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Full-information duration problem and its generalizations

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1 Introduction

Among the optimal stopping problems, a lot of generalizations of secretary problem have been considered. The problem dealt with generalization of duration and utility related to duration is introduced in this paper. In Section 2, we consider the simple problem where the utility is the total work done by the accepted applicant, which is related to a kind of duration. The objective is to find the stopping rule maximizing the total work. In this problem, we assume that the decision maker can observe the value (work rate) of the applicants, which is called full-information case.

In Ferguson, Hardwick and Tamaki (1993), the problem of maximizing the duration of owning the relatively best object is solved for various settings in both no-information case and full-information case. Tamaki, Pearce and Szajowski (1998) shows the optimal strategy for multiple stopping duration problem. In their problems, the duration is defined as the time period when the selected relatively best object remains to be relatively best, that is just before it becomes second-best. In other words, the decision maker is interested in the object only when it is a relatively best, and when the next relatively best arrives, the former relatively best is useless. Further generalization for no-information case is solved in Szajowski and Tamaki (2006). They call the problem shelf life problem, where the objective is to maximize the time period owning the relatively best or second-best. Our second problem introduced in Section 3 is a special case of the shelf life problem. We consider the generalization of the duration, which is defined as the time period of owning the relatively best or second-best object. However, the class of the stopping rule is restricted in stopping only at the relatively best applicant. We treat the problem in full-information case, and we have got the OLA (one-stage look-ahead) stopping rule. In order to show that the OLA stopping rule is optimal, we have to show two statements as sufficient condition. Now we could show one of the statements, the other remains unsolved, so we describe a part of the proof, and the conjecture.

2 Work Maximization Problem: Full-Information Case

Here we consider a simple utility maximization problem as a generalization of full-information secretary problem. Assume that the utility is the total work of the applicant. The work is defined as a product of value and time period.

The problem is described as follows: Fixed n applicants arrive sequentially in a random order. The decision maker (DM) has to decide whether to accept or reject the applicant after the interview. DM can observe the value of the applicant, which has uniform distribution, $U(0,1)$. The objective of DM is to maximizing the work that the accepted applicant will accomplish by the time $n+1$.

The optimal policy is solved by backward induction. Let u_i and w_i denote the value and the work of ith applicant, respectively. The total work of ith applicant is defined as the product of his/her value and time period, that is,

$$w_i(x) = u_i \times (n - i + 1). \tag{1}$$

Let V_j denote the expected duration when there are j stages to go and DM acts optimally. When X_j is observed with j stages to go, DM stops if $jX_j > V_{j-1}$.

$$V_j = E\max(jX_j, V_{j-1}) = \int_0^{V_{j-1}/V_j} V_{j-1}dx + \int_{V_{j-1}/V_j}^1 jxdx = \frac{1}{2}\left(j + \frac{V_{j-1}^2}{j}\right) \tag{2}$$

with initial condition $V_0 = 0$. If we let $A_j = V_{j-1}/j$, then we have

$$A_{j+1} = \frac{j}{j+1}\left(\frac{1+A_j^2}{2}\right) \tag{3}$$

with initial condition $A_1 = 0$.

Theorem 1 *Assume that n applicants arrive in random order. DM can observe the value of each applicant, which has uniform density between 0 and 1, $U(0,1)$. The objective is to maximize the total work of the accepted applicant. Then the optimal stopping time is given by*

$$\tau = \min\{j : X_j \geq A_j\},$$

where X_j is the value of the applicant who arrives when there remain j stages to go.

The numerical values of A_j is shown in Table 1.

Table 1: The numerical values of A_j

j	1	2	3	4	5	6	7	8	9	10
A_j	0	0.25	0.354	0.422	0.471	0.509	0.540	0.565	0.586	0.605
j	20	30	40	50	100	200	300	500	1000	5000
A_j	0.710	0.759	0.789	0.810	0.864	0.903	0.920	0.938	0.956	0.980

3 An Extension of Duration Problem

In this section, full-information case of the duration problem is generalized. The objective is to maximize the time period of owning the relatively best and the relatively second-best. Here we consider the class of the stopping rule restricted in that of only stopping at the relatively best object. The value of the applicants has the uniform distribution $U(0,1)$. Let X_i denote the value of the ith applicant.

Let $U_n(x)$ denote the expected duration of the relatively best whose rank remains within two when the time to go is n and DM accepts the relatively best applicant whose value x is the maximum value among that of the applicants arrived so far, that is $X_n = x$. $U_n(x)$ is given by

$$\begin{aligned} U_n(x) &= \sum_{k=2}^{n-1} k(k-1)(1-x^2)^2 x^{k-2} + nx^{n-1} + n(n-1)(1-x)x^{n-2} \\ &= 2\sum_{k=1}^{n-1} x^{k-1} - nx^{n-1}. \end{aligned}$$

The expected duration when DM does not accept the relatively best applicant whose value $X_n = x$ is the maximum value among that of the applicants arrived so far and accepts the next first relatively best applicant hereafter is given by

$$\sum_{k=1}^{n-1} x^{k-1} \int_x^1 U_{n-k}(y)dy = 2\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k-1} \frac{1-x^j}{j} - \sum_{k=1}^{n-1} x^{k-1} + (n-1)x^{n-1}.$$

Let $G_n(x)$ define

$$G_n(x) = U_n(x) - \sum_{k=1}^{n-1} x^{k-1} \int_x^1 U_{n-k}(y)dy,$$

then we got

$$\begin{aligned} G_n(x) &= 3\sum_{k=1}^{n} x^{k-1} - 2nx^{n-1} - 2\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} \frac{1-x^j}{j} \\ &= 3\sum_{k=1}^{n} x^{k-1} - 2nx^{n-1} - 2\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k-1} \frac{1}{j} + 2\sum_{k=1}^{n-1} x^k \sum_{j=1}^{k} \frac{1}{j}. \end{aligned}$$

Then OLA stopping region B is described as

$$B = \{(n,x) : G_n(x) \geq 0\},$$

where (n,x) is represented the state when when the time to go is n and the present applicant is the relatively best one whose value x is the maximum value among that of the applicants arrived so far, that is $X_n = x$. To show that the OLA stopping rule is optimal, it is sufficient to show the next two statements: (i) $G_n(x) \geq 0 \Longrightarrow G_{n-k}(x) \geq 0,\ k = 1,2,\ldots,$ and (ii) $G_n(x) \geq 0 \Longrightarrow G_n(y) \geq 0,\ y \geq x$. If both (i) and (ii) are shown, then it follows that $G_n(x) \geq 0 \Longrightarrow G_{n-k}(y) \geq 0$ for $k = 1,2,\ldots,,\ y \geq x$.

Here we have shown only the first statement.

Lemma 1 $G_{n+1}(x) \geq 0 \Longrightarrow G_n(x) \geq 0,\ k = 1,2,\ldots,$

Proof. $G_n(x)$ can be rewritten by

$$G_n(x) = \sum_{j=0}^{n-1} a_j^{(n)} x^j - 2nx^{n-1},$$

where

$$a_j^{(n)} = 3 - 2\sum_{l=1}^{n-j-1}\frac{1}{l} + 2\sum_{l=1}^{j}\frac{1}{l}.$$

It is easily shown that $a_j^{(n)}$ is increasing in j, and it follows that

$$a_j^{(n+1)} = a_j^{(n)} - \frac{2}{n-j}.$$

Then $G_n(x)$ has recursive expression as

$$\begin{aligned}
G_{n+1}(x) &= \sum_{j=0}^{n} a_j^{(n+1)}x^j - 2(n+1)x^n \\
&= \sum_{j=0}^{n-1}\left(a_j^{(n)} - \frac{2}{n-j}\right)x^j + a_n^{(n+1)}x^n - 2(n+1)x^n \\
&= G_n(x) + 2nx^{n-1} - \sum_{j=0}^{n-1}\frac{2}{n-j}x^j + \left(3 + \sum_{l=1}^{n}\frac{2}{l}\right)x^n - 2(n+1)x^n \\
&\equiv G_n(x) - f_n(x),
\end{aligned}$$

$G_{n+1}(x) \geq 0$ is equivalent to $G_n(x) \geq f_n(x)$. So it is sufficient to show $f_n(x) \geq 0$ for $0 < x < 1$. The proof is made by induction. Since

$$f_n(x) = \sum_{j=0}^{n-1}\frac{2}{n-j}x^j - \left(3 + \sum_{l=1}^{n}\frac{2}{l}\right)x^n + 2(n+1)x^n - 2nx^{n-1},$$

it is clear that

$$f_3(x) = \frac{4}{3}x^3 - 4x^2 + x + \frac{2}{3} \geq 0$$

for $0 < x < 1$ So when $n = 3$, $f_3(x) \geq 0$ for $0 < x < 1$.

Next, assume that $f_n(x) \geq 0$ $(0 < x < 1)$. Then

$$\begin{aligned}
f_{n+1}(x) &= xf_n(x) + 2\left\{\frac{1-x^{j+1}}{j+1} + (x^{j+1} - x^j)\right\} \\
&\geq xf_n(x) + 2\left\{x^j - x^{j+1} + (x^{j+1} - x^j)\right\} \\
&\geq xf_n(x) \\
&\geq 0.
\end{aligned}$$

Therefore, if $f_n(x) \geq 0$, then $f_{n+1}(x) \geq 0$.

By induction, we can show that $f_n(x) \geq 0$ for all $n(\geq 3)$ and $0 < x < 1$,

Finally, it follows that if $G_{n+1}(x) \geq 0$, then $G_n(x) \geq f_n(x) \geq 0$, that is,

$$G_{n+1}(x) \geq 0 \Longrightarrow G_n(x) \geq 0.$$

The proof is completed. □

It is necessary to show the second statement (ii) $G_n(x) \geq 0 \Longrightarrow G_n(y) \geq 0$, $y \geq x$ for the optimality of the OLA stopping rule. It remains unsolved, so we would like to conclude the paper with the conjecture about the optimal stopping rule.

Conjecture *For the full-information case of the duration problem where the objective is to maximize the duration of owning the relatively best or second-best, we assume that the class of stopping rule is restricted to that of stopping only at the relatively best. Then the optimal stopping rule is to accept the first applicant who has the maximum* $X_n = x \geq s_n$ *among the observed objects so far when the remaining time is* n*, where* $s_1 = 1$ *and* s_n*,* $n \geq 2$ *is the unique root of the equation*

$$3\sum_{k=1}^{n} x^{k-1} - 2nx^{n-1} - 2\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k-1} \frac{1}{j} + 2\sum_{k=1}^{n-1} x^{k} \sum_{j=1}^{k} \frac{1}{j} = 0.$$

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