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Probabilistic Interpretation Beyond Completely Monotone Capacities

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Abstract

A family H of subsets is called increasing if $B \in H$ whenever $A \subseteq B$ for some $A \in H$. The fundamental theorem of random sets, known as Choquet theorem, will be discussed in the light of random increasing family $\mathbf{H}$ of subsets. The distribution of $\mathbf{H}$ is determined by the completely monotone capacity $\varphi(A) = \mathbb{P}(A \in \mathbf{H})$ if $\mathbb{P}(A, B \in \mathbf{H}) = \mathbb{P}(A \cap B \in \mathbf{H})$ for every pair (A, B). Similarly the completely alternating capacity φ characterizes $\mathbf{H}$ for which $\mathbb{P}(A, B \in \mathbf{H}) = \mathbb{P}(A \in \mathbf{H}) + \mathbb{P}(B \in \mathbf{H}) - \mathbb{P}(A \cup B \in \mathbf{H})$. This paper presents our ongoing investigation in an effort of searching an extension of probabilistic interpretation of Choquet capacities.

1 Probabilistic interpretation of capacities

By $\mathcal{B}$ we denote the Boolean algebra of a finite set S, and introduce a natural partial ordering by the inclusion $\subseteq$. This poset $\mathcal{B}$ has the minimum element $\emptyset$ and the maximum element S, denoted respectively by $\hat{0}$ and $\hat{1}$. We call a nonnegative function φ on $\mathcal{B}$ a *capacity* if $a \subseteq b$ implies $\varphi(a) \leq \varphi(b)$ with $\varphi(\hat{0}) = 0$ and $\varphi(\hat{1}) = 1$ [i.e., $\varphi(\emptyset) = 0$ and $\varphi(S) = 1$].

Here we introduce successive difference functionals on capacities. For any $a \in \mathcal{B}$ and any sequence $b_1, b_2, \ldots$ of $\mathcal{B}$, we can define the following functionals recursively by

$$\nabla_{b_1}^{a} \varphi = \varphi(a) - \varphi(a \cap b_1)$$
$$\nabla_{b_1,\ldots,b_{n+1}}^{a} \varphi = \left(\nabla_{b_1,\ldots,b_n}^{a} - \nabla_{b_1,\ldots,b_n}^{a \cap b_{n+1}} \right) \varphi, \quad n = 1, 2, \ldots.$$

The definition above does not depend on the order of b_i's, and $\nabla_{b_1,\ldots,b_{n+1}}^{a} = \nabla_{b_1,\ldots,b_n}^{a}$ if $b_i = b_{n+1}$ for some $i \leq n$. Therefore, we can only define a distinct successive difference functional for every nonempty subset $B = \{b_1, \ldots, b_n\}$ of $\mathcal{B}$, and denote it simply by ∇_B^a. We call a capacity φ *completely monotone* if $\nabla_B^a \varphi \geq 0$ for any

$a \in \mathcal{B}$ and any nonempty subset B of $\mathcal{B}$. Let $\mathcal{B}_0 = \mathcal{B} \setminus \{\hat{0}\}$. Choquet [1] showed that if φ is completely monotone then a $\mathcal{B}_0$-valued random variable X satisfies

$$(1.1) \qquad \varphi(a) = \mathbb{P}(X \subseteq a) = \sum_{b \subseteq a} f(b) \quad \text{for } a \in \mathcal{B},$$

where $f(b) = \mathbb{P}(X = b)$ is the probability mass function of X. This representation is unique up to the probability mass function, and also suffices the property of complete monotonicity.

A subset H of $\mathcal{B}$ is called an *up-set* (or a hereditary set to the right) if $a \subseteq b$ and $a \in H$ imply $b \in H$. By $\mathcal{H}_0$ we denote the class of nonempty up-sets which do not contain the minimum element $\hat{0}$ of $\mathcal{B}$; that is, $\mathcal{H}_0$ is the class of all up-sets except for $\emptyset$ nor $\mathcal{B}$. We define the capacity χ_H by

$$\chi_H(a) = \begin{cases} 1 & \text{if } a \in H; \\ 0 & \text{if } a \notin H, \end{cases}$$

with $H \in \mathcal{H}_0$, and call it an *extreme capacity.* The entire class of capacities is a convex polytope (i.e., a bounded polyhedron) on the vector space of real-valued functions on $\mathcal{B}$, and that it consists of extreme points of the form χ_H (see [1]). Therefore, for any capacity φ we can find the representation

$$(1.2) \qquad \varphi(a) = \sum_{H \in \mathcal{H}_0} g(H)\, \chi_H(a), \quad a \in \mathcal{B}$$

where g is a probability mass function on $\mathcal{H}_0$.

Murofushi [2] pioneered the following greedy algorithm to construct the weight g in (1.2). Define a map $H(t) = \{a \in \mathcal{B} : \varphi(a) > t\}$ from $[0, 1)$ to $\mathcal{H}_0$, and observe that $H(t) = H_i$ for $t \in [r_{i-1}, r_i)$ for $i = 1, \dots, m$ with an increasing sequence $0 = r_0 < r_1 < \cdots < r_m = 1$ and a decreasing sequence $H_1 \supset \cdots \supset H_m$. Then set $g(H_i) = r_i - r_{i-1}$ for $i = 1, \dots, m$, and $g(K) = 0$ for every other $K \in \mathcal{H}_0$. It is easily checked that g satisfies (1.2).

We introduce an $\mathcal{H}_0$-valued random variable $\mathbf{H}$, and call it a *random up-set.* We can restate (1.2) by

$$(1.3) \qquad \varphi(a) = \mathbb{P}(a \in \mathbf{H}), \quad a \in \mathcal{B}$$

where $\mathbf{H}$ has the probability mass function $g(K) = \mathbb{P}(\mathbf{H} = K)$. However, as the following example shows, such a probabilistic interpretation is no longer unique in general.

Example 1.1. Let $\mathcal{B} = \{\hat{0}, \alpha, \beta, \gamma, \alpha\beta, \alpha\gamma, \beta\gamma, \hat{1}\}$ be the Boolean algebra with $\hat{0} = \emptyset$ and $\hat{1} = \alpha\beta\gamma$, and let φ be a capacity on $\mathcal{B}$ with $\varphi(\alpha\beta) = \varphi(\alpha\gamma) = \varphi(\beta\gamma) = \frac{1}{3}$ and $\varphi(\alpha) = \varphi(\beta) = \varphi(\gamma) = 0$; here we write $\alpha\beta$ for a subset $\{\alpha, \beta\}$. Then φ is completely monotone, and (1.1) holds for $f(\alpha\beta) = f(\alpha\gamma) = f(\beta\gamma) = \frac{1}{3}$. Let

$$\langle B \rangle = \{a \in \mathcal{B} : b \subseteq a \text{ for some } b \in B\}$$

be the up-set generated by a subset B of $\mathcal{B}$. The probability mass function g on $\mathcal{H}_0$ with $g(\langle\alpha\beta\rangle) = g(\langle\alpha\gamma\rangle) = g(\langle\beta\gamma\rangle) = \frac{1}{3}$ satisfies (1.3). On the other hand, the Murofushi's greedy algorithm determines $g(\langle\hat{1}\rangle) = \frac{1}{3}$ and $g(\langle\{\alpha\beta, \alpha\gamma, \beta\gamma\}\rangle) = \frac{2}{3}$ for which (1.3) also holds.

2 Möbius functions and successive differences

Let $\mathcal{P}$ be a finite poset. The Möbius function $\mu_{\mathcal{P}}$ is the unique function defined for every pair (a, b) such that $a \le b$, and satisfies $\mu_{\mathcal{P}}(a, a) = 1$ and $\sum_{b \le a} \mu_{\mathcal{P}}(b, a) = 0$ for each $a \in P$. Then we have

$$f(a) = \sum_{b \le a} \varphi(b)\mu_{\mathcal{P}}(b, a) \tag{2.4}$$

if and only if $\varphi(a) = \sum_{b \le a} f(b)$. Here f is called the Möbius inversion of φ.

Example 2.1. The results here and other arguments of this section are either taken from or inspired by Stanley [3]. Let $\mathcal{B}$ be a Boolean algebra of a finite set S. For $a \subseteq b$ we obtain the Möbius function $\mu_{\mathcal{B}}(a, b) = (-1)^{|b \setminus a|}$, where $|b \setminus a|$ denotes the number of elements in the set difference $b \setminus a$.

Let $a \in \mathcal{B}$ and a nonempty subset $B \subseteq \mathcal{B}$ be fixed. Then we can introduce the subposet

$$\mathcal{P}_B^a = \{\cap B' \cap a : B' \subseteq B\}$$

with the partial order $\le$ by inclusion, where

$$\cap B' = \begin{cases} \cap_{b \in B'} b & \text{if } B' \ne \emptyset; \\ \hat{1} & \text{if } B' = \emptyset. \end{cases}$$

In what follows we simply write $\mathcal{P}$ for $\mathcal{P}_B^a$ if there is no confusion with other posets in discussion. It is not difficult to see (e.g., Stanley [3]) that

$$\sum_{b \le a} \varphi(b)\mu_{\mathcal{P}}(b, a) = \sum_{B' \subseteq B} (-1)^{|B'|}\varphi(\cap B' \cap a) = \nabla_B^a \varphi$$

Here the element a is the maximum of $\mathcal{P}$.

Let φ be a capacity, and let f be the Möbius inversion of φ. We can argue the following interesting connection to the Möbius inversion from the successive difference ∇_B^a (or equivalently from the Möbius function $\mu_{\mathcal{P}}$). For $b \in \mathcal{P}$ we set $F(b)$ to be the summation of $f(x)$ over all x's satisfying $x \subseteq b$ and $x \not\subseteq b'$ for all $b' < b$ in $\mathcal{P}$. Then we have

$$\sum_{b' \le b} F(b') = \sum_{x \subseteq b} f(x) = \varphi(b)$$

and therefore,

$$F(a) = \sum_{b \le a} \varphi(b)\mu_{\mathcal{P}}(b, a) = \nabla_B^a \varphi \tag{2.5}$$

As a special case of (2.5) we obtain $f(a) = \nabla_B^a \varphi$ when $a = \{\alpha_1, \ldots, \alpha_k\}$ and $B = \{S \setminus \{\alpha_1\}, \ldots, S \setminus \{\alpha_k\}\}$. Hence we have shown the result of Choquet.

Proposition 2.2. *A capacity φ is completely monotone if and only if the Möbius inversion f is nonnegative.*

The Möbius inversion f of a capacity φ satisfies $f(\hat{0}) = 0$ and $\sum_{a \subseteq \hat{1}} f(a) = 1$, and it can be viewed as a probability mass function on $\mathcal{B}_0$ when f is nonnegative.

For any capacity φ we can define the dual capacity φ^* by $\varphi^*(a) = 1 - \varphi(\hat{1} \setminus a)$ for $a \in \mathcal{B}$. Then we can introduce the functional Δ_a^B by

$$\Delta_a^B \varphi = -\nabla_{B^*}^{\hat{1}\setminus a} \varphi^* \tag{2.6}$$

where $B^* = \{\hat{1} \setminus b : b \in B\}$. We can formulate it recursively by

$$\Delta_a^{b_1} \varphi = \varphi(a) - \varphi(a \cup b_1)$$
$$\Delta_a^{b_1,\ldots,b_{n+1}} \varphi = \left(\Delta_a^{b_1,\ldots,b_n} - \Delta_{a \cup b_{n+1}}^{b_1,\ldots,b_n}\right) \varphi, \quad n = 1, 2, \ldots.$$

We call φ *completely alternating* if $\Delta_a^B \varphi \le 0$ for any $a \in \mathcal{B}$ and any nonempty subset $B \subseteq \mathcal{B}$. It is clear from (2.6) that φ is completely alternating if and only if φ^* is completely monotone. By Proposition 2.2 we can see that if φ is completely alternating then a $\mathcal{B}_0$-valued random variable X satisfies

$$\varphi(a) = \mathbb{P}(X \cap a \ne \hat{0}) = \sum_{b \cap a \ne \hat{0}} f^*(b) \tag{2.7}$$

where $f^*(b) = \mathbb{P}(X = b)$ is the Möbius inversion of φ^*.

3 An extension of probabilistic interpretation

We introduce a partial order $\preceq$ on the class $\mathcal{H}$ of nonempty up-sets H as follows: For $K, H \in \mathcal{H}$ we define $K \preceq H$ if and only if $K \supseteq H$. Recall that a subset A of $\mathcal{B}$ is an antichain if none of pairs of A are comparable. We can obtain the Möbius function on $\mathcal{H}$ as follows; see Stanley [3]. For $K \preceq H$ we have

$$\mu_{\mathcal{H}}(K, H) = \begin{cases} (-1)^{|K \setminus H|} & \text{if } K \setminus H \text{ is an antichain;} \\ 0 & \text{otherwise.} \end{cases}$$

Let φ be a capacity. A nonempty up-set H is called *feasible* with respect to φ if $\Delta_{b \cap b'}^{\{b,b'\}} \varphi < 0$ for every pair (b, b') of H satisfying $b \cap b' \notin H$.

Definition 3.1. If a nonempty up-set H is not feasible, we can find a feasible up-set $K \preceq H$ greedily. First, set $K = H$, and repeat the following steps until K becomes feasible.

1. Choose a pair (b, b') of K such that $b \cap b' \notin K$ and $\Delta_{b\cap b'}^{\{b,b'\}}\varphi \geq 0$.
2. Set $K = \langle b \cap b' \rangle \cup K$. Here $\langle b \rangle = \{a \in \mathcal{B} : b \subseteq a\}$ is the up-set generated by $b \in \mathcal{B}$. Thus, $\langle b \cap b' \rangle \cup K$ is also an up-set.

Lemma 3.2. *For any nonempty up-set H Definition 3.1 generates the maximum feasible up-set K (in the poset $\mathcal{H}$) satisfying $K \preceq H$.*

Let H° denote the antichain of all the minimal elements of the unique feasible up-set $K \preceq H$ generated via Definition 3.1. For $H \in \mathcal{H}$ we define $\Phi(H) = -\Delta_{\hat{0}}^{H^\circ}\varphi$, and introduce the Möbius inversion g of Φ by

$$g(H) = \sum_{K \preceq H} \Phi(K)\mu_{\mathcal{H}}(K, H) \tag{3.8}$$

If g is nonnegative then the probability mass function $\mathbb{P}(\mathbf{H} = K) = g(K)$ on $\mathcal{H}_0$ satisfies

$$\mathbb{P}(a, b \in \mathbf{H}) = \varphi(a \cap b) - [\Delta_{a\cap b}^{a,b}\varphi]_- \quad \text{for } a, b \in \mathcal{B}_0, \tag{3.9}$$

where $[x]_- = \min\{x, 0\}$.

Theorem 3.3. *If an $\mathcal{H}_0$-valued random up-set* $\mathbf{H}$ *satisfies (3.9) with some capacity φ then the Möbius inversion g in (3.8) is nonnegative and uniquely determines the probability mass function of* $\mathbf{H}$.

If φ is completely monotone and satisfies (1.1) with some $\mathcal{B}_0$-valued random variable, then $\mathbf{H} = \langle X \rangle$ satisfies (3.9). If φ is completely alternating and satisfies (2.7) then $\mathbf{H} = \{a \in \mathcal{B} : X \cap a \neq \emptyset\}$ satisfies (3.9). Theorem 3.3 indicates that the random up-set $\mathbf{H}$ can be directly constructed via (3.8).

Theorem 3.3 also implies that there is a class of capacities capable of characterizing $\mathbf{H}$ uniquely in a way to extend Choquet theorem. Due to the nature of our research in progress we omit the proofs for Lemma 3.2 and Theorem 3.3.

References

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