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What is the origin of rotational motion in dissipative systems ?

散逸系における2次元スポット解の回転運動

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Spatially localized moving patterns such as traveling pulses and spots are fundamental objects arising in many reaction-diffusion systems, which display a large variety of dynamical behaviors [6, 4]. In two dimensions, traveling motion causes symmetry-breaking from the circular shape of a standing spot, and traveling velocity causes deformation to the elliptical shape. Recent developments in digital image analysis show that a head-tail asymmetry in cell shape determines the direction of motion [3]. Also, some sorts of interference wave pattern occurs during spontaneous cell migration. These biological experiments allow us to deduce the underlying mechanism of interplay between the spot locomotion and shape-change dynamics.

In this paper, we consider the spot dynamics near a codimension 2 singularity for reaction-diffusion systems in which the associated parameter values are located close to the drift and peanut bifurcation points. Drift instability originates in the translation-free mode and the associated deformation eigenvector represents a D^1 symmetry breaking from a disk shape. Peanut one is by D^2 symmetry breaking bifurcation, where D^n stands for the dihedral symmetry group. We show that such a codimension 2 singularity can induce rotational motion of traveling spots – that is, rotational spot (RS) motion – in a class of reaction-diffusion systems. The occurrence of such a motion is generic because the original partial differential equations (PDEs) can be reduced to finite-dimensional ordinary differential equations (ODEs) based on the method developed by [2], and the resulting ODEs take a normal form of 1:2 mode interaction of cubic type. The information about the original PDEs is renormalized in the coefficients of the reduced system.

We analyze the reduced ODEs, and show that there exists a solution in which both drift velocity vector and peanut deformation become time-periodic functions that correspond to the rotational solution to the original reaction-diffusion systems. We also discuss about the relationship between the global bifurcational structures of the original PDEs and the reduced ODEs, which sheds light on the origin of rotational motion.

A general setup for the PDE system in a neighborhood of codimension 2 bifurcation point $\boldsymbol{\lambda}^c = (\lambda_1^c, \lambda_2^c)$ reads, with a small parameter $\boldsymbol{\eta} = (\eta_1, \eta_2)$ as $\boldsymbol{\lambda} = \boldsymbol{\lambda}^c + \boldsymbol{\eta}$,

$$\boldsymbol{u}_t = \ D\triangle\boldsymbol{u} + F(\boldsymbol{u};\boldsymbol{\lambda}) \equiv \mathcal{L}(\boldsymbol{u};\boldsymbol{\lambda}^c) + \sum_{i=1}^{2} \eta_i \boldsymbol{g}_i(\boldsymbol{u}), \tag{1}$$

where $\boldsymbol{g}_i$ $(i = 1, 2)$ is N-dimensional vector-valued functions. Let $X := \{L^2(\mathbb{R})\}^N$, $\boldsymbol{u}(t, \boldsymbol{r}) = (u_1, \cdots, u_N)^T \in X$ be an N-dimensional vector and $F : \mathbb{R}^N \to \mathbb{R}^N$, D be a positive diagonal matrix. We assume that the nontrivial standing spot solution $S(\boldsymbol{r};\boldsymbol{\lambda})$ exists at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^c$, i.e., $\mathcal{L}(S;\boldsymbol{\lambda}^c) = 0$.

Let L be the linearized operator $L = \mathcal{L}'(S(\boldsymbol{r}, \boldsymbol{\lambda}^c))$. L has a codimension 2 singularity at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^c$ consisting of drift and peanut bifurcations in addition to the translation-free 0 eigenvalue; that is, there exist three types of eigenfunctions $\boldsymbol{\phi}_i(\boldsymbol{r})$, $\boldsymbol{\psi}_i(\boldsymbol{r})$ and $\boldsymbol{\xi}_i(\boldsymbol{r})$ $(i = 1, 2)$ such that $L\boldsymbol{\phi}_i = 0$, $\quad L\boldsymbol{\psi}_i = -\boldsymbol{\phi}_i$, and $\quad L\boldsymbol{\xi}_i = 0$, where $\boldsymbol{\phi}_i = \partial S/\partial x_i$ and $\boldsymbol{\psi}_i$ represents the deformation vector with Jordan form for the drift bifurcation.

Similar properties also hold for L^*. That is, there exist $\boldsymbol{\phi}_i^*$, $\boldsymbol{\psi}_i^*$, and $\boldsymbol{\xi}_i^*$ such that $L^*\boldsymbol{\phi}_i^* = 0$, $L^*\boldsymbol{\psi}_i^* = -\boldsymbol{\phi}_i^*$, and $L^*\boldsymbol{\xi}_i^* = 0$. Let $E = \mathrm{span}\{\boldsymbol{\phi}_i, \boldsymbol{\psi}_i, \boldsymbol{\xi}_i\}$ and the eigenfunctions be normalized by $\langle \boldsymbol{\psi}_i, \boldsymbol{\phi}_j \rangle_{L^2} = \langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j^* \rangle_{L^2} = 0$, and

$$\langle \boldsymbol{\phi}_i, \boldsymbol{\psi}_i^* \rangle_{L^2} = \langle \boldsymbol{\psi}_i, \boldsymbol{\phi}_i^* \rangle_{L^2} = \langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_i^* \rangle_{L^2} = \left\{ \begin{array}{ll} \pi & i = j, \\ 0 & i \neq j. \end{array} \right. \tag{2}$$

The motion of a spot solution $\boldsymbol{u}$ is essentially described by the two-dimensional vector functions of time t; $\boldsymbol{p} = (p_1, p_2)$ denotes the location of the spot; $\boldsymbol{q} = (q_1, q_2)$ denotes its velocity; and $\boldsymbol{s} = (s_1, s_2)$ denotes its deformation. For small $\boldsymbol{\eta}$, we can approximate a solution $\boldsymbol{u}$ by

$$\boldsymbol{U} \;= \tau(\boldsymbol{p}) \left\{ S(\boldsymbol{r}) + \textstyle\sum_{i=1}^{2} q_i \boldsymbol{\psi}_i(\boldsymbol{r}) + \sum_{i=1}^{2} s_i \boldsymbol{\xi}_i(\boldsymbol{r}) + \boldsymbol{\zeta}^\dagger \right\}, \tag{3}$$

where $\tau(\boldsymbol{p})$ is the translation operator with $(\tau(\boldsymbol{p})\boldsymbol{u})(\boldsymbol{r}) = \boldsymbol{u}(\boldsymbol{r} - \boldsymbol{p})$. The remaining term $\boldsymbol{\zeta}^\dagger$ belongs to $E^\perp$. More precisely, $\boldsymbol{\zeta}^\dagger = {q_1}^2\boldsymbol{\zeta}_1 + {q_2}^2\boldsymbol{\zeta}_2 + q_1q_2\boldsymbol{\zeta}_3 + {s_1}^2\boldsymbol{\zeta}_4 + {s_2}^2\boldsymbol{\zeta}_5 + s_1s_2\boldsymbol{\zeta}_6 + q_1s_1\boldsymbol{\zeta}_7 + q_2s_2\boldsymbol{\zeta}_8 + q_1s_2\boldsymbol{\zeta}_9 + q_2s_1\boldsymbol{\zeta}_{10} + \eta_1\boldsymbol{\zeta}_{11} + \eta_2\boldsymbol{\zeta}_{12}$ with $\boldsymbol{\zeta}_k (k = 1, \cdots 12) \in E^\perp$ are defined by solutions of

$$\left\{ \begin{array}{c} L\boldsymbol{\zeta}_1 + \dfrac{1}{2}F''(S){\boldsymbol{\psi}_1}^2 + \boldsymbol{\psi}_{1x_1} = \alpha\boldsymbol{\xi}_1, \\ L\boldsymbol{\zeta}_2 + \dfrac{1}{2}F''(S){\boldsymbol{\psi}_2}^2 + \boldsymbol{\psi}_{2x_2} = -\alpha\boldsymbol{\xi}_1, \\ L\boldsymbol{\zeta}_3 + F''(S)\boldsymbol{\psi}_1\boldsymbol{\psi}_2 + \boldsymbol{\psi}_{1x_2} + \boldsymbol{\psi}_{2x_1} = 2\alpha\boldsymbol{\xi}_2, \end{array} \right. \tag{4}$$

$$\left\{ \begin{array}{l} L\boldsymbol{\zeta}_4 + \dfrac{1}{2}F''(S){\boldsymbol{\xi}_1}^2 = 0, \\ L\boldsymbol{\zeta}_5 + \dfrac{1}{2}F''(S){\boldsymbol{\xi}_2}^2 = 0, \\ L\boldsymbol{\zeta}_6 + F''(S)\boldsymbol{\xi}_1\boldsymbol{\xi}_2 = 0, \end{array} \right. \tag{5}$$

$$\left\{ \begin{array}{l} L\boldsymbol{\zeta}_7 + F''(S)\boldsymbol{\psi}_1\boldsymbol{\xi}_1 + \boldsymbol{\xi}_{1x_1} = \beta\boldsymbol{\psi}_1 + \beta'\boldsymbol{\phi}_1, \\ L\boldsymbol{\zeta}_8 + F''(S)\boldsymbol{\psi}_2\boldsymbol{\xi}_2 + \boldsymbol{\xi}_{2x_2} = \beta\boldsymbol{\psi}_1 + \beta'\boldsymbol{\phi}_1, \\ L\boldsymbol{\zeta}_9 + F''(S)\boldsymbol{\psi}_1\boldsymbol{\xi}_2 + \boldsymbol{\xi}_{2x_1} = \beta\boldsymbol{\psi}_2 + \beta'\boldsymbol{\phi}_2, \end{array} \right. \tag{6}$$

$$\begin{cases} L\boldsymbol{\zeta}_{10} + F''(S)\boldsymbol{\psi}_2\boldsymbol{\xi}_1 + \boldsymbol{\xi}_{1x_2} = -\beta\boldsymbol{\psi}_2 - \beta'\boldsymbol{\phi}_2, \\ L\boldsymbol{\zeta}_{11} + \boldsymbol{g}_1(S) = 0, \\ L\boldsymbol{\zeta}_{12} + \boldsymbol{g}_2(S) = 0, \end{cases} \tag{7}$$

where α, β, and β' are constants satisfying the following conditions:

$$\begin{cases} \langle F''(S)\boldsymbol{\psi}_1\boldsymbol{\psi}_2 + \boldsymbol{\psi}_{1x_2} + \boldsymbol{\psi}_{2x_1} - 2\alpha\boldsymbol{\xi}_2, \boldsymbol{\xi}_2^*\rangle_{L^2} = 0, \\ \langle F''(S)\boldsymbol{\psi}_1\boldsymbol{\xi}_2 + \boldsymbol{\xi}_{2x_1} - \beta\boldsymbol{\psi}_2 - \beta'\boldsymbol{\phi}_2, \boldsymbol{\phi}_2^*\rangle_{L^2} = 0, \\ \langle F''(S)\boldsymbol{\psi}_1\boldsymbol{\xi}_2 + \boldsymbol{\xi}_{2x_1} - \beta\boldsymbol{\psi}_2 - \beta'\boldsymbol{\phi}_2, \boldsymbol{\psi}_2^*\rangle_{L^2} = 0. \end{cases} \tag{8}$$

Substituting (3) into (1) and taking the inner product with the adjoint eigenfunctions, we obtain the principal part by the following system:

$$\begin{cases} \dot{z}_0 = z_1 - \beta'\overline{z}_1 z_2, \\ \dot{z}_1 = M_1|z_1|^2 z_1 + M_2|z_2|^2 z_1 + M_3 z_1 + \beta\overline{z}_1 z_2, \\ \dot{z}_2 = N_1|z_2|^2 z_2 + N_2|z_1|^2 z_2 + N_3 z_2 + \alpha {z_1}^2. \end{cases} \tag{9}$$

Here we introduce the complex variables $z_0 = p_1 + ip_2$, $z_1 = q_1 + iq_2$, and $z_2 = s_1 + is_2$. Note that $\boldsymbol{\zeta}^\dagger$ is necessary for computations of cubic terms in (9). The constants M_i and N_i $(i = 1 \cdots 3)$ are obtained from the model system (1). The details are shown in [5].

The dynamics of (9) are essentially governed by the last two equations, exactly the same as the normal form obtained in the study of resonance patterns in a bilayer fluid under $O(2)$-symmetry operations [1]. It is natural that the relationship between drift and peanut deformations viewed from a circular shape is analogous to the 1:2 mode interactions. Letting $z_1 = Qe^{i\phi}$ and $z_2 = Se^{i\psi}$, we rewrite (9) as

$$\begin{cases} \dot{Q} = (M_1 Q^2 + M_2 S^2 + M_3)Q + \beta Q S \cos\theta, \\ \dot{S} = (N_1 S^2 + N_2 Q^2 + N_3)S + \alpha Q^2 \cos\theta, \\ \dot{\theta} = -\left(2\beta S + \dfrac{\alpha Q^2}{S}\right)\sin\theta, \end{cases} \tag{10}$$

where we set $\theta = \psi - 2\phi$. In addition to the trivial standing disk (SD) spot of $Q = S = 0$, we have the fixed points of (10) with $|\cos\theta| = 1$ as

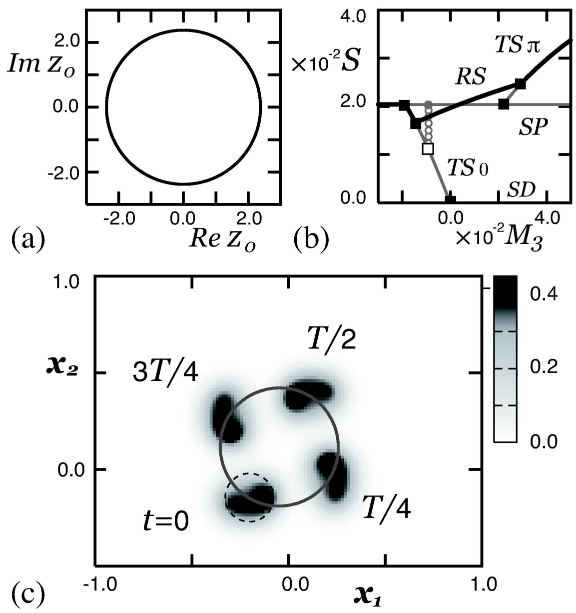

Figure 1: (a) 1:2 mode interaction in a rotational spot (RS) motion for ODE of (9). (b) Bifurcation diagram of spot solutions for the ODEs of (10), where N_3 is fixed to 0.1. Stable RS motion appears via pitchfork bifurcations and connects between the TS_0 and TS_π branches. (c) Rotational spot (RS) motion in the PDE system: A spot moves in a counterclockwise direction as observed in four superimposed snapshots. The trajectory of the the centroid of v-component distribution is depicted by the solid line.

the standing peanut (SP) spot of $Q = 0$ and $S^2 = -N_3/N_1$. Hereafter we use (M_3, N_3) as the new bifurcation parameter set.

The traveling spot solution of (11) bifurcates from the SD spot at $M_3 = 0$ and from the SP spot at $M_3 - M_2N_3/N_1 \pm \beta(-N_3/N_1)^{1/2} = 0$.

$$\begin{cases} M_1Q^2 + M_2S^2 + M_3 \pm \beta S = 0, \\ (N_1S^2 + N_2Q^2 + N_3)S \pm \alpha Q^2 = 0, \end{cases} \tag{11}$$

where the traveling spot TS_0 with $\cos\theta = 1$ (resp. TS_π with $\cos\theta = -1$) corresponds to a propagation direction parallel (resp. perpendicular) to the long axis of the deformed shape.

The solution of (11) becomes unstable when the coefficient of the angle equation of (10) is positive. That is, the following solutions of

(12) with $|\cos\theta| \neq 1$ emanate via pitchfork bifurcation,

$$\begin{cases} Q^2 = \left(-\dfrac{2\beta}{\alpha}\right) S^2 = \left(-\dfrac{2\beta}{\alpha}\right) \dfrac{N_3 + 2M_3}{K}, \\ \cos^2\theta = \dfrac{(N_3(M_2 - 2\beta M_1/\alpha) - M_3(N_1 - 2\beta N_2/\alpha))^2}{\beta^2 (N_3 + 2M_3) K}, \end{cases} \tag{12}$$

where $K = 4\beta M_1/\alpha - 2M_2 - N_1 + 2\beta N_2/\alpha$. Accordingly, we solve the slave part in (9) as $z_0 = (2/\alpha\beta)^{1/2}(\beta' S e^{i\theta_0} - 1)e^{i\beta S \sin\theta t}/\sin\theta$, where θ_0 is constant. This allows the occurrence of RS motion with radius $|z_0|^2 = 2((\beta' S)^2 - 1)/(\alpha\beta\sin^2\theta)$ for $\cos\theta_0 = (\beta' S)^{-1}$. Since the phase speed $\dot\psi = 2\dot\phi = 2\beta S\sin\theta$ becomes zero at the pitchfork bifurcation point of $|\cos\theta| = 1$, where Q and θ are continuous, clockwise and counterclockwise rotational motions with an infinite radius are equally possible to emanate from a straight motion.

As a representative model system fitting our framework, we employ the following activator-substrate-inhibitor reaction diffusion system:

$$\begin{cases} u_t = & D_u \triangle u - \dfrac{uv^2}{1 + f_2 w} + f_0(1 - u), \\ v_t = & D_v \triangle v + \dfrac{uv^2}{1 + f_2 w} - (f_0 + f_1)v, \\ \tau w_t = & D_w \triangle w + f_3(v - w). \end{cases} \tag{13}$$

As shown in Fig.1(c), by numerical simulations of (13), we find the RS motion, i.e., its trajectory of centroid of v-component distribution draws a circle. A spot maintains the shape and rotates with constant velocity. The details are shown in [5].

In summary, we have studied the spot dynamics near the drift-peanut codimension 2 singularity. Such instabilities are detected in a class of three-component reaction diffusion systems. Their PDE dynamics can be reduced to finite dimensional ODEs. Bifurcation leading to the onset of RS motion of traveling spots in two dimensions is analytically investigated in close analogy to the normal form of 1:2 resonance patterns.

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