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MAXIMAL MEASURE ALGEBRAS IN $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$

ILIJAS FARAH

Let $\mathcal{Z}_0$ denote the ideal of asymptotic density zero subsets of $\mathbb{N}$,

$$\mathcal{Z}_0 = \{X \subseteq \mathbb{N} : \lim_{n \to \infty} |X \cap n|/n = 0\}$$

and let $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ denote the quotient Boolean algebra. It is known that $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ includes a measure algebra of Maharam character $2^{\aleph_0}$ as a subalgebra ([3], see also [2] and [1]). In this note I will show that the existence of a maximal measure subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ of character strictly smaller than $2^{\aleph_0}$ is relatively consistent with ZFC, answering a question of David Fremlin.

Let $\mathcal{S}_\kappa$ be the forcing for adding $\kappa$ side-by-side Sacks reals, with countable support. Let $\mathcal{D}$ be the family of all subsets of $\mathbb{N}$ that have density. Let $\mathcal{Z}_0$ be the ideal of sets of asymptotic density zero. Recall that $I_n = [2^n, 2^{n+1})$ and $\nu_n(A) = |A \cap I_n|/2^n$, then $d^*(A) = \lim sup_{n \to \infty} \nu_n(A)$ and $d(A) = \lim_{n \to \infty} \nu_n(A)$, if it exists. A family of sets $\mathcal{A}$ is $\epsilon$-independent with respect to $\mu$ if for every finite $F \subseteq \mathcal{A}$ and every $p : F \to \{\pm 1\}$ we have $|\mu(\cap_{A \in F} A^{p(A)}) - 2^{-|F|}| \leq \epsilon$. Here $\mu$ can be a measure or a convex mean.

If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ and $m \in \mathbb{N}$ then we say that $\mathcal{A}$ is $\epsilon$-independent at $m$ if it is $\epsilon$-independent with respect to $\nu_m$.

A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is a maximal stochastically independent family with respect to $d$ if it is included in $\mathcal{D}$, stochastically independent with respect to $d$, and maximal with respect to these properties.

**Lemma 1.** A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is stochastically independent (with respect to $d$) if and only if for every finite $F \subseteq \mathcal{A}$ and every $\epsilon > 0$ there exists $n$ such that $F$ is $\epsilon$-independent at every $m \geq n$. \qed

Fix an uncountable cardinal $\kappa$.

**Lemma 2.** Assume CH. Then there is a family $\{A_\alpha : \alpha < \omega_1\}$ that is maximal stochastically independent with respect to $d$ such that in the extension by $\mathcal{S}_\kappa$ it remains maximal.

**Proof.** Let $(P_\alpha, \hat{r}_\alpha)$ $(\alpha < \omega_1)$ enumerate all pairs such that $P_\alpha$ is a condition in $\mathcal{S}_{\omega_1}$ and $\hat{r}_\alpha$ is a name for a subset of $\mathbb{N}$. We construct

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I would like to thank David Fremlin for conversations on this topic, and in particular for providing a direct proof of Theorem 3 from Lemma 2.
$A_{\alpha}$ ($\alpha < \omega_1$) by recursion. Assume $A_{\delta} = \{A_{\alpha} : \alpha < \delta\}$ has been constructed. Consider $(P_{\delta}, \dot{r}_{\delta})$. If $P_{\delta}$ does not force that $A_{\delta} \cup \{\dot{r}_{\delta}\}$ is stochastically independent, choose any $A_{\delta}$ such that $A_{\delta} \cup \{A_{\delta}\}$ is stochastically independent.

Otherwise, find a fusion sequence $Q_t$ ($t \in T$) of conditions extending $P_{\delta}$ indexed by a perfect tree $T \subseteq 2^{\omega_1}$ and $\{n_i : i \in \mathbb{N}\}$ as follows. Let $T_k$ be the $k$-th level of $T$, and re-enumerate $A_{\alpha}$ as $\{A_i : i \in \mathbb{N}\}$. Write $A_k = \{A_i^k : i \leq k\}$.

1. $(i + 1)n_i < n_{i+1}$, $n_1 = 1$.
2. $T$ branches only at the $n_i + 1$-st level for $i \in \mathbb{N}$, and only once. Thus $|T_{n_i}| = i$.
3. $Q_s \subseteq Q_t$ if $s \subseteq t$.
4. The fusion $Q = \bigcup_{f \in [T]} \cap_{n=1}^{\infty} Q_{frn}$ is a condition in $S_{\omega_1}$.
5. If $t \in T_{n_i}$, then $Q_t$ forces that $A_i^t \cup \{\dot{r}\}$ is $2^{-i}$-independent at every $m \geq n_{i+1}$.
6. If $t \in T_{n_i}$, then $Q_t$ decides $\dot{r} \cap I_m$ for all $m < n_{i+1}$.

This can be accomplished by using the standard means. That such sequences can be found is the only property of $S_\kappa$ that we shall need.

Enumerate each $T_{n_i}$ as $t_i^1, \ldots, t_i^i$, and write $t_j^i = t_i^i$ for $j > i$. Now pick $A_{\delta}$ so that for all $i$ and $j < n_{i+1} - n_i$ we have

7. $A_{\delta} \cap I_{n_i+j} = u_j^i$, where $Q_{t_j^i} \models \dot{r} \cap I_{n_i+j} = u_j^i$.

Then for every $i$ the family $A_i^t \cup \{A_{\delta}\}$ is $2^{-i}$-independent at each $m \geq n_{i+1}$, hence $A_{\delta} \cup \{A_{\delta}\}$ is stochastically independent.

It remains to prove that $A = \{A_{\alpha} : \alpha < \omega_1\}$ is maximal in the extension by $S_\kappa$. We need to prove that for every name $\dot{r}$ for a subset of $\mathbb{N}$ and every condition $P$, $P$ does not force that $A \cup \{\dot{r}\}$ is independent. Assume otherwise. We may assume $\kappa = \omega_1$, by picking an elementary submodel $M$ of a sufficiently large $H_\lambda$ such that $M$ is closed under $\omega$-sequences, of size $\aleph_1$, and large enough, and intersecting $S_\kappa$ with $M$.

Fix $\delta$ such that $(P_{\delta}, \dot{r}_{\delta}) = (P, \dot{r})$. We claim that $Q$ as in (4) forces that $\{A_{\delta}, \dot{r}\}$ is not independent. Otherwise some $R \subseteq Q$ decides $i$ such that $\{A_{\delta}, \dot{r}\}$ are $1/4$-independent at all $m \geq n_i$. But some $Q_t$, for $t \in T_{n_i}$, is compatible with $R$, and by (7) it forces that $A_{\delta} \cap I_m = \dot{r} \cap I_m$ for some $m \geq n_i$, a contradiction. \qed

**Theorem 3.** Assume CH. Then there is a subalgebra $B$ of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ such that $(B, d)$ is a measure algebra of Maharam character $\aleph_1$ and in the extension by $S_\kappa$ it is a maximal subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ with this property.

After preliminary lemmas, we give two proofs of this theorem. The first one is shorter and it uses Lemma 2, while the second one provides
a more robust object and involves an extension of the proof of Lemma 2 that may be of an independent interest.

Lemma 4. If $A \in \mathcal{D}$ and $f: A \to \mathbb{N}$ is a strictly increasing surjection, then $d^*(f(B)) = d^*(B)d(A)$.

Proof. Let $A = \{n_i : i \in \mathbb{N}\}$ be its increasing enumeration, and let $g: \mathbb{N} \to \mathbb{N}$ be such that $g(m) = |A \cap m|$. and let $B = \{n_i : i \in C\}$. Then $d^*(B) = \lim \sup_j |B \cap j|/j = \lim \sup_j |B \cap j|/g(j) \cdot g(j)/j$. But $\lim_j g(j)/j = d(A)$, and $\lim \sup_j |B \cap j|/g(j) = d^*(C)$. □

A proof of Theorem 3 using Lemma 2. By Lemma 2, in the extension by $\mathcal{S}_\kappa$ there is a maximal stochastically independent family $\mathcal{A}$ of size $\aleph_1$. By [4, §491], $\mathcal{A}$ generates a subalgebra $\mathcal{B}_0$ of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ that is isomorphic to a measure algebra of character $\aleph_1$, and the measure on $\mathcal{B}$ is given by $d$. Let $\mathcal{B}'$ be a maximal subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ that includes $\mathcal{B}$ and such that $(\mathcal{B}', d)$ is a measure algebra.

Let $\mathcal{B}_A$ denote the factor algebra, $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$. Assume there is a nonzero $A \in \mathcal{B}_0$ such that $(\mathcal{B}_0)\mathcal{A} = \mathcal{B}'_A$. Let $A = \{n_i : i \in \mathbb{N}\}$ be its increasing enumeration. The map $\Phi: \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})$ defined by

\[\Phi(A) = \bigcap_{i=0}^{n-1} A_{i}^{s(i)}\]

satisfies the formula $d(A)d^*(\Phi(B)) = d^*(B)$, by Lemma 4. Therefore it sends $(\mathcal{B}_0)\mathcal{A}$ to a subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ that is its maximal measure subalgebra of Maharam character $\aleph_1$.

We may therefore assume that for every nonzero $A \in \mathcal{B}_0$ the relative Maharam type of $\mathcal{B}_A$ over $(\mathcal{B}_0)\mathcal{A}$ is infinite. By [3, §333], there is a partition of unity $A_i$ ($i \in \mathbb{N}$) such that each $\mathcal{B}_A$ is relatively Maharam homogeneous and atomless. Therefore by applying Maharam’s theorem we may find $A \in \mathcal{B}\setminus \mathcal{B}_0$ such that $\mathcal{A} \cup \{A\}$ is stochastically independent, contradicting the maximality of $\mathcal{A}$. □

Lemma 5. Assume $A_0, \ldots, A_{n-1}$ are stochastically independent in some atomless measure space $(X, \mu)$ and $B$ is a measurable set of measure $1/2$ such that for every Boolean combination $C$ of $A_0, \ldots, A_{n-1}$ we have $\mu(B \Delta C) \geq \varepsilon$ for some $\varepsilon > 0$. Then there is $A_n$ stochastically independent with $A_0, \ldots, A_{n-1}$ and such that $\mu(A_n \cap B) \leq 1/2 - \varepsilon$.

Proof. Let $C_s = \bigcap_{i=0}^{n-1} A_{i}^{s(i)}$ for $s: n \to \{\pm 1\}$. Choose $A_n$ so that $\mu(A_n \cap C_s) = 1/2$ and $\mu(A_n \cap C_s \cap B)$ is minimal for all $s$. □

A proof of Theorem 3 using the proof of Lemma 2. Let $(P_\alpha, \hat{r}_\alpha) (\alpha < \omega_1)$ enumerate all pairs such that $P_\alpha$ is a condition in $\mathcal{S}_{\omega_1}$ and $\hat{r}_\alpha$ is a name for a subset of $\mathbb{N}$. We construct $A_\alpha (\alpha < \omega_1)$ by recursion. Assume $A_\delta = \{A_\alpha : \alpha < \delta\}$ has been constructed. Consider $(P_\delta, \hat{r}_\delta)$.
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If $P_\delta$ forces that $\dot{r}$ belongs to the measure algebra generated by $A_\delta$, or if it does not force that $d(\dot{r}) = 1/2$, then choose any $A_\delta$ such that $A_\delta \cup \{A_\delta\}$ is stochastically independent.

Otherwise, some $P \leq P_\delta$ forces that $\dot{r}$ does not belong to the measure algebra generated by $A_\delta$. If in the forcing extension for every $m$ there is a Boolean combination $C_m$ of elements of $A_\delta$ such that $d(C_m \Delta \dot{r}) \leq 2^{-m}$, then $D = \bigcup_m \bigcap_{n=m}^\infty C_m$ satisfies $d(D \Delta \dot{r}) = 0$. Therefore we may extend $P$ further to decide a rational number $\varepsilon > 0$ such that for every finite Boolean combination $C$ of elements of $A_\delta$ we have $d(C \Delta \dot{r}) \geq \varepsilon$.

Find a fusion sequence $Q_t$ ($t \in T$) of conditions extending $P$ indexed by a perfect tree $T \subseteq 2^{<\mathbb{N}}$ and $\{n_i : i \in \mathbb{N}\}$ as follows. Let $T_k$ be the $k$-th level of $T$, and re-enumerate $A_\alpha$ as $\{A'_i : i \in \mathbb{N}\}$. Write $A'_k = \{A'_i : i \leq k\}$.

\begin{enumerate}
  \item $2(i+1)n_i < n_{i+1}$, $n_1 = 1$,
  \item $T$ branches only at the $n_i + 1$-st level for $i \in \mathbb{N}$, and only once. Thus $|T_{n_i}| = i$.
  \item $Q_t \leq Q_s$ if $s \subseteq t$.
  \item the fusion $Q = \bigcup_{f \in [T]} \bigcap_{n=1}^\infty Q_{frn}$ is a condition in $\mathcal{S}_{\omega_1}$.
  \item If $t \in T_{n_i}$, then for every Boolean combination $C$ of elements of $A'_i$, the condition $Q_t$ forces that
    \[
    \max(\nu_m(C \Delta \dot{r}), \nu_m(d(C \setminus \dot{r}))) \geq \varepsilon/2
    \]
    for every $m \geq n_{i+1}$.
  \item $A'_i$ is $2^{-i-1}$-independent at each $m \geq n_{i+1}$.
  \item If $t \in T_{n_i}$, then $Q_t$ decides $\dot{r} \cap I_m$ for all $m < n_{i+1}$.
\end{enumerate}

This can be accomplished by using the standard means. That such sequences can be found is the only property of $\mathcal{S}_\alpha$ that we shall need.

Enumerate each $T_{n_i}$ as $t'_1, \ldots t'_i$, and write $t'_{j+1} = t'_i$ for $j > i$. Now pick $A_\delta$ so that for all $i$ and $j < n_{i+1} - n_i$ we have (let $m = n_i + j$)

\begin{enumerate}
  \item $A_\delta$ is $2^{-i}$-independent with $A'_i$ at $m$, and
  \item If $j < i$, then $Q_{t'_{j+1}} \Vdash \nu_m(A_\delta \cap \dot{r}) \leq 1/2 - \varepsilon/2 + 2^{-i}$,
  \item If $j \geq i$, then $Q_{t'_{j+1}} \Vdash \nu_m(A_\delta \setminus \dot{r}) \leq 1/2 - \varepsilon/2 + 2^{-i}$.
\end{enumerate}

This can be achieved by using a discrete version of Lemma 5. Then for every $i$ the family $A'_i \cup \{A_\delta\}$ is $2^{-i}$-independent at each $m \geq n_{i+1}$, hence $A_\delta \cup \{A_\delta\}$ is stochastically independent.

It remains to prove that the algebra $B$ generated by $A = \{A_\alpha : \alpha < \omega_1\}$ is maximal in the extension by $\mathcal{S}_\alpha$. We will prove that for every name $\dot{r}$ for a subset of $\mathbb{N}$ and every condition $P$, $P$ either forces that $\dot{r}$ belongs to $B$ or some extension of $P$ forces that $A_\delta \cap \dot{r} \notin \mathcal{D}$. 

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Assume otherwise, and let \((P, \dot{r})\) be a pair such that \(P\) forces that \(\dot{r}\) does not belong to \(B\) and that \(\dot{r} \cap A_\delta\) is in \(\mathcal{D}\) for all \(\delta\). We may assume that \(P \models d(\dot{r}) = 1/2\) and \(\kappa = \omega_1\).

Fix \(\delta\) such that \((P_\delta, \dot{r}_\delta) = (P, \dot{r})\). Let \(\varepsilon > 0\) be as in the construction of \(A_\delta\). Then \(Q\) as in (11) forces that \(d(A_\delta \cap \dot{r})\) is not defined. Otherwise some \(R \leq Q\) forces that for some rational \(a \in [0, 1]\) we have \(|d(A_\delta \cap \dot{r}) - a| < \varepsilon/8\). By extending \(R\) further, we may decide \(i\) such that for all \(m \geq n_i\)

\[(18) \ R \models |\nu_m(A_\delta \cap \dot{r}) - a| < \varepsilon/8.\]

We may assume that \(i\) is large enough so that \(2^{-i} < \varepsilon/4\). But some \(Q_t\) for \(t \in T_{n_i}\) is compatible with \(R\), and by (16) and (17) it forces that there are \(m\) and \(m'\) greater than \(n_i\) such that

\[R \models |\nu_m(A_\delta \cap \dot{r}) - \nu_{m'}(A_\delta \cap \dot{r})| \geq 2\varepsilon/2 - 2^{-i+1} > \varepsilon/2.\]

But this contradicts (18).

\[\square\]

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