Probabilistic Interpretation Beyond Completely Monotone Capacities

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Abstract

A family $H$ of subsets is called increasing if $B \in H$ whenever $A \subseteq B$ for some $A \in H$. The fundamental theorem of random sets, known as Choquet theorem, will be discussed in the light of random increasing family $H$ of subsets. The distribution of $H$ is determined by the completely monotone capacity $\varphi(A) = \mathbb{P}(A \in H)$ if $\mathbb{P}(A, B \in H) = \mathbb{P}(A \cap B \in H)$ for every pair $(A, B)$. Similarly the completely alternating capacity $\varphi$ characterizes $H$ for which $\mathbb{P}(A, B \in H) = \mathbb{P}(A \in H) + \mathbb{P}(B \in H) - \mathbb{P}(A \cup B \in H)$. This paper presents our ongoing investigation in an effort of searching an extension of probabilistic interpretation of Choquet capacities.

1 Probabilistic interpretation of capacities

By $\mathcal{B}$ we denote the Boolean algebra of a finite set $S$, and introduce a natural partial ordering by the inclusion $\subseteq$. This poset $\mathcal{B}$ has the minimum element $\emptyset$ and the maximum element $S$, denoted respectively by $\hat{0}$ and $\hat{1}$. We call a nonnegative function $\varphi$ on $\mathcal{B}$ a capacity if $a \subseteq b$ implies $\varphi(a) \leq \varphi(b)$ with $\varphi(\emptyset) = 0$ and $\varphi(\hat{1}) = 1$ [i.e., $\varphi(\emptyset) = 0$ and $\varphi(S) = 1$].

Here we introduce successive difference functionals on capacities. For any $a \in \mathcal{B}$ and any sequence $b_1, b_2, \ldots$ of $\mathcal{B}$, we can define the following functionals recursively by

$$
\nabla_{b_1}^a \varphi = \varphi(a) - \varphi(a \cap b_1)
$$

$$
\nabla_{b_1, \ldots, b_{n+1}}^a \varphi = \left( \nabla_{b_1, \ldots, b_n}^a - \nabla_{b_1, \ldots, b_{n+1}}^a \right) \varphi, \quad n = 1, 2, \ldots
$$

The definition above does not depend on the order of $b_i$’s, and $\nabla_{b_1, \ldots, b_{n+1}}^a = \nabla_{b_1, \ldots, b_n}^a$ if $b_i = b_{n+1}$ for some $i \leq n$. Therefore, we can only define a distinct successive difference functional for every nonempty subset $B = \{b_1, \ldots, b_n\}$ of $\mathcal{B}$, and denote it simply by $\nabla_B^a$. We call a capacity $\varphi$ completely monotone if $\nabla_B^a \varphi \geq 0$ for any
$a \in \mathcal{B}$ and any nonempty subset $B$ of $\mathcal{B}$. Let $\mathcal{B}_0 = \mathcal{B} \setminus \{\emptyset\}$. Choquet [1] showed that if $\varphi$ is completely monotone then a $\mathcal{B}_0$-valued random variable $X$ satisfies

$$(1.1) \quad \varphi(a) = \mathbb{P}(X \subseteq a) = \sum_{b \subseteq a} f(b) \quad \text{for } a \in \mathcal{B},$$

where $f(b) = \mathbb{P}(X = b)$ is the probability mass function of $X$. This representation is unique up to the probability mass function, and also suffices the property of complete monotonicity.

A subset $H$ of $\mathcal{B}$ is called an up-set (or a hereditary set to the right) if $a \subseteq b$ and $a \in H$ imply $b \in H$. By $\mathcal{H}_0$ we denote the class of nonempty up-sets which do not contain the minimum element $\emptyset$ of $\mathcal{B}$; that is, $\mathcal{H}_0$ is the class of all up-sets except for $\emptyset$ nor $\mathcal{B}$. We define the capacity $\chi_H$ by

$$\chi_H(a) = \begin{cases} 1 & \text{if } a \in H; \\ 0 & \text{if } a \notin H, \end{cases}$$

with $H \in \mathcal{H}_0$, and call it an extreme capacity. The entire class of capacities is a convex polytope (i.e., a bounded polyhedron) on the vector space of real-valued functions on $\mathcal{B}$, and that it consists of extreme points of the form $\chi_H$ (see [1]). Therefore, for any capacity $\varphi$ we can find the representation

$$(1.2) \quad \varphi(a) = \sum_{H \in \mathcal{H}_0} g(H) \chi_H(a), \quad a \in \mathcal{B}$$

where $g$ is a probability mass function on $\mathcal{H}_0$.

Murofushi [2] pioneered the following greedy algorithm to construct the weight $g$ in (1.2). Define a map $H(t) = \{a \in \mathcal{B} : \varphi(a) > t\}$ from $[0, 1)$ to $\mathcal{H}_0$, and observe that $H(t) = H_i$ for $t \in [r_{i-1}, r_i)$ for $i = 1, \ldots, m$ with an increasing sequence $0 = r_0 < r_1 < \cdots < r_m = 1$ and a decreasing sequence $H_1 \supset \cdots \supset H_m$. Then set $g(H_i) = r_i - r_{i-1}$ for $i = 1, \ldots, m$, and $g(K) = 0$ for every other $K \in \mathcal{H}_0$. It is easily checked that $g$ satisfies (1.2).

We introduce an $\mathcal{H}_0$-valued random variable $\mathbf{H}$, and call it a random up-set. We can restate (1.2) by

$$(1.3) \quad \varphi(a) = \mathbb{P}(a \in \mathbf{H}), \quad a \in \mathcal{B}$$

where $\mathbf{H}$ has the probability mass function $g(K) = \mathbb{P}(\mathbf{H} = K)$. However, as the following example shows, such a probabilistic interpretation is no longer unique in general.

Example 1.1. Let $\mathcal{B} = \{\emptyset, \alpha, \beta, \gamma, \alpha\beta, \alpha\gamma, \beta\gamma, \hat{1}\}$ be the Boolean algebra with $\emptyset = 0$ and $\hat{1} = \alpha\beta\gamma$, and let $\varphi$ be a capacity on $\mathcal{B}$ with $\varphi(\alpha\beta) = \varphi(\alpha\gamma) = \varphi(\beta\gamma) = \frac{1}{3}$ and $\varphi(\alpha) = \varphi(\beta) = \varphi(\gamma) = 0$; here we write $\alpha\beta$ for a subset $\{\alpha, \beta\}$. Then $\varphi$ is completely monotone, and (1.1) holds for $f(\alpha\beta) = f(\alpha\gamma) = f(\beta\gamma) = \frac{1}{3}$. Let $\langle \mathcal{B} \rangle = \{a \in \mathcal{B} : b \subseteq a \text{ for some } b \in \mathcal{B} \}$.
be the up-set generated by a subset $B$ of $\mathcal{B}$. The probability mass function $g$ on $\mathcal{H}_0$ with $g(\langle \alpha\beta \rangle) = g(\langle \alpha\gamma \rangle) = g(\langle \beta\gamma \rangle) = \frac{1}{3}$ satisfies (1.3). On the other hand, the Murofushi's greedy algorithm determines $g(\langle \hat{1} \rangle) = \frac{1}{3}$ and $g(\{\langle \alpha\beta, \alpha\gamma, \beta\gamma \rangle\}) = \frac{2}{3}$ for which (1.3) also holds.

### 2 Möbius functions and successive differences

Let $\mathcal{P}$ be a finite poset. The Möbius function $\mu_{\mathcal{P}}$ is the unique function defined for every pair $(a, b)$ such that $a \leq b$, and satisfies $\mu_{\mathcal{P}}(a, a) = 1$ and $\sum_{b \leq a} \mu_{\mathcal{P}}(b, a) = 0$ for each $a \in \mathcal{P}$. Then we have

$$(2.4) \quad f(a) = \sum_{b \leq a} \varphi(b) \mu_{\mathcal{P}}(b, a)$$

if and only if $\varphi(a) = \sum_{b \leq a} f(b)$. Here $f$ is called the Möbius inversion of $\varphi$.

**Example 2.1.** The results here and other arguments of this section are either taken from or inspired by Stanley [3]. Let $\mathcal{B}$ be a Boolean algebra of a finite set $S$. For $a \subseteq b$ we obtain the Möbius function $\mu_{\mathcal{B}}(a, b) = (-1)^{|b \setminus a|}$, where $|b \setminus a|$ denotes the number of elements in the set difference $b \setminus a$.

Let $a \in \mathcal{B}$ and a nonempty subset $B \subseteq \mathcal{B}$ be fixed. Then we can introduce the subposet

$${\mathcal{P}}_B^a = \{ \cap B' \cap a : B' \subseteq B \}$$

with the partial order $\leq$ by inclusion, where

$$\cap B' = \begin{cases} \cap_{b \in B'} b & \text{if } B' \neq \emptyset; \\ 1 & \text{if } B' = \emptyset. \end{cases}$$

In what follows we simply write $\mathcal{P}$ for $\mathcal{P}_B^a$ if there is no confusion with other posets in discussion. It is not difficult to see (e.g., Stanley [3]) that

$$\sum_{b \leq a} \varphi(b) \mu_{\mathcal{P}}(b, a) = \sum_{B' \subseteq B} (-1)^{|B'|} \varphi(\cap B' \cap a) = \nabla_B^a \varphi$$

Here the element $a$ is the maximum of $\mathcal{P}$.

Let $\varphi$ be a capacity, and let $f$ be the Möbius inversion of $\varphi$. We can argue the following interesting connection to the Möbius inversion from the successive difference $\nabla_B^a$ (or equivalently from the Möbius function $\mu_{\mathcal{P}}$). For $b \in \mathcal{P}$ we set $F(b)$ to be the summation of $f(x)$ over all $x$'s satisfying $x \subseteq b$ and $x \not\subset b'$ for all $b' < b$ in $\mathcal{P}$. Then we have

$$\sum_{b' \leq b} F(b') = \sum_{x \subseteq b} f(x) = \varphi(b)$$
and therefore,

\[(2.5) \quad F(a) = \sum_{b \leq a} \varphi(b) \mu_{\mathcal{P}}(b, a) = \nabla_B^a \varphi \]

As a special case of (2.5) we obtain \( f(a) = \nabla_B^a \varphi \) when \( a = \{\alpha_1, \ldots, \alpha_k\} \) and \( B = \{S \setminus \{\alpha_1\}, \ldots, S \setminus \{\alpha_k\}\} \). Hence we have shown the result of Choquet.

**Proposition 2.2.** A capacity \( \varphi \) is completely monotone if and only if the Möbius inversion \( f \) is nonnegative.

The Möbius inversion \( f \) of a capacity \( \varphi \) satisfies \( f(\emptyset) = 0 \) and \( \sum_{a \subseteq \emptyset} f(a) = 1 \), and it can be viewed as a probability mass function on \( \mathcal{B}_0 \) when \( f \) is nonnegative.

For any capacity \( \varphi \) we can define the dual capacity \( \varphi^* \) by \( \varphi^*(a) = 1 - \varphi(\hat{1} \setminus a) \) for \( a \in B \). Then we can introduce the functional \( \Delta_a^B \) by

\[(2.6) \quad \Delta_a^B \varphi = -\nabla_{B^*}^{\hat{1}\setminus a} \varphi^* \]

where \( B^* = \{\hat{1} \setminus b : b \in B\} \). We can formulate it recursively by

\[\Delta_a^{b_1} \varphi = \varphi(a) - \varphi(a \cup b_1) \]

\[\Delta_a^{b_1, \ldots, b_{n+1}} \varphi = \left(\Delta_a^{b_1, \ldots, b_n} - \Delta_a^{b_1, \ldots, b_n} \right) \varphi, \quad n = 1, 2, \ldots.\]

We call \( \varphi \) completely alternating if \( \Delta_a^B \varphi \leq 0 \) for any \( a \in B \) and any nonempty subset \( B \subseteq B \). It is clear from (2.6) that \( \varphi \) is completely alternating if and only if \( \varphi^* \) is completely monotone. By Proposition 2.2 we can see that if \( \varphi \) is completely alternating then a \( \mathcal{B}_0 \)-valued random variable \( X \) satisfies

\[(2.7) \quad \varphi(a) = \mathbb{P}(X \cap a \neq \hat{0}) = \sum_{b \cap a \neq \hat{0}} f^*(b) \]

where \( f^*(b) = \mathbb{P}(X = b) \) is the Möbius inversion of \( \varphi^* \).

## 3 An extension of probabilistic interpretation

We introduce a partial order \( \preceq \) on the class \( \mathcal{H} \) of nonempty up-sets \( H \) as follows:

For \( K, H \in \mathcal{H} \) we define \( K \preceq H \) if and only if \( K \supseteq H \). Recall that a subset \( A \) of \( B \) is an antichain if none of pairs of \( A \) are comparable. We can obtain the Möbius function on \( \mathcal{H} \) as follows; see Stanley [3]. For \( K \preceq H \) we have

\[\mu_{\mathcal{H}}(K, H) = \begin{cases} (-1)^{|K \setminus H|} & \text{if } K \setminus H \text{ is an antichain;} \\ 0 & \text{otherwise.} \end{cases}\]

Let \( \varphi \) be a capacity. A nonempty up-set \( H \) is called feasible with respect to \( \varphi \) if \( \Delta_a^{(b, b')} \varphi < 0 \) for every pair \( (b, b') \) of \( H \) satisfying \( b \cap b' \notin H \).
Definition 3.1. If a nonempty up-set \( H \) is not feasible, we can find a feasible up-set \( K \preceq H \) greedily. First, set \( K = H \), and repeat the following steps until \( K \) becomes feasible.

1. Choose a pair \((b, b')\) of \( K \) such that \( b \cap b' \notin K \) and \( \Delta_{b \cap b}^{(b, b')} \varphi \geq 0 \).
2. Set \( K = (b \cap b') \cup K \). Here \((b) = \{a \in B : b \subseteq a\}\) is the up-set generated by \( b \in B \). Thus, \((b \cap b') \cup K \) is also an up-set.

Lemma 3.2. For any nonempty up-set \( H \) Definition 3.1 generates the maximum feasible up-set \( K \preceq H \) (in the poset \( \mathcal{H} \)) satisfying \( K \preceq H \).

Let \( H^\circ \) denote the antichain of all the minimal elements of the unique feasible up-set \( K \preceq H \) generated via Definition 3.1. For \( H \in \mathcal{H} \) we define \( \Phi(H) = -\Delta_{\hat{0}}^{H^\circ} \varphi \), and introduce the Möbius inversion \( g \) of \( \Phi \) by

\[
(3.8) \quad g(H) = \sum_{K \preceq H} \Phi(K) \mu_{\mathcal{H}}(K, H)
\]

If \( g \) is nonnegative then the probability mass function \( \mathbb{P}(H = K) = g(K) \) on \( \mathcal{H}_0 \) satisfies

\[
(3.9) \quad \mathbb{P}(a, b \in H) = \varphi(a \cap b) - [\Delta_{a \cap b}^{a, b} \varphi]_\leq \quad \text{for } a, b \in B_0,
\]

where \([x]_\leq = \min\{x, 0\}\).

Theorem 3.3. If an \( \mathcal{H}_0 \)-valued random up-set \( H \) satisfies (3.9) with some capacity \( \varphi \) then the Möbius inversion \( g \) in (3.8) is nonnegative and uniquely determines the probability mass function of \( H \).

If \( \varphi \) is completely monotone and satisfies (1.1) with some \( B_0 \)-valued random variable, then \( H = \langle X \rangle \) satisfies (3.9). If \( \varphi \) is completely alternating and satisfies (2.7) then \( H = \{a \in B : X \cap a \neq \emptyset\} \) satisfies (3.9). Theorem 3.3 indicates that the random up-set \( H \) can be directly constructed via (3.8).

Theorem 3.3 also implies that there is a class of capacities capable of characterizing \( H \) uniquely in a way to extend Choquet theorem. Due to the nature of our research in progress we omit the proofs for Lemma 3.2 and Theorem 3.3.

References

