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Generalized Variational Relation Problems With Applications

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Abstract. In this paper, we first obtain an existence theorem of the solutions for a variational relation problem. An existence theorem for a variational inclusion problem, a KKM theorem will be established as particular cases. Some applications concerning a saddle point problem with constraints, existence of a common fixed point for two mappings and an optimization problem with constraints, will be given in the last section of the paper.

1 Introduction and preliminaries

If X and Y are topological spaces, a multivalued mapping (or simply, a mapping) $T : X \multimap Y$ is said to be: (i) *upper semicontinuous* (in short, usc) (respectively, *lower semicontinuous* (in short, lsc)) if for every closed subset B of Y the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; (ii) *continuous* if it is usc and lsc; (iii) *closed* if its graph (that is, the set $GrT = \{(x,y) \in X \times Y : \ y \in T(x), \ x \in X \ \}$) is a closed subset of $X \times Y$; (iv) *compact* if $T(X)$ is contained in a compact subset of Y.

For a mapping $T : X \multimap Y$ and $y \in Y$, the set $T^{-}(y) = \{x \in X : y \in T(x)\}$ (respectively, $T^{*}(y) = \{x \in X : y \notin T(x)\}$) is called the fiber (respectively, the cofiber) of T on y.

Let X be a nonempty convex subset of a real locally convex Hausdorff topological vector space, $T : X \multimap X$, $Q : X \multimap X$ be multivalued mappings and $R(x,y)$ be a relation linking $x \in X$ and $y \in X$. In this paper, we study the following variational relation problem:

(**VR**) Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $R(\bar{x}, y)$ holds for all $y \in Q(\bar{x})$.

Such problems are called variational relation problems and have been studied for the first time by Luc [1] and Khanh and Luc [2] and Lin et al. [3, 4]. The relation R is often determined by equalities and inequalities of real functions or by inclusion and intersection of multivalued mappings. Typical examples of variational relation problems are the following problems:

(i) Variational inclusion problem:

Let Z be a vector space. Given a multivalued mapping $F : X \times X \multimap Z$, the variational relation R is defined as follows

$$R(x, y) \text{ holds iff } 0 \in F(x, y).$$

Then (VR) becomes

Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, and $0 \in F(\bar{x}, y)$ for all $y \in Q(\bar{x})$.

This is a variational inclusion problem studied in [5-7] which generalizes several models of [8].

(ii) Equilibrium problems:

Let Z be a topological vector space and $F : X \times X \multimap Z$, $C : X \multimap Z$. The variarional relation R is defined as

$$R(x, y) \text{ holds iff } F(x, y)\rho C(x),$$

where $F(x, y)\rho C(x)$ represents one of the following relations $F(x, y) \cap C(x) \neq \emptyset$, $F(x, y) \subseteq C(x)$, $F(x, y) \cap \mathrm{int}(-C(x)) \neq \emptyset$, $F(x, y) \subseteq Z \setminus -(\mathrm{int}(C(x)))$. Then (VR) becomes

Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, and $F(\bar{x}, y)\rho C(x)$ for all $y \in Q(\bar{x})$.

This is a typical generalized vector set-valued equilibrium problem studied in [8-11].

(iii) Differential inclusion problem:

Let $C[0,1]$ be the space of continuous functions on the interval $[0,1]$, and $C^1[0,1]$ be the space of continuous differentiable functions on the interval $[0,1]$. Let $X \subseteq C^1[0,1]$ be a nonempty compact convex set and $F : X \times X \multimap C[0,1]$. We define a relation R as follows:

$$R(x,y) \text{ holds iff } \frac{dx}{dt} \in F(x,y).$$

Then (VR) is formulated as follows:

Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, and $\frac{d\bar{x}}{dt} \in F(\bar{x},y)$ for all $y \in Q(\bar{x})$.

This is a differential inclusion problem studied in [9] and many other papers.

(iv) Ekeland's variational principle:

Given a nonempty compact convex subset X of a Banach space, and a function $f : X \to \mathbb{R}$, we define a relation R as follows:

$$R(x,y) \text{ holds iff } f(y) + ||x-y|| \geq f(x).$$

Then (VR) is formulated as follows:

Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, and $f(y) + ||\bar{x} - y|| \geq f(\bar{x})$ for all $y \in Q(\bar{x})$.

(v) Optimization problem:

Given a nonempty convex subset of a real locally convex Hausdorff topological vector space and a function $f : X \to \mathbb{R}$, we define a relation R as follows:

$$R(x,y) \text{ holds iff } f(y) \geq f(x).$$

Then (VR) is formulated as follows:

Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, and $f(y) \geq f(\bar{x})$ for all $y \in Q(\bar{x})$.

This problem is known as constrained extreme problem.

Motivated by the previous considerations, an existence theorem for problem (VR), obtained in the next section, will be our main result. An existence theorem for a variational inclusion problem, a KKM theorem and several equilibrium theorems will be established as particular cases.

Some applications concerning a saddle point problem with constraints, existence of a common fixed point for two mappings and an optimization problem with constraints, will be given in the last section of the paper.

2 Main result

In order to establish the main result, we need the following two lemmas:

Lemma 2.1. [9-10] Let X be a topological space, Y be a topological vector space and $S, T : X \multimap Y$ be two mappings. If S is usc with nonempty compact values and T is closed, then $S + T$ is a closed mapping.

Lemma 2.2. Let X be a topological space and Y be a Hausdorff topological vector space. If $f : X \to \mathbb{R}$ is a continuous function and $T : X \multimap Y$ a compact closed mapping, then the mapping $fT : X \multimap Y$ defined by $(fT)(x) = f(x)T(x)$ is closed.

Definition 2.1. [12] For a subset K of a vector space E and $x \in E$, the *outward set* of K at x is denoted and defined as follows:

$\mathbf{O}(K; x) = \bigcup_{\lambda \geq 1}(\lambda x + (1 - \lambda)K)$.

Definition 2.2. Let X be a convex subset of a topological vector space E. $F : X \multimap E$ is said to be a KKM mapping w.r.t. itself if $F(\mathrm{co}(A)) \subseteq \cup_{x \in A} F(x)$ for each finite subset A of X.

Theorem 2.1. Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E, $T : X \multimap X$, $Q : X \multimap X$, be multivalued mappings and R be a variational relation linking points of X, satisfying the following conditions:

(i) T is usc with nonempty compact convex values;

(ii) Q is nonempty convex valued;

(iii) for each $x \in X$, the set $\{y \in X : R(x, y)$ does not hold $\}$ is convex;

(iv) for each $y \in X$, the set $Q^-(y) \cap \{x \in X : R(x, y)$ does not hold $\}$ is open in X;

(v) for each $x \in X$ and $y \in \mathbf{O}(T(x); x) \cap Q(x)$, $R(x, y)$ holds.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $R(\bar{x}, y)$ holds for all $y \in Q(\bar{x})$.

Denote by S_R the set of all $\bar{x} \in X$ satisfying the conclusion of Theorem 2.1.

Proposition 2.1. If condition (iv) in Theorem 2.1 is replaced by the following condition:

(iv') Q has open fibers and the set $\{(x, y) \in X \times X : R(x, y)$ holds$\}$ is closed in $X \times X$.

Then S_R is nonempty and compact.

Theorem 2.2. Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E, Z be a vector space and $T : X \multimap X$, $Q : X \multimap X$ and $F : X \times X \multimap Z$ be multivalued mappings satisfying conditions (i) and (ii) in Theorem 2.1 and:

(iii') for each $x \in X$, the set $\{y \in X : 0 \notin F(x, y)\}$ is convex;

(iv') for each $y \in X$, the set $Q^-(y) \cap \{x \in X : 0 \notin F(x, y)\}$ is open in X;

(v') for each $x \in X$ and $y \in \mathbf{O}(T(x); x) \cap Q(x)$, $0 \in F(x, y)$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $0 \in F(\bar{x}, y)$ for all $y \in Q(\bar{x})$.

Remark 2.1. Theorem 2.1 is different from any result in [8-10]. It is not a generalization of any result in [8-10]. The proof of Theorem 2.1 is also different from any results in [8-10].

As a simple consequence of Theorem 2.2, we have the following KKM theorem and minimax element theorem.

Theorem 2.3. Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E, $T : X \multimap X$ and $G : X \multimap X$ be multivalued mappings satisfying the following conditions:

(i) T is an u.s.c. multivalued map with nonempty closed convex values;

(ii) G is a KKM mapping w.r.t. itself;

(iii) for each $y \in X$, $G(y)$ is closed;

(iv) for each $x \in X$ and $y \in \mathbf{O}\big(T(x); x\big)$, $x \in G(y)$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x}) \cap [\bigcap_{y \in X} G(y)]$.

Theorem 2.4. Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E and $T, Q, H : X \multimap X$ be multivalued mappings satisfying the following conditions:

(i) T is an usc multivalued map with nonempty closed convex values;

(ii) for each $y \in X$, $Q^{-}(y) \cap H^{-}(y)$ is open in X;

(iii) for each $x \in X$, $H(x)$ and $Q(x)$ are convex;

(iv) for each $x \in X$ and $y \in \mathbf{O}\big(T(x); x\big) \cap Q(x)$, $y \notin H(x)$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, and $H(\bar{x}) \cap Q(\bar{x}) = \emptyset$.

Theorem 2.5. The Kakutani-Fan-Glicksberg fixed point theorems and Theorems 2.1, 2.2, 2.3 and 2.4 are equivalent.

3 Applications

The following lemma is a part of Berge's maximum theorem [13].

Lemma 3.1. Let X and Y be topological spaces, X be aompact, $S : Y \multimap X$ be a continuous mapping with nonempty compact values, and $\varphi : X \times Y \to \mathbb{R}$ be a continuous function. Then the mapping $T : Y \multimap X$ defined by

$$T(y) = \{x \in S(y) : \varphi(x, y) = max_{x' \in S(y)} \varphi(x', y)\}$$

is u.s.c. with nonempty compact values.

Theorem 3.1. Let X be a nonempty compact convex subset of a normed space E, $S : X \multimap X$ be a continuous mapping with nonempty compact convex values and $Q : X \multimap X$ be a mapping with nonempty convex values and open (in X) fibers. Let $\varphi : X \times X \to \mathbb{R}$ be a continuous function satisfying the following conditions:

(i) for each $x \in X$, the function $\varphi(x, \cdot)$ is quasiconvex;

(ii) for each $y \in X$, the function $\varphi(\cdot, y)$ is quasiconcave;

(iii) for each $x \in X$ and $y \in \mathbf{O}\big(S(x); x\big) \cap Q(x)$, $\varphi(x, x) \leq \varphi(x, y)$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and $\varphi(x, \bar{x}) \leq \varphi(\bar{x}, \bar{x}) \leq \varphi(\bar{x}, y)$, for all $(x, y) \in S(\bar{x}) \times Q(\bar{x})$.

The next application is a common fixed point theorem for two mappings.

Theorem 3.2. Let X be a nonempty compact convex subset of a real normed space, $T : X \multimap X$ be a u.s.c. mapping with nonempty compact convex values and $Q : X \multimap X$ be a mapping with nonempty convex values and open (in X) fibers. If $\mathbf{O}\big(T(x); x\big) \cap Q(x) \setminus \{x\} = \emptyset$ for all $x \in X$, then there exists a $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})) \cap Q(\bar{x}) \neq \emptyset$.

Example 3.1. Let $X = [-2,2]$ and the mappings $T, Q : [-2,2] \multimap [-2,2]$ defined by

$$T(x) = \begin{cases} \left[\frac{x}{2}, -\frac{x^2}{4}\right] & \text{if } x \in [-2,0)\,, \\ \left[\frac{x^2}{4}, \frac{x}{2}\right] & \text{if } x \in [0,2]\,. \end{cases} \quad Q(x) = \begin{cases} (x,0] & \text{if } x \in [-2,0)\,, \\ \{0\} & \text{if } x = 0, \\ [0,x) & \text{if } x \in (0,2]\,. \end{cases}$$

Note that T is usc with nonempty closed convex values. One can easily check that

$$Q^-(y) = \begin{cases} [-2,y) & \text{if } y \in [-2,0)\,, \\ [-2,2] & \text{if } y = 0 \\ (y,2] & \text{if } y \in (0,2]\,. \end{cases}$$

and

$$\mathbf{O}\big(T(x);x\big) \cap Q(x) = \begin{cases} [-2,x] & \text{if } x \in [-2,0)\,, \\ \{0\} & \text{if } x = 0, \\ [x,2] & \text{if } x \in (0,2)\,. \end{cases}$$

Hence Q has open fibers in X and $\mathbf{O}\big(T(x);x\big) \cap Q(x) \setminus \{x\} = \emptyset$ for all $x \in [-2,2]$. The mappings T and Q satisfy all the requirements of Theorem 3.2 and by this theorem T and Q have a common fixed point. Let us observe that the unique common fixed point is $x_0 = 0$.

The last application of Theorem 2.1 is an existence theorem for the solution of a quasivector optimization problem, connected to Pareto optimization. Let X be a nonempty compact convex of a normed space E, Z be a norm space and C be a proper, closed, pointed and convex cone of Z.

For a function $\varphi : X \to Z$ we define the subdifferential of φ in $x \in X$, denoted by $\partial\varphi(x)$, as

$$\partial\varphi(x) = \{u \in L(E,Z)^* : \varphi(y) - \varphi(x) - \langle u, y - x\rangle \in C, \forall y \in X\},$$

where $L(E,Z)^*$ and $\langle u, x\rangle$ denote the space of linear continuous function from E into Z and the evaluation of $u \in (E,Z)^*$ at $x \in E$, respectively.

Theorem 3.3. Let X, Z, C and φ be as above, $T : X \multimap X$ be a u.s.c. mapping with nonempty compact convex values and Q be a mapping with nonempty convex values and open (in X) fibers. Suppose that:

(i) $\partial\varphi$ is a u.s.c. mapping with nonempty compact convex values;

(ii) for each $x \in X$ and $y \in T(x) \cap \mathbf{O}(Q(x); x)$, $\varphi(y) - \varphi(x) \notin \mathrm{int}(C)$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $\varphi(y) - \varphi(\bar{x}) \notin -\mathrm{int}(C)$, for all $y \in Q(\bar{x})$.

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