MAXIMAL MEASURE ALGEBRAS IN $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$

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Let $\mathcal{Z}_0$ denote the ideal of asymptotic density zero subsets of $\mathbb{N}$,

$$\mathcal{Z}_0 = \{ X \subseteq \mathbb{N} : \lim_{n \to \infty} |X \cap n|/n = 0 \}$$

and let $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ denote the quotient Boolean algebra. It is known that $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ includes a measure algebra of Maharam character $2^{\aleph_0}$ as a subalgebra ([3], see also [2] and [1]). In this note I will show that the existence of a maximal measure subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ of character strictly smaller than $2^{\aleph_0}$ is relatively consistent with ZFC, answering a question of David Fremlin.

Let $\mathcal{S}_\kappa$ be the forcing for adding $\kappa$ side-by-side Sacks reals, with countable support. Let $\mathcal{D}$ be the family of all subsets of $\mathbb{N}$ that have density. Let $\mathcal{Z}_0$ be the ideal of sets of asymptotic density zero. Recall that $I_n = [2^n, 2^{n+1})$ and $\nu_n(A) = |A \cap I_n|/2^n$, then $d^*(A) = \limsup_{n \to \infty} \nu_n(A)$ and $d(A) = \lim_{n \to \infty} \nu_n(A)$, if it exists. A family of sets $\mathcal{A}$ is $\varepsilon$-independent with respect to $\mu$ if for every finite $F \subseteq \mathcal{A}$ and every $p: F \to \{ \pm 1 \}$ we have $|\mu(\bigcap_{A \in F} A^{p(A)}) - 2^{-|F|}| \leq \varepsilon$. Here $\mu$ can be a measure or a convex mean.

If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ and $m \in \mathbb{N}$ then we say that $\mathcal{A}$ is $\varepsilon$-independent at $m$ if it is $\varepsilon$-independent with respect to $\nu_m$.

A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is a maximal stochastically independent family with respect to $d$ if it is included in $\mathcal{D}$, stochastically independent with respect to $d$, and maximal with respect to these properties.

Lemma 1. A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is stochastically independent (with respect to $d$) if and only if for every finite $F \subseteq \mathcal{A}$ and every $\varepsilon > 0$ there exists $n$ such that $F$ is $\varepsilon$-independent at every $m \geq n$. \hfill $\Box$

Fix an uncountable cardinal $\kappa$.

Lemma 2. Assume CH. Then there is a family $\{ A_\alpha : \alpha < \omega_1 \}$ that is maximal stochastically independent with respect to $d$ such that in the extension by $\mathcal{S}_\kappa$ it remains maximal.

Proof. Let $(P_\alpha, \hat{r}_\alpha) (\alpha < \omega_1)$ enumerate all pairs such that $P_\alpha$ is a condition in $\mathcal{S}_{\omega_1}$ and $\hat{r}_\alpha$ is a name for a subset of $\mathbb{N}$. We construct

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I would like to thank David Fremlin for conversations on this topic, and in particular for providing a direct proof of Theorem 3 from Lemma 2.
$A_\alpha$ ($\alpha < \omega_1$) by recursion. Assume $\mathcal{A}_\delta = \{A_\alpha : \alpha < \delta\}$ has been constructed. Consider $(P_\delta, \hat{r}_\delta)$. If $P_\delta$ does not force that $\mathcal{A}_\delta \cup \{\hat{r}_\delta\}$ is stochastically independent, choose any $A_\delta$ such that $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

Otherwise, find a fusion sequence $Q_t$ ($t \in T$) of conditions extending $P_\delta$ indexed by a perfect tree $T \subseteq 2^{\omega_1}$ and $\{n_i : i \in \mathbb{N}\}$ as follows. Let $T_k$ be the $k$-th level of $T$, and re-enumerate $\mathcal{A}_\alpha$ as $\{A'_i : i \in \mathbb{N}\}$. Write $A'_k = \{A'_i : i \leq k\}$.

1. $(i+1)n_i < n_{i+1}$, $n_1 = 1$,
2. $T$ branches only at the $n_i + 1$-st level for $i \in \mathbb{N}$, and only once. Thus $|T_{n_i}| = i$.
3. $Q_t \leq Q_s$ if $s \subseteq t$.
4. the fusion $Q = \bigcup_{f \in [T]} \bigcap_{n=1}^{\infty} Q_{f|n}$ is a condition in $\mathcal{S}_{\omega_1}$.
5. If $t \in T_{n_i}$, then $Q_t$ forces that $\mathcal{A}'_i \cup \{\hat{r}\}$ is $2^{-i}$-independent at every $m \geq n_i + 1$.
6. If $t \in T_{n_i}$, then $Q_t$ decides $\hat{r} \cap I_m$ for all $m < n_i + 1$.

This can be accomplished by using the standard means. That such sequences can be found is the only property of $\mathcal{S}_\kappa$ that we shall need.

Enumerate each $T_{n_i}$ as $t_1, \ldots, t_i$, and write $t_j^i = t_j^i$ for $j > i$. Now pick $\mathcal{A}_\delta$ so that for all $i$ and $j < n_i + 1 - n_i$ we have

7. $A_\delta \cap I_{n_i+j} = u_j^i$, where $Q_{t_j^i} \models \hat{r} \cap I_{n_i+j} = u_j^i$.

Then for every $i$ the family $\mathcal{A}'_i \cup \{A_\delta\}$ is $2^{-i}$-independent at each $m \geq n_i + 1$, hence $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

It remains to prove that $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is maximal in the extension by $\mathcal{S}_\kappa$. We need to prove that for every name $\hat{r}$ for a subset of $\mathbb{N}$ and every condition $P$, $P$ does not force that $\mathcal{A} \cup \{\hat{r}\}$ is independent. Assume otherwise. We may assume $\kappa = \omega_1$, by picking an elementary submodel $M$ of a sufficiently large $H_\lambda$ such that $M$ is closed under $\omega$-sequences, of size $\aleph_1$, and large enough, and intersecting $\mathcal{S}_\kappa$ with $M$.

Fix $\delta$ such that $(P_\delta, \hat{r}_\delta) = (P, \hat{r})$. We claim that $Q$ as in (4) forces that $\{A_\delta, \hat{r}\}$ is not independent. Otherwise some $R \leq Q$ decides $i$ such that $\{A_\delta, \hat{r}\}$ are $1/4$-independent at all $m \geq n_i$. But some $Q_t$, for $t \in T_{n_i}$ is compatible with $R$, and by (7) it forces that $A_\delta \cap I_m = \hat{r} \cap I_m$ for some $m \geq n_i$, a contradiction. 

**Theorem 3.** Assume $CH$. Then there is a subalgebra $\mathcal{B}$ of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ such that $(\mathcal{B}, d)$ is a measure algebra of Maharam character $\aleph_1$ and in the extension by $\mathcal{S}_\kappa$ it is a maximal subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ with this property.

After preliminary lemmas, we give two proofs of this theorem. The first one is shorter and it uses Lemma 2, while the second one provides
a more robust object and involves an extension of the proof of Lemma 2 that may be of an independent interest.

**Lemma 4.** If \( A \in \mathcal{D} \) and \( f : A \to \mathbb{N} \) is a strictly increasing surjection, then \( d^*(f(B)) = d^*(B)d(A) \).

**Proof.** Let \( A = \{ n_i : i \in \mathbb{N} \} \) be its increasing enumeration, and let \( g : \mathbb{N} \to \mathbb{N} \) be such that \( g(m) = |A \cap m| \). and let \( B = \{ n_i : i \in C \} \).

Then \( d^*(B) = \limsup |B \cap j|/j = \limsup |B \cap j|/g(j) \cdot g(j)/j \). But \( \lim_j g(j)/j = d(A) \), and \( \limsup |B \cap j|/g(j) = d^*(C) \). \( \square \)

A proof of Theorem 3 using Lemma 2. By Lemma 2, in the extension by \( \mathcal{S}_\kappa \) there is a maximal stochastically independent family \( \mathcal{A} \) of size \( \mathfrak{N}_1 \). By [4, §491], \( \mathcal{A} \) generates a subalgebra \( \mathcal{B}_0 \) of \( \mathcal{P}(\mathbb{N})/\mathcal{Z}_0 \) that is isomorphic to a measure algebra of character \( \mathfrak{N}_1 \), and the measure on \( \mathcal{B} \) is given by \( d \). Let \( \mathcal{B}' \) be a maximal subalgebra of \( \mathcal{P}(\mathbb{N})/\mathcal{Z}_0 \) that includes \( \mathcal{B} \) and such that \( (\mathcal{B}', d) \) is a measure algebra.

Let \( \mathcal{B}_A \) denote the factor algebra, \( \mathcal{B}_A = \{ B \cap A : B \in \mathcal{B} \} \). Assume there is a nonzero \( A \in \mathcal{B}_0 \) such that \( (\mathcal{B}_0)_A = \mathcal{B}'_A \). Let \( A = \{ n_i : i \in \mathbb{N} \} \) be its increasing enumeration. The map \( \Phi : \mathcal{P}(A) \to \mathcal{P}(\mathbb{N}) \) defined by

\[
\Phi(\{ n_j : j \in C \}) = C
\]

satisfies the formula \( d(A)d^*(\Phi(B)) = d^*(B) \), by Lemma 4. Therefore it sends \( (\mathcal{B}_0)_A \) to a subalgebra of \( \mathcal{P}(\mathbb{N})/\mathcal{Z}_0 \) that is its maximal measure subalgebra of Maharam character \( \mathfrak{N}_1 \).

We may therefore assume that for every nonzero \( A \in \mathcal{B}_0 \) the relative Maharam type of \( \mathcal{B}_A \) over \( (\mathcal{B}_0)_A \) is infinite. By [3, §333], there is a partition of unity \( A_i \) (\( i \in \mathbb{N} \)) such that each \( \mathcal{B}_A \) is relatively Maharam homogeneous and atomless. Therefore by applying Maharam's theorem we may find \( A \in \mathcal{B} \setminus \mathcal{B}_0 \) such that \( \mathcal{A} \cup \{ A \} \) is stochastically independent, contradicting the maximality of \( \mathcal{A} \). \( \square \)

**Lemma 5.** Assume \( A_0, \ldots, A_{n-1} \) are stochastically independent in some atomless measure space \((X, \mu) \) and \( B \) is a measurable set of measure \( 1/2 \) such that for every Boolean combination \( C \) of \( A_0, \ldots, A_{n-1} \) we have \( \mu(B \Delta C) \geq \epsilon \) for some \( \epsilon > 0 \). Then there is \( A_n \) stochastically independent with \( A_0, \ldots, A_{n-1} \) and such that \( \mu(A_n \cap B) \leq 1/2 - \epsilon \).

**Proof.** Let \( C_s = \bigcap_{i=0}^{n-1} A_i^{s(i)} \) for \( s : n \to \{ \pm 1 \} \). Choose \( A_n \) so that \( \mu(A_n \cap C_s) = 1/2 \) and \( \mu(A_n \cap C_s \cap B) \) is minimal for all \( s \). \( \square \)

A proof of Theorem 3 using the proof of Lemma 2. Let \( (P_\alpha, \dot{r}_\alpha) \) (\( \alpha < \omega_1 \)) enumerate all pairs such that \( P_\alpha \) is a condition in \( \mathcal{S}_{\omega_1} \) and \( \dot{r}_\alpha \) is a name for a subset of \( \mathbb{N} \). We construct \( A_\alpha \) (\( \alpha < \omega_1 \)) by recursion. Assume \( \mathcal{A}_\delta = \{ A_\alpha : \alpha < \delta \} \) has been constructed. Consider \( (P_\delta, \dot{r}_\delta) \).
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If $P_\delta$ forces that $\dot{r}$ belongs to the measure algebra generated by $\mathcal{A}_\delta$, or if it does not force that $d(\dot{r}) = 1/2$, then choose any $A_\delta$ such that $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

Otherwise, some $P \leq P_\delta$ forces that $\dot{r}$ does not belong to the measure algebra generated by $\mathcal{A}_\delta$. If in the forcing extension for every $m$ there is a Boolean combination $C_m$ of elements of $\mathcal{A}_\delta$ such that $d(C_m \Delta \dot{r}) \leq 2^{-m}$, then $D = \bigcup_m \bigcap_{n=m}^\infty C_m$ satisfies $d(D \Delta \dot{r}) = 0$. Therefore we may extend $P$ further to decide a rational number $\varepsilon > 0$ such that for every finite Boolean combination $C$ of elements of $\mathcal{A}_\delta$ we have $d(C \Delta \dot{r}) \geq \varepsilon$.

Find a fusion sequence $Q_t$ $(t \in T)$ of conditions extending $P$ indexed by a perfect tree $T \subseteq 2^{<\mathbb{N}}$ and $\{n_i : i \in \mathbb{N}\}$ as follows. Let $T_k$ be the $k$-th level of $T$, and re-enumerate $\mathcal{A}_\alpha$ as $\{A'_i : i \in \mathbb{N}\}$. Write $\mathcal{A}'_k = \{A'_i : i \leq k\}$.

(8) $2(i + 1)n_i < n_{i+1}$, $n_1 = 1$,
(9) $T$ branches only at the $n_i + 1$-st level for $i \in \mathbb{N}$, and only once. Thus $|T_{n_i}| = i$.
(10) $Q_t \leq Q_s$ if $s \subseteq t$.
(11) the fusion $Q = \bigcup_{f \in [T]} \bigcap_{n=1}^\infty Q_f | n$ is a condition in $\mathcal{S}_{\omega_1}$.
(12) If $t \in T_{n_i}$, then for every Boolean combination $C$ of elements of $\mathcal{A}'_i$, the condition $Q_t$ forces that

$$\max(\nu_m(C \Delta \dot{r}), \nu_m(d(C \setminus \dot{r}))) \geq \varepsilon/2$$

for every $m \geq n_{i+1}$.

(13) $\mathcal{A}'_i$ is $2^{-i-1}$-independent at each $m \geq n_{i+1}$.
(14) If $t \in T_{n_i}$, then $Q_t$ decides $\dot{r} \cap I_m$ for all $m < n_{i+1}$.

This can be accomplished by using the standard means. That such sequences can be found is the only property of $\mathcal{S}_\alpha$ that we shall need.

Enumerate each $T_{n_i}$ as $t'_1, \ldots t'_i$, and write $t'_j = t'_i$ for $j > i$. Now pick $A_\delta$ so that for all $i$ and $j < n_{i+1} - n_i$ we have (let $m = n_i + j$)

(15) $A_\delta$ is $2^{-i}$-independent with $\mathcal{A}'_i$ at $m$, and
(16) If $j < i$, then $Q_{t'_i} \models \nu_m(A_\delta \cap \dot{r}) \leq 1/2 - \varepsilon/2 + 2^{-i}$,
(17) If $j \geq i$, then $Q_{t'_i} \models \nu_m(A_\delta \setminus \dot{r}) \leq 1/2 - \varepsilon/2 + 2^{-i}$.

This can be achieved by using a discrete version of Lemma 5. Then for every $i$ the family $\mathcal{A}'_i \cup \{A_\delta\}$ is $2^{-i}$-independent at each $m \geq n_{i+1}$, hence $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

It remains to prove that the algebra $\mathcal{B}$ generated by $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is maximal in the extension by $\mathcal{S}_\alpha$. We will prove that for every name $\dot{r}$ for a subset of $\mathbb{N}$ and every condition $P$, $P$ either forces that $\dot{r}$ belongs to $\mathcal{B}$ or some extension of $P$ forces that $A_\delta \cap \dot{r} \notin \mathcal{D}$. 
Assume otherwise, and let \((P, \dot{r})\) be a pair such that \(P\) forces that \(\dot{r}\) does not belong to \(B\) and that \(\dot{r} \cap A_{\delta}\) is in \(\mathcal{D}\) for all \(\delta\). We may assume that \(P \models d(\dot{r}) = 1/2\) and \(\kappa = \omega_1\).

Fix \(\delta\) such that \((P_{\delta}, \dot{r}_{\delta}) = (P, \dot{r})\). Let \(\varepsilon > 0\) be as in the construction of \(A_{\delta}\). Then \(Q\) as in (11) forces that \(d(A_{\delta} \cap \dot{r})\) is not defined. Otherwise some \(R \leq Q\) forces that for some rational \(a \in [0, 1]\) we have \(|d(A_{\delta} \cap \dot{r}) - a| < \varepsilon/8\). By extending \(R\) further, we may decide \(i\) such that for all \(m \geq n_i\)

\[ R \models |\nu_m(A_{\delta} \cap \dot{r}) - a| < \varepsilon/8. \]

We may assume that \(i\) is large enough so that \(2^{-i} < \varepsilon/4\). But some \(Q_t\) for \(t \in T_{n_i}\) is compatible with \(R\), and by (16) and (17) it forces that there are \(m\) and \(m'\) greater than \(n_i\) such that

\[ R \models |\nu_m(A_{\delta} \cap \dot{r}) - \nu_{m'}(A_{\delta} \cap \dot{r})| \geq 2\varepsilon/2 - 2^{-i+1} > \varepsilon/2. \]

But this contradicts (18). \(\square\)

REFERENCES


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