H}) & \xrightarrow{\text{induction}} & \text{Comp}(S \uparrow^{G})
\end{array}
\]

From now, assume that \( S \) is \( G \)-invariant, that is, for any \( x \in G \), \( S \otimes x \cong S \) as \( \mathcal{R}Q \)-modules. For \( \sigma \in \overline{G} \), let \( x_{\sigma} \) be an element of \( \sigma \). We have

\[
S \uparrow^{G} = \bigoplus_{\sigma \in \overline{G}} S \otimes x_{\sigma}.
\]

We set

\[
E_{\sigma} = \{ \psi \in E_{\overline{G}} \mid \psi(S \otimes 1) \subseteq S \otimes x_{\sigma}\}.
\]

By the assumption \( E_{\sigma} \) contains an invertible element \( \psi_{\sigma} \). We have

\[
E_{\sigma}E_{\tau} = E_{\sigma\tau} \quad (\forall \sigma, \tau \in \overline{G}), \quad E_{\overline{H}} = \bigoplus_{\sigma \in \overline{H}} E_{\sigma} \quad (Q \leq H \leq G).
\]

That is, \( E_{\overline{H}} \) is a crossed product of \( \overline{H} \) over \( E_{\overline{1}} \).

Let a subgroup \( H \) be fixed. Set

\[
l_{\delta} = m(V_{\delta}, S \uparrow^{H}) = m(P_{\delta}, E_{\overline{H}}) \quad (\forall \delta \in \mathcal{P}(E_{\overline{H}})).
\]

We also set \( G = \bigcup_{i=1}^{|G:H|} y_{i}H \) and \( \psi_{i} = \psi_{y_{i}Q} \). Since \( \psi_{1} \) is invertible, we have the following

\[
E_{\overline{G}} = \bigoplus_{i=1}^{|G:H|} \psi_{i}E_{\overline{H}} \cong \bigoplus_{\delta \in \mathcal{P}(E_{\overline{H}})} l_{\delta}(\bigoplus_{i=1}^{|G:H|} \psi_{i}P_{\delta})
\]

as \( E_{\overline{H}} \)-modules. Hence

\[
\Psi : E_{\overline{G}} \otimes_{E_{\overline{H}}} (S \uparrow^{H}) \to S \uparrow^{G} \quad (\psi \otimes (s \otimes h) \mapsto \psi(s \otimes h))
\]

is an isomorphism of \( \mathcal{R}H \)-modules (cf. Theorem A in [1]). Let

\[
E_{\overline{G}} = \bigoplus_{s} P_{s}
\]

be a decomposition of \( E_{\overline{G}} \) into indecomposable \( E_{\overline{H}} \)-modules, where \( v = |G:H| \sum_{\delta \in \mathcal{P}(E_{\overline{H}})} l_{\delta} \).

The isomorphism \( \Psi \) induces a decomposition of \( S \uparrow^{G} \) into \( \mathcal{R}H \)-modules:

\[
S \uparrow^{G} = \bigoplus_{s=1}^{v} P_{s}(S \uparrow^{H}), \quad P_{s} \otimes_{E_{\overline{H}}} S \uparrow^{H} \cong P_{s}(S \uparrow^{H}).
\]

We note that if \( P_{s} \) and \( P_{t} \) are isomorphic, then it is clear that \( P_{s} \otimes_{E_{\overline{H}}} S \uparrow^{H} \cong P_{t} \otimes_{E_{\overline{H}}} S \uparrow^{H} \), and hence \( P_{s}(S \uparrow^{H}) \cong P_{t}(S \uparrow^{H}) \). Moreover, if \( P_{s} \cong \psi_{i}P_{\delta} \), then \( P_{s}(S \uparrow^{H}) \cong V_{\delta} \). Hence we have the following.
Proposition 5 For any $\epsilon \in \mathcal{P}(E_{\overline{G}})$,
\[
V_\epsilon \downarrow_H \cong \bigoplus_{\delta \in \mathcal{P}(E_{\overline{H}})} m(P_\delta, P_\epsilon \downarrow E_{\overline{H}})V_\delta.
\]

A correspondence between $\text{Comp}(S \uparrow^G)^A$ and $\text{Comp}(S \uparrow^C)$

In this section we assume Hypothesis 1 and let $Q$ and $S$ be as in the previous section. Moreover we assume $Q \subseteq C$. Then $A$ acts on $\overline{G}$. Since $|A| = 1$, $C_{\overline{G}}(A) = \overline{C}$. Since $Q \subseteq C$, the induced module $S \uparrow^G$ is $A$-invariant, in fact, $S \uparrow^G$ becomes an $\mathcal{R}(G \times A)$-module by the following action of $A$ on $S \uparrow^G$:

\[
(s \otimes x)a = s \otimes x^a \quad (s \in S, \ x \in G, \ a \in A).
\]

And we have

\[
(mx)a = (ma)x^a \quad (m \in S \uparrow^G, \ x \in G, a \in A).
\]

Moreover, $A$ acts on $E_{\overline{G}}$ via $\mathcal{R}$-algebra automorphisms as follows:

\[
\psi^a(m) = \psi(ma^{-1})a \quad (\psi \in E_{\overline{G}}, \ m \in S \uparrow^G, a \in A).
\]

If $\psi \in E_\sigma$, then

\[
\psi^a(s \otimes 1) = \psi(s \otimes 1)a \in S \otimes (x_\sigma)^a,
\]

where $x_\sigma \in \sigma$. Therefore

\[
(E_\sigma)^a = E_{\sigma^a} \quad (\sigma \in \overline{G}, a \in A).
\]

Lemma 1 For $\epsilon \in \mathcal{P}(E_{\overline{G}})$ and $a \in A$, we have

\[
(V_\epsilon)^a \cong V_{\epsilon^a}.
\]

In particular $\epsilon$ is $A$-invariant if and only if $V_\epsilon$ is $A$-invariant.

Proof. We can set $V_\epsilon = e(S \uparrow^G)$ ( $e \in \epsilon$). From the action of $A$ on $E_{\overline{G}}$, for $a \in A$, $e^a(S \uparrow^G) = e(S \uparrow^G)a = (V_\epsilon)a$. Therefore

\[
v^a \in (V_\epsilon)^a \rightarrow va \in (V_\epsilon)a
\]

is an isomorphism of $\mathcal{R}G$-modules (see (8)).

From now on we assume $S$ is indecomposable. Let $Q \leq H \leq G$. Then $J(E_{(1)})E_H = E_HJ(E_{(1)})$ is an ideal of $E_H$. We set

\[
\overline{E}_H = E_H/J(E_{(1)})E_H.
\]
\[ \bar{E}_\sigma = (E_\sigma + J(E_{\{i\}})E_G)/J(E_{\{i\}})E_G \quad (\forall \sigma \in \bar{G}). \]

We can regard \( \bar{E}_H \) as a \( k \)-subalgebra of \( \bar{E}_G \). Since \( k \) is algebraically closed, \( \bar{E}_G \) is a twisted group algebra of \( G \) over \( k \). As \( E_{\{i\}} \) is \( A \)-invariant, \( A \) acts on \( \bar{E}_G \) via \( k \)-algebra automorphisms. Moreover we have

\[ (\bar{E}_\sigma)^a = \bar{E}_{\sigma^a} \quad (\sigma \in \bar{G}, \ a \in A). \]

Therefore \( \bar{G} \), \( A \) and \( \bar{E}_G \) satisfies Hypothesis 2.

**Lemma 2** With the above notations, assume \( \bar{G} \) is \( p \)-solvable. There exists a bijection

\[ \pi(E_G, A) : \mathcal{P}(E_G)^A \rightarrow \mathcal{P}(E_C) \]

which satisfies the following (i) and (ii). For \( \epsilon \in \mathcal{P}(E_G)^A \), set \( \epsilon' = \pi(E_G, A)(\epsilon) \).

(i) If \( B \leq A \), then \( \pi(E_G, A) = \pi(E_{\overline{G}(B)}, A/B)\pi(E_G, B) \).

(ii) Assume \( A \) is an \( r \)-group for a prime \( r \). Then \( \epsilon' \) is a unique element of \( \mathcal{P}(E_C) \) such that \( r \not| m(P_\epsilon, P_\epsilon \downarrow_{E_G}) \). Moreover \( \epsilon \) is a unique element of \( \mathcal{P}(E_G)^A \) such that \( r \not| m(P_\epsilon, P_\epsilon \otimes_{E_G} E_G) \), and we have also \( m(P_\epsilon, P_\epsilon \downarrow_{E_G}) \equiv m(P_\epsilon, P_\epsilon \otimes_{E_G} E_G) \pmod{r} \).

**Proof.** In our proof we will use lifting of idempotents ([6], Theorem 3.2) repeatedly. Let \( Q \leq U \leq G \). Since \( J(E_{\{i\}})E_G \) is contained in \( J(E_\bar{G}) \), the canonical homomorphism from \( E_G \) onto \( \bar{E}_G \) induces a bijection between \( \mathcal{P}(E_G) \) and \( \mathcal{P}(\bar{E}_G) \). For \( \delta \in \mathcal{P}(E_G) \), we denote by \( \delta \) the corresponding point of \( \bar{E}_G \). When \( U \) is \( A \)-invariant, \( \delta \) is \( A \)-invariant if and only if \( \delta \) is \( A \)-invariant. Therefore by using the bijection \( \pi(E_G, A) \) obtained in Proposition 3 for the twisted group algebra \( \bar{E}_G \), we can define the bijection \( \pi(E_G, A) : \mathcal{P}(E_G)^A \rightarrow \mathcal{P}(E_C) \) as follows

\[ \pi(E_G, A)(\epsilon) = \pi(\bar{E}_G, A)(\bar{\epsilon}). \]

From Proposition 3, (i), \( \pi(E_G, A) \) satisfies (i). Now it is easy to see that

\[ m(P_\delta, P_\epsilon \downarrow_{E_G}) = m(P_\delta, P_\epsilon \downarrow_{E_G}), \]

\[ m(P_\epsilon, P_\delta \otimes_{E_G} E_G) = m(P_\epsilon, P_\delta \otimes_{E_G} E_G) \]

because \( P_\delta \otimes_{E_G} E_G \cong (P_\delta \otimes_{E_G} E_G)/(P_\delta \otimes_{E_G} E_G)J(E_{\{i\}}) \). Hence from Proposition 3, (ii), (ii) holds. \( \blacksquare \)

Let \( Q \leq U \leq G \) with \( U \) \( A \)-invariant. We denote by \( \text{Comp}(S \uparrow_U) \) the isomorphism classes of indecomposable components of \( S \uparrow_U \). From (4), (6), Lemmas 1 and 2, the following holds.

**Proposition 6** With the above notations, assume \( \bar{G} \) is \( p \)-solvable. There exists a bijection

\[ \pi(\bar{G}, A; S) : \text{Comp}(S \uparrow_G)^A \rightarrow \text{Comp}(S \uparrow_C) \]

which satisfies the following (i) and (ii). For \( [V] \in \text{Comp}(S \uparrow_G)^A \), set \( [V'] = \pi(\bar{G}, A; S)([V]) \).

(i) If \( B \leq A \), then \( \pi(\bar{G}, Q; S) = \pi(C_G(B), A/B; S)\pi(\bar{G}, B; S) \).

(ii) Assume \( A \) is an \( r \)-group for a prime \( r \). Then \( V' \) is a unique indecomposable component of \( V \downarrow_G \) with the multiplicity prime to \( r \). Moreover \( V \) is a unique \( A \)-invariant indecomposable component of \( V' \uparrow_G \) with the multiplicity prime to \( r \), and we have also

\[ m(V', V \downarrow_G) \equiv m(V, V' \uparrow_G) \pmod{r}. \]
4 Proof of Theorem 1

We assume Hypothesis 2. Let $K \leq G$ and $X$ be an $\mathcal{R}K$-module. We have

$$X^a \uparrow^G \cong (X \uparrow^G)^a \ (l^a \otimes_{H^a} x \mapsto (l \otimes_L x^{a^{-1}})^a).$$

Therefore if an indecomposable $\mathcal{R}G$-module $X$ has a vertex $D$, then $X^a$ has a vertex $D^a$.

Let $Q \leq C$. If an indecomposable $\mathcal{R}G$-module $V$ has a vertex $Q$, then $V^a$ has a vertex $Q$. We denote by $g_{NC(Q)}$ the Green correspondence from $\text{Ind}(\mathcal{R}N_G(Q)|Q)$ onto $\text{Ind}(\mathcal{R}G|Q)$. If $V'$ is the Green correspondent of an indecomposable $\mathcal{R}G$-module $V$, $V'^a$ is the Green correspondent of $V^a$. In particular $V$ is $A$-invariant if and only if $V'$ is $A$-invariant. Let $S$ be an indecomposable $\mathcal{R}Q$-module and set $T = N_G(Q, S)$, the stabilizer of $S$ in $N_G(Q)$. Then there is a natural bijection compatible with the action of $A$ between $\text{Comp}(S \uparrow^T)$ and $\text{Ind}(S \uparrow^{N_G(Q)})$ (3, Corollary 4.6.8). Assume $Q$ is a vertex of $S$. We denote by $\text{Ind}(\mathcal{R}G||S)$ the set of isomorphism classes of indecomposable $\mathcal{R}G$-modules with a $Q$-source $S$. Hence there is a natural bijection compatible with the action of $A$ between $\text{Ind}(\mathcal{R}G||S)$ and $\text{Comp}(\mathcal{R}T||S)$ by Green correspondence. Now suppose that $Q \leq H \leq G$. For $M \in \text{Comp}(S \uparrow^T)$ and $L \in \text{Comp}(S \uparrow^{T\cap H})$, set $V = g_{NC(Q)}(M \uparrow^{N_G(Q)})$ and $W = g_{NH(Q)}(L \uparrow^{NH(Q)})$. By a property of Green correspondence we can see

$$m(V, W \uparrow^G) = m(M, L \uparrow^T),$$

(9)

$$m(W, V \downarrow^H) = m(L, M \downarrow_{H \cap T}).$$

Proof. At first we give a remark. By Hypothesis 1, if $x \in N_G(Q)$, then $x = cy$ ($c \in N_C(Q), y \in C_G(Q)$) by a theorem of Schur-Zassenhaus. Therefore if $\mathcal{R}Q$-modules $S_1$ and $S_2$ are $N_C(Q)$-conjugate, then those are $N_C(Q)$-conjugate.

Now let $[V] \in \text{Ind}(\mathcal{R}G|Q)^A$ and $S$ be a $Q$-source of $V$. Set $T = N_G(Q, S)$. By a property of Green correspondence there is a unique $M \in \text{Comp}(S \uparrow^T)^A$ such that $V$ is a component of $M \uparrow^G$, that is, $V$ is the Green correspondent of $M \uparrow^{N_G(Q)}$. Let $[M'] = \pi(T, A : S)([M])$ and $V' = g_{NC(Q)}(M \uparrow^{N_G(Q)})$ where $T = T/Q$. By the above remark, the map

$$[V] \in \text{Ind}(\mathcal{R}G|Q)^A \mapsto [V'] \in \text{Ind}(\mathcal{R}C|Q)$$

is a bijection. We denote it by $\pi(G, A; Q)$. By Proposition, (i), (i) holds.
Now assume \( A \) is an \( r \)-group. Then the above \( M \) is an \( A \)-invariant unique indecomposable component of \( M' \uparrow^T \) with the multiplicity prime to \( r \) by Proposition 6, (ii). Therefore, from (9), we see \( V \) is a unique \( A \)-invariant indecomposable component of \( V' \uparrow^G \) with the multiplicity prime to \( r \) and with vertex \( Q \). (In fact we have \( m(V, V' \uparrow^G) = m(M, M' \uparrow^T) \)) On the other hand \( m(V', V \downarrow_C) = m(M', M \downarrow_{C \cap T}) \) from (10). Hence \( m(V, V' \uparrow^G) \equiv m(V', V \downarrow_C) \) (mod \( r \)) by Proposition 6, (ii). Now suppose that

\[ \text{vx}(	V) = C, \text{r} \not| m(	V, V \downarrow_C) \]

for an indecomposable \( RC \)-module \( \tV \). Moreover let \( \tS \) be a \( Q \)-source of \( \tV \). Then by Mackey decomposition, \( \tS \) and \( S \) are \( NG(Q) \)-conjugate, and hence we may assume \( \tS = S \). By Proposition 6, (ii) again, we see \( [\tV] = [V'] \). This completes the proof. \( \blacksquare \)

**Question:** Assume \( A \) is solvable. Let \( \beta \in \text{IBr}(G)^A \) and \( \beta' \) be the Uno correspondent of \( \beta \). Suppose \( \text{vx}(\beta) = Q \leq C \). Then \( \text{vx}(\beta') = Q \) by [4], Theorem. Let \( V_\beta \) be a \( kG \)-module with Brauer character \( \beta \). When is \( \pi(G, A; Q)(V_\beta) = V_{\beta'} \)?

**References**


