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Kyoto University
On the Glauberman-Watanabe correspondence for $p$-blocks of a $p$-nilpotent group with a cyclic defect group

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1

Let $p$ be a prime. Let $(K, O, k)$ be a $p$-modular system where $O$ is a complete discrete valuation ring having an algebraically closed residue field $k$ of characteristic $p$ and having a quotient field $K$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article. We use the notation $-$ for the reduction modulo $J(O)$. Let $R \in \{K, O, k\}.

Below, for groups $H_1$ and $H_2$, an $R[H_1 \times H_2]$-module $X$ and an $(RH_1, RH_2)$-bimodule $X$ will be identified in the usual way, namely $(h_1, h_2) \cdot x = h_1 \cdot x \cdot h_2^{-1}$ where $h_1 \in H_1$, $h_2 \in H_2$ and $x \in X$. For a common subgroup $D$ of $H_1$ and $H_2$, denote by $\Delta D = \{(u, u) | u \in D\}$ a diagonal subgroup of $H_1 \times H_2$. Let $R' \in \{O, k\}$. For a $p$-group $P$, an $R'$-free $R'P$-module $T$ is called an endo-permutation module if $\text{End}_{R'}(T)$ has an $P$-invariant $R'$-basis ([1]).

Let $q$ be a prime such that $q \neq p$. Let $S = <s>$ be a cyclic group of order $q$. Let $\mu \in O$ be a fixed non-trivial $q$-th root of unity.

Let $G$ be a finite group such that $q \nmid |G|$. Assume that $S$ acts on $G$. Then with this action, we can consider the semi-direct product of $G$ and $S$, denoted by $GS$. Denote by $GS$ the centralizer $C_G(S)$ of $S$ in $G$. When $q$ is odd, for $\theta \in \text{Irr}(G)^S$, there is a unique extension $\hat{\theta} \in \text{Irr}(GS)$ of $\theta$, a unique character $\pi(G, S)(\theta) \in \text{Irr}(GS)$ and a unique sign $\epsilon_\theta$ such that $\hat{\theta}(cs) = \epsilon_\theta \pi(G, S)(\theta)(c)$ where $c \in GS$. When $q = 2$, for $\theta \in \text{Irr}(G)^S$ and a chosen sign $\epsilon_\theta$, there is a unique extension $\hat{\theta} \in \text{Irr}(GS)$ of $\theta$ and a unique character $\pi(G, S)(\theta) \in \text{Irr}(GS)$ such that $\hat{\theta}(cs) = \epsilon_\theta \pi(G, S)(\theta)(c)$ for $c \in GS$. The character $\pi(G, S)(\theta)$ is called the Glauberman correspondence of $\theta$, see [3]. For $t \in \mathbb{Z}$, let $\lambda^t \hat{\theta} \in \text{Irr}(GS)$ be the extension of $\theta \in \text{Irr}(G)^S$ such that $\lambda^t \hat{\theta}(gs) = \mu^t \hat{\theta}(gs)$ where $g \in G$.

Let $b$ be an $S$-invariant ($p$-)block of $G$ having an $S$-centralized defect group $D$. Denote by $w(b)$ the Glauberman-Watanabe corresponding block of $b$, that is, the block of $GS$ with a defect group $D$ such that $\text{Irr}(w(b)) = \{\pi(G, S)(\theta) | \theta \in \text{Irr}(b) = \text{Irr}(b)^S\}$. For $t \in \mathbb{Z}$, let $b_t$ be the block of $GS$ such that $\text{Irr}(b_t) = \{\lambda^t \hat{\theta} | \theta \in \text{Irr}(b)\}$ (under appropriate choices of signs $\epsilon_\theta$ when $q = 2$), and let $e_t$ be the block of $S$ corresponding to the representation of $S$ determined by $s \mapsto \mu^t$. Let
\[ b_r = \sum_{t=0}^{q-1} \epsilon_t b_{t+r} \quad \text{for} \quad 0 \leq r \leq q - 1. \] (1)

Then \( b = \sum_{r=0}^{q-1} b_r \) is an orthogonal idempotent decomposition of \( b \) in \( \mathcal{O}G^{s}b \) and so \( b_r \mathcal{O}G \) is a direct summand of the \( \mathcal{O}[G^S \times G] \)-module \( \mathcal{O}Gb \), and the following equation of the generalized characters of \( G^S \times G \) holds, see [6] and [7]:

\[ \chi_{b_0} \mathcal{O}G - \chi_{b} \mathcal{O}G = \sum_{\theta \in \text{Irr}(b)} \epsilon_{\theta} \pi(G, S)(\theta) \otimes_{\mathcal{K}} \check{\theta} \quad \text{for} \quad 1 \leq l \leq q - 1, \] (2)

where \( \chi_{b_0} \mathcal{O}G \) is a character corresponding to a \( \mathcal{K}[G^S \times G] \)-module \( b_r \mathcal{K}G \) and \( \check{\theta} \) is a \( \mathcal{K} \)-dual of \( \theta \). (Below, denote by \( \check{b} \) the block containing \( \check{\theta} \) for \( \theta \in \text{Irr}(b) \).)

Equation (2) gives immediately the following Watanabe’s result, see [9]:

The map determined by \( \theta \mapsto \epsilon_{\theta} \pi(G, S)(\theta) \) where \( \theta \in \text{Irr}(b) \), induces a perfect isometry \( Z\text{Irr}(b) \simeq Z\text{Irr}(w(b)) \) between the Glauberman-Watanabe corresponding blocks.

And, as noted by Okuyama in [6], raised the following question:

Is the left hand side of equation (2) is a “shadow” of a complex of \( \mathcal{O}G^{S}w(b), \mathcal{O}Gb \)-bimodule which induces a derived equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^Sw(b) \)?

In fact, we have the following:

**Theorem 1.1.** With the above notations, moreover assume that \( G \) is \( p \)-nilpotent and \( D \) is cyclic. Then there is a two term complex \( C^\bullet \) of \( \mathcal{O}G^S(w(b), \mathcal{O}Gb) \)-bimodule satisfying the following:

(1) \( b_0 \mathcal{O}G \) is in degree 0 and \( b_1 \mathcal{O}G \) is in degree 1 or \(-1\).

(2) \( C^\bullet \) induces a derived equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^S(w(b)) \).

Further, \( C^\bullet \) is quasi-isomorphic to a one term complex consisting of the bimodule \( M \) satisfying the following \( (M \) is in degree 0 if \( \epsilon_b = 1 \) and \( M \) is in degree 1 or \(-1\) if \( \epsilon_b = -1 \) where \( \epsilon_b = \epsilon_\theta \) for \( \theta \in \text{Irr}(b) \), which depends only on \( b \)):

(a) \( M \) induces a Morita equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^S(w(b)) \).

(b) \( M \) has a vertex \( \Delta DD \) and an endo-permutation source.

\( C^\bullet \) in Theorem 1.1 induces above Watanabe’s perfect isometry, see the condition in Theorem 1.1(1) and question (2), and \( M \) in Theorem 1.1 induces the Glauberman correspondence of characters belonging to \( b \) and \( w(b) \). The existence of \( M \) as in Theorem 1.1 is a particular case of the result of Harris-Linckelman for \( p \)-solvable case and of Watanabe for \( p \)-nilpotent blocks, see [5] and [10]. See also [4] for the existence of a derived equivalence between blocks with cyclic defect groups inducing prescribed perfect isometry.
Below, with the assumptions in Section 1, \( G \) and \( b \) are such that:

**Condition 2.1.** \( G \) is a \( p \)-nilpotent group with an \( S \)-centralized cyclic Sylow \( p \)-subgroup \( P \) of order \( p^\alpha \), that is, \( G = KP = K \times P \) where \( K = O_p'(G) \). \( b \) is a \( P \)-invariant block of \( K \), hence a block of \( G \) with a defect group \( D = P \).

In fact, by the Fong's first reduction as described in [5, Section 5] and Theorem 2.2 and 2.3 below, Theorem 1.1 above can be shown.

Denote by \( P_i \) the unique subgroup of \( P \) with the order \( p^i \) for \( i \) such that \( 0 \leq i \leq \alpha \). Recall that the image \( Br_{P_i}(b) \) of the Brauer homomorphism \( Br_{P_i} \) of \( b \) is primitive in \( Z(kC_K(P_i)) \) and hence is a block of \( C_G(P_i) = C_K(P_i)P \), and let \( Br_{P_i}(b) \) be the corresponding block over \( \mathcal{O} \). Note that \( b = Br_{P_0}(b) \). Idempotents \( Br_{P_i}(b_r) \in (\mathcal{O}C_G(P_i)Br_{P_i}(b))^{C_{G^S}(P_i)} \) (see (1) in Section 1) are defined similarly. Denote by \( M^i_j \) the unique trivial source \( \mathcal{O}[C_{G^S}(P_i) \times C_{G}(P_i)] \)-module in \( w(Br_{P_i}(b)) \times Br_{P_i}(b) \) with vertex \( \Delta P_j \) for \( j \) such that \( 0 \leq j \leq \alpha \). Let \( M^i_0 = M^i_\alpha \). Let \( \epsilon_{Br_{P_i}(b)} = \epsilon_{\chi_i} \) where \( \chi_i \in \text{Irr}(C_G(P_i) | Br_{P_i}(b)) \). Note that \( \epsilon_{Br_{P_i}(b)} \) depends only on \( Br_{P_i}(b) \).

**Theorem 2.2.** The following are equivalent for a fixed \( i \) where \( 0 \leq i \leq \alpha \):

1. \( \epsilon_{Br_{P_i}(b)} = \epsilon_{Br_{P}(b)} \) for any \( h \) such that \( i \leq h \leq \alpha \).

2. The unique simple \( k(C_K(P_i) \times C_K(P_i))\Delta P \)-module in \( w(Br_{P_i}(b)) \times Br_{P_i}(b) \) is a trivial source module.

3. \( M^i_\alpha \) is a unique indecomposable direct summand of \( \mathcal{O}C_G(P_i)Br_{P_i}(b) \downarrow_{C_{G^S}(P_i) \times C_{G}(P_i)} \mathcal{O}C_{G^S}(P_i) \times C_{G}(P_i) \) with a multiplicity not divisible by \( q \).

4. (a) \( Br_{P_i}(b)_{0} \mathcal{O}C_G(P_i) \simeq M^i_\alpha \oplus Br_{P_i}(b)_{0} \mathcal{O}C_G(P_i) \) if \( \epsilon_{Br_{P}(b)} = 1 \).

   (b) \( Br_{P_i}(b)_{0} \mathcal{O}C_G(P_i) \simeq M^i_\alpha \oplus Br_{P_i}(b)_{0} \mathcal{O}C_G(P_i) \) if \( \epsilon_{Br_{P}(b)} = -1 \).

5. \( M^i_\alpha \) induces a Morita equivalence between \( \mathcal{O}C_G(P_i) Br_{P_i}(b) \) and \( \mathcal{O}C_{G^S}(P_i) \mathcal{O}Br_{P_i}(b) \).

6. \( \mathcal{O}C_G(P_i) Br_{P_i}(b) \) and \( \mathcal{O}C_{G^S}(P_i) \mathcal{O}Br_{P_i}(b) \) are Puig equivalent.

The conditions of Theorem 2.2 above always holds for \( i = \alpha \). If the conditions of Theorem 2.2 holds for \( i = 0 \), that is, \( OGb \) and \( OG^S w(b) \) are Puig equivalent, then, by the conditions of Theorem 2.2(4) and (5), we can construct a desired two term complex \( C^* \) as in Theorem 1.1 with \( M = M^\alpha \).
Below, we consider the case where \( \mathcal{O}Gb \) and \( \mathcal{O}G^{s}w(b) \) are not Puig equivalent. Then there is some \( \beta \) as in Theorem 2.3 below, see, for example, conditions of Theorem 2.2(1) and Theorem 2.3(1).

Since \( (K^{s}\times K)\Delta P \) is \( p \)-nilpotent, sources of simple \( k(K^{s}\times K)\Delta P \)-modules are endo-permutation modules (Dade [2]). Since \( \Delta P \) is cyclic, indecomposable endo-permutation \( k\Delta P \)-modules with vertex \( \Delta P \) are the modules of the following form (Dade [2]):

\[
\Omega_{\Delta P}\inf_{\Delta(P/P_{1})}^{\Delta(P/P_{1})}\Omega_{\Delta(P/P_{2})}^{\Delta(P/P_{2})}\cdots\inf_{\Delta(P/P_{\alpha-1})}^{\Delta(P/P_{\alpha-1})}\Omega_{\Delta(P/P_{\alpha-2})}^{\Delta(P/P_{\alpha-2})}(k),
\]

where \( \Omega \) means Heller translate and \( a_{i} \in \{0, 1\} \).

**Theorem 2.3.** Let \( \beta \) be such that \( 0 \leq \beta \leq \alpha - 1 \). The following conditions on \( \beta \) are equivalent:

1. \( e_{\mathfrak{B}r_{P_{h}}(b)} \neq e_{\mathfrak{B}r_{P_{h}}(b)} \) and \( e_{\mathfrak{B}r_{P_{h}}(b)} = e_{\mathfrak{B}r_{P_{h}}(b)} \) for any \( h \) such that \( \beta + 1 \leq h \leq \alpha \).

2. \( a_{h} = 1 \) and \( a_{h} = 0 \) for any \( h \) such that \( \beta + 1 \leq h \leq \alpha \) where \( a_{i} \)'s are 0 or 1 describing a source of the unique simple \( k(K^{s}\times K)\Delta P \)-module in \( w(b) \times \tilde{b} \) as above (when \( p = 2 \), let \( a_{\alpha - 1} = 0 \)).

3. \( \mathcal{O}C_{G}(P_{\beta})\mathfrak{B}r_{P_{\beta}}(b) \) and \( \mathcal{O}C_{G^{s}}(P_{\beta})w(\mathfrak{B}r_{P_{\beta}}(b)) \) are not Puig equivalent and \( \mathcal{O}C_{G}(P_{h})\mathfrak{B}r_{P_{h}}(b) \) and \( \mathcal{O}C_{G^{s}}(P_{h})w(\mathfrak{B}r_{P_{h}}(b)) \) are Puig equivalent for any \( h \) such that \( \beta + 1 \leq h \leq \alpha \).

4. The multiplicity of \( M^{\beta} \) in \( \mathcal{O}Gb \downarrow_{G^{s} \times G} \) is not divisible by \( q \).

5. \( M^{\alpha} \) and \( M^{\beta} \) are only indecomposable direct summands of \( \mathcal{O}Gb \downarrow_{G^{s} \times G} \) with multiplicities not divisible by \( q \).

6. (a) \( b_{1}\mathcal{O}G \oplus M^{\alpha} \simeq b_{0}\mathcal{O}G \oplus M^{\beta} \) if \( e_{\mathfrak{B}r_{P}(b)} = 1 \).

   (b) \( b_{0}\mathcal{O}G \oplus M^{\alpha} \simeq b_{1}\mathcal{O}G \oplus M^{\beta} \) if \( e_{\mathfrak{B}r_{P}(b)} = -1 \).

7. (a) When \( e_{b}e_{\mathfrak{B}r_{P}(b)} = -1 \), there is an epimorphism \( \Phi : M^{\beta} \rightarrow M^{\alpha} \) such that \( N = \text{Ker} \Phi \) induces a Morita equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^{s}w(b) \).

   (b) When \( e_{b}e_{\mathfrak{B}r_{P}(b)} = 1 \), there is an epimorphism \( \Phi : M^{\alpha} \rightarrow M^{\beta} \) such that \( N = \text{Ker} \Phi \) induces a Morita equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^{s}w(b) \).

If \( \mathcal{O}Gb \) and \( \mathcal{O}G^{s}w(b) \) are not Puig equivalent, then, by the conditions of Theorem 2.3(6) and (7), we can construct a desired two term complex \( C^{\bullet} \) as in Theorem 1.1 with \( M = N \). Note that a source of \( \overline{N} \) is a source of the unique simple \( k(K^{s}\times K)\Delta P \)-module in \( w(b) \times \tilde{b} \), and an \( \mathcal{O} \)-lift of an endo-permutation module is an endo-permutation module.
In fact, a source of the module inducing the concerned Morita equivalence between $kG\overline{b}$ and $kG^S w(\overline{b})$ and "signs of the local blocks" $\epsilon_{\mathfrak{B}r_{P_i}(b)}$ are related as follows:

**Proposition 2.4.** The following conditions on $\alpha$ numbers $a_i \in \{0, 1\}$ $(0 \leq i \leq \alpha - 1)$ are equivalent when $p$ is odd:

1. A source of the unique simple $k(K^S \times K)\Delta P$-module in $w(\overline{b}) \times \tilde{b}$ has the following form:

$$\Omega_{P_0}^{a_0} \text{Inf}_{\Delta(P/P_1)}^{\Delta(P/P_1)} \Omega_{\Delta(P/P_2)}^{a_1} \text{Inf}_{\Delta(P/P_2)}^{\Delta(P/P_2)} \cdots \text{Inf}_{\Delta(P/P_{\alpha-2})}^{\Delta(P/P_{\alpha-2})} \Omega_{\Delta(P/P_{\alpha-2})}^{a_{\alpha-2}}(k).$$

2. $\epsilon_{\mathfrak{B}r_{P_i}(b)} = (-1)^{a_i} \epsilon_{\mathfrak{B}r_{P_{i+1}}(b)}$ for any $i$ such that $0 \leq i \leq \alpha - 1$.

**Proposition 2.5.** The following conditions on $\alpha - 1$ numbers $a_i \in \{0, 1\}$ $(0 \leq i \leq \alpha - 2)$ are equivalent when $p = 2$:

1. A source of the unique simple $k(K^S \times K)\Delta P$-module in $w(\overline{b}) \times \tilde{b}$ has the following form:

$$\Omega_{P_0}^{a_0} \text{Inf}_{\Delta(P/P_1)}^{\Delta(P/P_1)} \Omega_{\Delta(P/P_2)}^{a_1} \text{Inf}_{\Delta(P/P_2)}^{\Delta(P/P_2)} \cdots \text{Inf}_{\Delta(P/P_{\alpha-2})}^{\Delta(P/P_{\alpha-2})} \Omega_{\Delta(P/P_{\alpha-2})}^{a_{\alpha-2}}(k).$$

2. $\epsilon_{\mathfrak{B}r_{P_i}(b)} = (-1)^{a_i} \epsilon_{\mathfrak{B}r_{P_{i+1}}(b)}$ for any $i$ such that $0 \leq i \leq \alpha - 2$.

**References**


